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OPTIMAL APPROXIMATION OF FOURIER INTEGRALS BY THE PHI-FUNCTION METHOD

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This paper investigates the construction of an optimal quadrature formula for Fourier integrals of $f(x)$ function using the method of φ-functions. The function $f(x)$ belongs to the Hilbert space, which consists of absolutely continuous functions with quadratically integrable first-order derivatives. We explore the problem of optimality in the sense of Sard, aiming to minimize the supremum of the absolute error $|R_n(f)|$ over all functions in. The method of φ-functions provides a means to derive the coefficients A_k that achieve this minimal error. Our work details the φ-function method and its application to constructing optimal quadrature formulas, presenting a systematic approach to minimizing the quadrature error.

Keywords: Hilbert space, phi-function method, the optimal quadrature formula, error of the quadrature formula.

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1 Introduction

Numerical calculation of integrals of highly oscillating functions is one of the more important problems of numerical analysis, because such integrals are encountered in applications in many branches of mathematics as well as in other science such as quantum physics, flow mechanics and electromagnetism. Main examples of strongly oscillating integrands are encountered in different transformation, for example, the Fourier transformation and Fourier-Bessel transformation. Standard methods of numerical integration frequently require more computational works and they cannot be successfully applied.

For integrals with different type highly oscillating functions many special effective methods such as Filon-type method, Clenshaw-Curtis-Filon type method, Levin type methods, modified Clenshaw-Curtis method, generalized quadrature rule, Gauss-Laguerre quadrature are worked out (see, for example, [1–3, 6], for more review see, for instance, [4, 5] and references therein).

This paper is devoted to construction of optimal quadrature formulas for approximate calculation of Fourier integrals in a Hilbert space of non-periodic functions by φ function method.

Given the known geometric value, the problem of finding the numerical value of integrals is often called quadrature and cubature, respectively. Various quadrature and cubature methods allow the calculation of the integral using a finite number of values of the integrated function. These methods are universal and can be used where other calculation methods fail. Currently, in the theory of constructing quadrature and cubature formulas, there are the following main approaches: *algebraic, probabilistic, numerical theoretic and functional.* Since the research in this work relates to the latter approach, we will provide an overview of the results in this area.

Note that there are several methods for constructing optimal formulas by minimizing the norm of the error functional over coefficients at fixed nodes: the spline method, the method of φ-functions, and the Sobolev method. A.Sard [21, 22], L.F.Meyers [20], G.Coman [11, 12], I.J.Schoenberg [23, 24], S.D.Silliman [25], and P.Köhler [18] utilized the spline method to construct optimal quadrature formulas in various spaces. A.Ghizzetti and A.Ossicini [13], F.Lanzara [19], T.Catinaş and G.Coman [10] employed the method of φ-functions for this purpose.

It is necessary to consider the problem of approximate calculation of the integral

$$I(\varphi) = \int_a^b e^{2\pi i\omega x}\varphi(x)dx, \tag{1}$$

with real ω. This type of integrals are called *highly oscillating integrals* for large values of ω. In most cases, it is impossible to get the exact values of such integrals. Thus, they can be approximately calculated using the formulas of numerical integration.

Recently, in [7, 9] the problem of construction of optimal quadrature formulas in the sense of Sard for numerical calculation of integrals (1) with integer ω was studied in Hilbert spaces $L_2^{(m)}$ and $W_2^{(m,m-1)}$.

It is worth noting that in recent years, Kh.M. Shadimetov, G.V. Milovanovič, A.R. Hayotov, and N.D. Boltaev [8], as well as A.R. Hayotov, C.-O. Lee, and S. Jeon [14–16], and S.S. Babaev [17], have conducted scientific research in the Hilbert spaces $L_2^{(m)}$ and $W_2^{(m,m-1)}$. Their work focuses on developing optimal quadrature formulas for the approximate calculation of strongly oscillatory integrals and their practical applications. As a result, they have achieved high-resolution reconstruction of Computed Tomography images in both industrial and medical fields under laboratory conditions.

2 Problems of optimal approximation of Fourier integrals

In this work, we study the construction of an optimal quadrature formula using the method of φ-functions. In this regard, consider quadrature formulas of the form

$$\int_a^b e^{2\pi i\omega x}\cdot f(x)dx = \sum_{k=0}^{n} A_k f(x_k) + R_n(f), \tag{2}$$

where A_k and x_k are coefficients and nodes of the quadrature formula. Let the nodes of the formula be located on the segment $[a, b]$ as follows

$$a = x_0 < x_2 < \ldots < x_n = b, \tag{3}$$

and $R_n(f)$ is the residual of formula (2).

Suppose that the integrand function $f(x)$ is from the space $W_{2,\sigma}^{(1,0)}(a, b)$, where $W_{2,\sigma}^{(1,0)}(a, b)$ is the Hilbert space of absolutely continuous functions that are quadratically integrable with the first-order derivative on the interval $[a, b]$. The scalar product of any two functions $f(x)$ and $g(x)$ from this space is defined by the following formula

$$\langle f(x), g(x)\rangle = \int_a^b (f'(x) + \sigma f(x))(g'(x) + \sigma g(x))dx, \tag{4}$$

where $\sigma \in \mathbb{R}$ and $\sigma \neq 0$. This space is provided with the corresponding norm

$$\|f(x)\|_{W_{2,\sigma}^{(1,0)}} = \left(\int_a^b (f'(x) + \sigma f(x))^2 dx\right)^{1/2}. \tag{5}$$

One of the important problems in the theory of quadrature formulas is the problem of the optimality of quadrature formulas relative to the error of this formula. In this paper, we will consider the problem of optimality of a formula in the sense of Sard. We use the one-to-one correspondence between quadrature formulas and φ - functions in this.

For convenience, we introduce the multi-index notations

$$A = (A_0, A_1, \dots, A_n) \text{ and } X = (x_0, x_1, \dots, x_n). \tag{6}$$

Definition 1. The quadrature formula (2) is called *optimal in the sense of Nikolsky* in space $W_{2,\sigma}^{(1,0)}$, if the value

$$F_n(W_{2,\sigma}^{(1,0)}, A, X) = \sup_{f \in W_{2,\sigma}^{(1,0)}} |R_n(f)| \tag{7}$$

reaches its smallest value relative to A and X, where A and X are defined in (6).

Definition 2. The quadrature formula (2) is called *optimal in the sense of Sard* in the space $W_{2,\sigma}^{(1,0)}$ if the quantity

$$F_n(W_{2,\sigma}^{(1,0)}, A) = \sup_{f \in W_{2,\sigma}^{(1,0)}} |R_n(f)|, \tag{8}$$

reaches its smallest value relative to A for fixed X, where A and X are defined in (6).

In this work, we solve the problem of constructing an optimal quadrature formula of the form (2) in the sense of Sard in the space $W_{2,\sigma}^{(1,0)}(a,b)$, i.e., let us find such coefficients of the formula (2) that satisfy the condition (8). In this case, we use the method of φ-functions.

Next, the rest of this work is organized as follows. Section 3 describes the method of φ-functions for constructing quadrature formulas of the form (2) in the space $W_{2,\sigma}^{(1,0)}$ and provides the relationship between the coefficients and φ-functions. Section 4 is devoted to obtaining φ-functions that give the smallest error value of a quadrature formula of the form (2). Using the obtained φ-functions, the coefficients of the optimal quadrature formula of the form (2) are calculated.

3 Method of φ - functions for constructing quadrature formulas in the space $W_{2,\sigma}^{(1,0)}$

In this section, we explain the method of φ - functions for constructing optimal quadrature formulas of the form (2) in the sense of Sard in the space $W_{2,\sigma}^{(1,0)}$.

Let functions $f(x)$ be from the space $W_{2,\sigma}^{(1,0)}(a,b)$ and for a given positive integer n the nodes of the quadrature formula under consideration are located as in (6). Then for each subinterval $[x_{k-1}, x_k]$, $k = 1, 2, \dots, n$, consider the functions φ_k, $k = 1, 2, \dots, n$ having the following property

$$\varphi_k'(x) - \sigma\varphi_k(x) = e^{2\pi i\omega x}, \quad k = 1, 2, \dots, n. \tag{9}$$

Then the function φ is defined as follows

$$\varphi|_{[x_{k-1}, x_k]} = \varphi_k(x), \quad k = 1, 2, \dots, n. \tag{10}$$

That is, the restriction of the function φ to the interval $[x_{k-1}, x_k]$ is equal to φ_k.

Let us introduce the following notation

$$I(f) := \int_a^b e^{2\pi i\omega x} \cdot f(x)dx, \tag{11}$$

$$Q_n(f) := \sum_{k=0}^{n} A_k f(x_k). \tag{12}$$

Now, using the property of additivity of a definite integral, taking into account the equality (9), from (11), we have

$$\begin{aligned}
I(f) &:= \int_a^b e^{2\pi i\omega x} \cdot f(x)dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} (\varphi_k'(x) - \sigma\varphi_k(x))f(x)dx = \\
&= \sum_{k=1}^{n} \Big(\int_{x_{k-1}}^{x_k} \varphi_k'(x)f(x)dx - \sigma \int_{x_{k-1}}^{x_k} \varphi_k(x)f(x)dx \Big) = \\
&= \sum_{k=1}^{n} \Big(\varphi_k(x)f(x)\Big|_{x_{k-1}}^{x_k} - \int_{x_{k-1}}^{x_k} \varphi_k(x)f'(x)dx - \sigma \int_{x_{k-1}}^{x_k} \varphi_k(x)f(x)dx \Big) = \\
&= \sum_{k=1}^{n} \Big(\varphi_k(x)f(x)\Big|_{x_{k-1}}^{x_k} - \int_{x_{k-1}}^{x_k} \varphi_k(x)(f'(x) + \sigma f(x)) \Big) dx = \\
&= \varphi_n(x_n)f(x_n) + \sum_{k=1}^{n-1} \varphi_k(x_k)f(x_k) - \sum_{k=1}^{n-1} \varphi_{k+1}(x_k)f(x_k) - \varphi_1(x_0)f(x_0) - \\
&\quad - \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k(x)\Big(f'(x) + \sigma f(x) \Big) dx.
\end{aligned}$$

From here we have

$$I(f) : \;=\; A_0 f(x_0) + \sum_{k=1}^{n-1} A_k f(x_k) + A_n f(x_n) + R_n[f]. \tag{13}$$

Consequently from (13), we get

$$\begin{aligned}
A_0 &= -\varphi_1(x_0), \\
A_k &= \varphi_k(x_k) - \varphi_{k+1}(x_k), \quad k = 1, 2, \ldots, n-1, \\
A_n &= \varphi_n(x_n)
\end{aligned} \tag{14}$$

and the error of the formula (2) has the form

$$R_n[f] \;=\; -\sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k(x)\Big(f'(x) + \sigma f(x) \Big) dx = -\int_a^b \varphi(x)\Big(f'(x) + \sigma f(x) \Big). \tag{15}$$

Remark 1. Knowing the function φ from (14) we can find the coefficients A_k, $k = 0, 1, \ldots, n$. This method of constructing a quadrature formula is called the method of φ - functions.

Remark 2. From the expression (15) it is clear that the quadrature formula (2) is exact on functions that are a solution to the equation

$$f'(x) + \sigma f(x) = 0. \tag{16}$$

Further, in the next section, we are engaged in calculating the coefficients of the optimal quadrature formula of the form (2) in the space $W_{2,\sigma}^{(1,0)}(a,b)$.

4 The optimality problem for a quadrature formula of the form (2)

In this section, we will discuss the problem of optimality of a quadrature formula of the form (2) in the space $W_{2,\sigma}^{(1,0)}(a,b)$.

Using the Cauchy-Schwartz inequality for the absolute value of the error (15) of the quadrature formula (2) we have the following

$$\begin{aligned} |R_n(f)| &\leqslant \|f'(x)+\sigma f(x)\|_{L_2(a,b)} \left(\int_a^b \varphi^2(x)dx\right)^{1/2} \\ &= \|f(x)\|_{W_{2,\sigma}^{(1,0)}} \|\varphi(x)\|_{L_2(a,b)}. \end{aligned} \tag{17}$$

It should be noted that here the task of constructing an optimal quadrature formula of the form (2) in the sense of Sard in the space $W_{2,\sigma}^{(1,0)}(a,b)$ is the task of finding the coefficients $A=(A_0,A_1,\dots,A_n)$ (for fixed nodes $X=(x_0,x_1,\dots,x_n)$ satisfying the condition (3)) giving the smallest value to the quantity

$$F_n(A)=\int_a^b \varphi^2(x)dx. \tag{18}$$

In turn, this problem is equivalent to finding functions $\varphi_k(x)$, $k=1,2,\dots,n$, satisfying the equation (9) and giving the smallest value to the quantity (18) on each interval $[x_{k-1},x_k]$, $k=1,2,\dots,n$.

Next, from the beginning we will find the functions $\varphi_k(x)$, $k=1,2,\dots,n$ that give the smallest value to the quantity (18) and then using the formulas (14) we will calculate coefficients A_k, $k=0,1,\dots,n$ of the optimal quadrature formula (2).

4.1 Finding functions φ_k

Now we are engaged in finding the functions φ_k on each interval $[x_{k-1},x_k]$ for $k= =1,2,\dots,n$, which are the solution to the equation

$$y'-\sigma y=e^{2\pi i\omega x}. \tag{19}$$

We will seek a solution to this equation in the form of the product $y=uy_1$ of the functions $u(x)$ and $y_1(x)$, where y_1 is the solution to the corresponding homogeneous equation

$$y'-\sigma y=0. \tag{20}$$

It is easy to check that one of the solutions to the equation (20) has the form

$$y_1(x)=e^{\sigma x}. \tag{21}$$

Now we can solve the equation (19). By assumption, the solution to the equation (19) has the form

$$y(x)=uy_1, \tag{22}$$

where $y_1(x)$ is defined by the equality (21). Then we just need to find the function $u(x)$. To find it, we first calculate the first-order derivative of the unknown function $y(x)$ in (22). Then we have

$$y' = u'y_1 + uy_1'. \tag{23}$$

Substituting (22) into (19), taking into account (21), we get

$$u'e^{\sigma x} + u\sigma e^{\sigma x} - u\sigma e^{\sigma x} = e^{2\pi i\omega x}.$$

From here

$$u' = e^{(2\pi i\omega-\sigma)x}.$$

Integrating both sides of the last equality with respect to x, and we find

$$u(x) = \frac{1}{2\pi i\omega - \sigma} \cdot e^{(2\pi i\omega-\sigma)x} + C.$$

This means, taking into account the last equality and (21), to solve (22) the equation (19) we obtain

$$y = \frac{1}{2\pi i\omega - \sigma} \cdot e^{2\pi i\omega x} + Ce^{\sigma x}. \tag{24}$$

Hence, on each interval $[x_{k-1}, x_k]$, $k = 1, 2, \ldots, n$ we take functions $\varphi_k(x)$ in the form (24), t.e.

$$\varphi_k(x) = \frac{1}{2\pi i\omega - \sigma} \cdot e^{2\pi i\omega x} + C_1^{(k)}e^{\sigma x}, \quad x \in [x_{k-1}, x_k],\ k = 1, 2, \ldots, n. \tag{25}$$

From here we conclude that to find the functions $\varphi_k(x)$ we need to find such coefficients $C_1^{(k)}$, $k = 1, 2, \ldots, n$, which give the smallest values to the quantity (18) on each of the intervals $[x_{k-1}, x_k]$ for $k = 1, 2, \ldots, n$. Next, we will find $C_1^{(k)}$ such that the integral of the square of the function $\varphi_k(x)$ defined by the equality (25) on the interval $[x_{k-1}, x_k]$ takes lowest value. In this regard, consider the following functions

$$\mathcal{F}_k(C_1^{(k)}) = \int\limits_{x_{k-1}}^{x_k} \varphi_k^2(x)dx, \quad k = 1, 2, \ldots, n.$$

Then from here, taking into account (25), we have

$$\mathcal{F}_k(C_1^{(k)}) = \int\limits_{x_{k-1}}^{x_k} \left(\frac{1}{2\pi i\omega - \sigma} \cdot e^{2\pi i\omega x} + C_1^{(k)}e^{\sigma x}\right)^2 dx =$$

$$= \int\limits_{x_{k-1}}^{x_k} \left(\frac{e^{2\pi i\omega x}}{2\pi i\omega - \sigma}\right)^2 dx + 2C_1^{(k)}\frac{1}{2\pi i\omega - \sigma}\int\limits_{x_{k-1}}^{x_k} e^{2\pi i\omega x} \cdot e^{\sigma x}dx +$$

$$+(C_1^{(k)})^2 \int\limits_{x_{k-1}}^{x_k} e^{2\sigma x}dx,\ k = 1, 2, \ldots, n.$$

Then calculating the first order derivatives of the function $\mathcal{F}_k(C_1^{(k)})$ with respect to $C_1^{(k)}$ and equating it to zero, we have

$$2C_1^{(k)}\int\limits_{x_{k-1}}^{x_k} e^{2\sigma x}dx + \frac{2}{2\pi i\omega - \sigma}\int\limits_{x_{k-1}}^{x_k} e^{2\pi i\omega x} \cdot e^{\sigma x}dx = 0,\ k = 1, 2, \ldots, n.$$

From the last equality we get the following

$$C_1^{(k)} = -\frac{2\sigma\left(e^{(2\pi i\omega+\sigma)x_{k-1}}-e^{(2\pi i\omega+\sigma)x_k}\right)}{(2\pi i\omega-\sigma)(2\pi i\omega+\sigma)\left(e^{2\sigma x_{k-1}}-e^{2\sigma x_k}\right)},\ k=1,2,\dots,n. \tag{26}$$

It is easy to check that this value $C_1^{(k)}$ gives the smallest value of the function $\mathcal{F}_k(C_1^{(k)})$ on the interval $[x_{k-1},x_k]$. Then, taking into account (26), from (25) we have

$$\varphi_k(x) = \frac{1}{2\pi i\omega-\sigma}\cdot e^{2\pi i\omega x}-\frac{2\sigma\left(e^{(2\pi i\omega+\sigma)x_{k-1}}-e^{(2\pi i\omega+\sigma)x_k}\right)}{(2\pi i\omega-\sigma)(2\pi i\omega+\sigma)\left(e^{2\sigma x_{k-1}}-e^{2\sigma x_k}\right)}e^{\sigma x}, \tag{27}$$
$$x\in[x_{k-1},x_k],\ k=1,2,\dots,n.$$

4.2 Calculation of coefficients of the optimal quadrature formula of the form (2)

Now, using (27), from the formulas (14) we calculate the coefficients A_k, $k=0,1,\dots,n$ of the optimal quadrature formula of the form (2).

First, let's calculate A_0. Then from (14), taking into account $\varphi_1(x)$, we have

$$A_0 = -\varphi_1(x_0) = -\frac{e^{2\pi i\omega x_0}}{2\pi i\omega-\sigma}+\frac{2\sigma e^{\sigma x_0}\left(e^{(2\pi i\omega+\sigma)x_0}-e^{(2\pi i\omega+\sigma)x_1}\right)}{(2\pi i\omega-\sigma)(2\pi i\omega+\sigma)\left(e^{2\sigma x_0}-e^{2\sigma x_1}\right)}. \tag{28}$$

Now let's calculate the coefficients A_k, $k=1,2,\dots,n-1$. Then from (14), taking into account $\varphi_k(x)$ for $k=1,2,\dots,n-1$, we have

$$A_k=\varphi_k(x_k)-\varphi_{k+1}(x_k)=$$
$$=\frac{2\sigma e^{\sigma x_k}}{(2\pi i\omega)^2-(\sigma)^2}\left(\frac{e^{(2\pi i\omega+\sigma)x_k}-e^{(2\pi i\omega+\sigma)x_{k+1}}}{e^{2\sigma x_k}-e^{2\sigma x_{k+1}}}-\frac{e^{(2\pi i\omega+\sigma)x_{k-1}}-e^{(2\pi i\omega+\sigma)x_k}}{e^{2\sigma x_{k-1}}-e^{2\sigma x_k}}\right). \tag{29}$$

Finally, let's calculate the last coefficient A_n. Then, from (14), taking into account (27), we obtain

$$A_n = \varphi_n(x_n) = \frac{e^{2\pi i\omega x_n}}{2\pi i\omega-\sigma}-\frac{2\sigma e^{\sigma x_n}\left(e^{(2\pi i\omega+\sigma)x_{n-1}}-e^{(2\pi i\omega+\sigma)x_n}\right)}{(2\pi i\omega-\sigma)(2\pi i\omega+\sigma)\left(e^{2\sigma x_{n-1}}-e^{2\sigma x_n}\right)}. \tag{30}$$

Thus, summing up the results of (28), (29) and (30), we obtain the following main theorem of this work.

Theorem 1. In the space $W_{2,\sigma}^{(1,0)}(a,b)$ for each fixed positive integer n, there is a unique quadrature formula that is optimal in the sense of Sard of the form

$$\int_a^b e^{2\pi i\omega x}\cdot f(x)dx=\sum_{k=0}^{n}A_kf(x_k)+R_n(f),$$

with coefficients

$$A_0=-\frac{e^{2\pi i\omega x_0}}{2\pi i\omega-\sigma}+\frac{2\sigma e^{\sigma x_0}\left(e^{(2\pi i\omega+\sigma)x_0}-e^{(2\pi i\omega+\sigma)x_1}\right)}{(2\pi i\omega-\sigma)(2\pi i\omega+\sigma)\left(e^{2\sigma x_0}-e^{2\sigma x_1}\right)},$$

$$A_k=\frac{2\sigma e^{\sigma x_k}}{(2\pi i\omega)^2-(\sigma)^2}\left(\frac{e^{(2\pi i\omega+\sigma)x_k}-e^{(2\pi i\omega+\sigma)x_{k+1}}}{e^{2\sigma x_k}-e^{2\sigma x_{k+1}}}-\frac{e^{(2\pi i\omega+\sigma)x_{k-1}}-e^{(2\pi i\omega+\sigma)x_k}}{e^{2\sigma x_{k-1}}-e^{2\sigma x_k}}\right),$$
$$k=1,2,\dots,n-1,$$

$$A_n=\frac{e^{2\pi i\omega x_n}}{2\pi i\omega-\sigma}-\frac{2\sigma e^{\sigma x_n}\left(e^{(2\pi i\omega+\sigma)x_{n-1}}-e^{(2\pi i\omega+\sigma)x_n}\right)}{(2\pi i\omega-\sigma)(2\pi i\omega+\sigma)\left(e^{2\sigma x_{n-1}}-e^{2\sigma x_n}\right)},$$

for fixed nodes x_k, $k = 0, 1, \dots, n$ satisfying the inequality $a = x_0 < x_1 < \dots < x_k = b$.

Remark 1. It should be noted that for $[a, b] = [0, 1]$ and $x_k = kh$, where $k = 0, 1, \dots, N$, $h = 1/N$ from Theorem 1 we obtain the result of the work [**?**].

5 Conclusion

In this work, we constructed an optimal quadrature formula in the space $W_{2,\sigma}^{(1,0)}(a, b)$, where $W_{2,\sigma}^{(1,0)}(a, b)$ is the Hilbert space of absolutely continuous functions whose first-order derivatives are square-integrable on the interval $[a, b]$. Here the quadrature sum consists of a linear combination of the values $f(x_k)$ of the function $f(x)$ at the nodes $x_k \in [a, b]$, where $a = x_0 < x_1 < ... < x_n = b$. The error of the quadrature formula under consideration is estimated from above using the product of the norm of the integrand and the L_2 norm of the particular φ function from the space $W_{2,\sigma}^{(1,0)}(a, b)$. Moreover, this φ function is determined by an unknown factor on each subinterval. The optimal quadrature formula is obtained by choosing these factors, which provide the smallest value of the L_2-norm of the φ function. In this work, we found such a φ - function. Explicit coefficients for optimal quadrature found using φ-function. The resulting quadrature formula is exact for the functions $e^{\sigma x}$ and $e^{-\sigma x}$. In particular, well-known results are obtained from the results of this work.

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ОПТИМАЛЬНОЕ АППРОКСИМАЦИЯ ИНТЕГРАЛОВ ФУРЬЕ МЕТОДОМ ФИ-ФУНКЦИЙ

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Данная работа посвящена построению оптимальной квадратурной формулы для интегралов Фурье функции $f(x)$ с использованием метода φ-функций. Функция $f(x)$ принадлежит гильбертовому пространству, которое состоит из абсолютно непрерывных функций с квадратично интегрируемыми производными первого порядка. Мы рассматриваем проблему оптимальности в смысле Сарда, стремясь минимизировать супремум абсолютной ошибки $|R_n(f)|$ для всех функций в. Метод φ-функций предоставляет способ получения коэффициентов A_k, которые достигают этой минимальной ошибки. Наша работа описывает метод φ-функций и его применение к построению оптимальных квадратурных формул, предлагая систематический подход к минимизации квадратурной ошибки.

Ключевые слова: Гильбертово пространство, метод фи-функций, оптимальная квадратурная формула, погрешность формулы.

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