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A MASS-CONSERVING MODEL OF POLLUTANT TRANSPORT IN THE ATMOSPHERE OVER COMPLEX TERRAIN

^{1*}Nasrullayev P.A., ¹Juraboyeva O.S., ²Samanova G.B.

*uzbekpy@gmail.com

¹Digital Technologies and Artificial Intelligence Development Research Institute, 17A, Buz-2, Tashkent, 100125 Uzbekistan;

²Tashkent State Medical University, 2, Farobiy str., Tashkent, 100109 Uzbekistan

A mathematical model and a verified numerical method are developed for the transport of passive and settling pollutants in the atmospheric boundary layer over complex terrain. The advection–diffusion equation is written in the terrain-following coordinate $\sigma = (z - h)/(H_{top} - h)$ in fully conservative (Jacobian-weighted) form. The contravariant vertical velocity is diagnosed from the discrete continuity equation, so that the kinematic condition at the terrain surface is satisfied exactly and no mass can cross the ground. The horizontal wind, prescribed from a single-point forecast or reanalysis through a logarithmic profile in the height above ground, is made mass-consistent by a column-flux normalisation under a rigid lid. Turbulent diffusivities follow Monin–Obukhov similarity with a parabolic decay towards the top of the layer, and the stability parameter is obtained from the hourly solar radiation and air temperature. The finite-volume discretisation uses shared face fluxes, first-order upwinding, and an explicit step chosen from a local positivity bound. It conserves mass to round-off, preserves a uniform field exactly, and keeps concentrations non-negative without clipping. Verification includes an analytical point-source solution (relative L_2 error 1.1%, $R = 0.9999$, $FAC2 = 1$ on the finest grid), grid and time-step refinement, and a closed global budget with residual below 2×10^{-12} . It is also shown that the widely used simplification of a horizontal wind ($w = 0$) in the σ -equation violates the kinematic surface condition: over a 200 m ridge it removes more than 60% of the emitted mass within 15 min and produces a spurious lee-side attenuation of two to three orders of magnitude. With the conservative formulation, a ridge changes near-ground concentrations by 2–51%, the largest effect occurring for an elevated source under stable stratification, whereas atmospheric stability changes them by more than an order of magnitude. The model is coupled with the Copernicus digital elevation model and ERA5 reanalysis data and applied to the city of Tashkent.

Keywords: atmospheric boundary layer, pollutant dispersion, terrain-following coordinate, finite-volume method, mass conservation, Monin–Obukhov similarity, Copernicus DEM

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1 Introduction

Air quality in cities located in foothills and intermontane valleys is controlled by two groups of factors that act on different time scales: the orography, which is fixed, and the meteorological regime, which changes within hours through the diurnal cycle of stability, calm periods, and wind turning [1, 2]. Practical regulatory tools such as AERMOD [3]

and CALPUFF [4] parameterise terrain effects within Gaussian or puff formulations. Lagrangian particle models [5] and comprehensive Eulerian chemical-transport systems [6] resolve the transport explicitly, but they require three-dimensional meteorological fields produced by a mesoscale model and substantial computing resources. Laboratory and theoretical studies of stratified flow over hills [7] and of neutral flow over low hills [8] show that the near-surface flow either passes over or is deflected around an obstacle depending on stability. Near-ground concentrations are therefore sensitive both to the terrain shape and to the source height relative to the terrain.

A standard way of representing an irregular lower boundary in an Eulerian model is a terrain-following vertical coordinate [9–11], which maps the physical domain onto a rectangular computational domain. The mapping brings a Jacobian into the transport equation and a contravariant vertical velocity into its advective part. Both have to be treated consistently: the tracer equation must be written in Jacobian-weighted flux form for the discrete mass to be conserved, and the contravariant velocity must vanish at the ground for the terrain surface to be impermeable. The latter condition is easily lost in simplified models in which the wind is assumed horizontal in physical space ($w = 0$). The transformed vertical velocity then does not vanish at the surface and tracer mass leaves the domain through the ground. As shown below, this defect is not small: it can dominate the solution over a ridge of moderate height.

Turbulent exchange in the surface layer is described by Monin–Obukhov similarity [12] with the Businger–Dyer universal functions [13, 14]; above the surface layer the eddy diffusivity is reduced towards the top of the boundary layer by a polynomial (O’Brien-type) profile [15]. The surface heat flux that controls the Obukhov length can be estimated from routine data such as global radiation [16]. The input data for such a model are now available openly and globally: the ERA5 reanalysis provides hourly near-surface meteorology [17], the Copernicus digital elevation model (DEM) provides the terrain [18], and both can be retrieved through a single programming interface [19]. Within the regional research school, pollutant dispersion in the atmosphere has been modelled by operator splitting [20], with account for the capture of particles by vegetation [21], and conservative finite-volume schemes have been developed for related transport problems [22]. These studies, however, consider flat terrain.

The present paper develops a terrain-following dispersion model whose conservation properties are established rather than assumed, and couples it with open elevation and reanalysis data. The contributions are: (i) a conservative σ -coordinate formulation in which the contravariant vertical velocity is diagnosed from continuity, so that the ground is exactly impermeable, together with a mass-consistent wind obtained from single-point meteorological data by a column-flux normalisation; (ii) a finite-volume scheme with exact discrete conservation, exact preservation of a uniform field, and a positivity-preserving adaptive time step, for which the corresponding discrete identities are stated; (iii) a verification suite reported within the paper, including an analytical solution, grid and time-step refinement, and closure of the global budget, together with a quantitative demonstration of the error caused by the $w = 0$ simplification; and (iv) the coupling with the Copernicus DEM and ERA5 and an application to Tashkent. The nomenclature is collected at the end of the paper.

2 Problem statement

Consider the domain $\Omega = \{0 < x < L_x, 0 < y < L_y, h(x, y) < z < H_{top}\}$, where x points east, y north, z upwards, $h(x, y)$ is the terrain height measured from the lowest

point of the domain, and H_{top} is the height of the upper boundary. The concentration $C(x, y, z, t)$ of a pollutant satisfies the advection–diffusion equation in conservative form [1, 2]

$$\begin{aligned} \frac{\partial C}{\partial t} + \frac{\partial(uC)}{\partial x} + \frac{\partial(vC)}{\partial y} + \frac{\partial((w - w_g)C)}{\partial z} \\ = \frac{\partial}{\partial x}\left(K_x \frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial C}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial C}{\partial z}\right) + S - \lambda C, \end{aligned} \quad (1)$$

where u, v, w are the wind components, w_g is the gravitational settling velocity, K_x, K_y, K_z are the eddy diffusivities, S is the source term, and λ is a first-order decay rate. For spherical particles of diameter d_p and density ρ_p the settling velocity is given by the Stokes law with the Cunningham correction [1], $w_g = \rho_p g d_p^2 C_c / (18\mu)$, $C_c = 1 + (2\ell_a/d_p)(1.257 + 0.4\exp(-1.1d_p/(2\ell_a)))$, where μ is the dynamic viscosity of air and ℓ_a the mean free path of air molecules; $w_g = 0$ for gases.

The terrain-following transformation [9, 10]

$$\sigma = \frac{z - h(x, y)}{D(x, y)}, \quad D(x, y) = H_{top} - h(x, y), \quad 0 \leq \sigma \leq 1, \quad (2)$$

maps Ω onto the rectangular box $(0, L_x) \times (0, L_y) \times (0, 1)$; D is the Jacobian of the transformation and $z_{ag} = \sigma D$ is the height above ground. Since the terrain does not change in time, the coordinate surfaces $\sigma = \text{const}$ are fixed. The rate of change of σ following an air parcel defines the contravariant vertical velocity

$$\dot{\sigma} = \frac{d\sigma}{dt} = \frac{1}{D} \left[w - (1 - \sigma) \left(u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} \right) \right], \quad W = D\dot{\sigma}. \quad (3)$$

Transforming (1) with the chain rule and multiplying by D gives the Jacobian-weighted conservative equation

$$\begin{aligned} \frac{\partial(DC)}{\partial t} + \frac{\partial(DuC)}{\partial x} + \frac{\partial(DvC)}{\partial y} + \frac{\partial((W - w_g)C)}{\partial \sigma} \\ = \frac{\partial}{\partial x}\left(DK_x \frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial y}\left(DK_y \frac{\partial C}{\partial y}\right) + \frac{\partial}{\partial \sigma}\left(\frac{K_z}{D} \frac{\partial C}{\partial \sigma}\right) + DS - \lambda DC. \end{aligned} \quad (4)$$

The advective part of (4) is exact. In the diffusive part the cross-derivative metric terms, which are of order $|\nabla h|$ relative to the retained terms, are neglected, so horizontal diffusion acts along the σ -surfaces; this is the usual approximation for moderate slopes. For $h \equiv 0$ one has $D = H_{top}$, $W = H_{top}w$, and (4) reduces to (1).

The ground is impermeable: at $z = h$ the air moves along the surface, $w = u \partial h / \partial x + v \partial h / \partial y$, which by (3) is equivalent to $W = 0$ at $\sigma = 0$. For an incompressible flow the continuity equation in the transformed coordinates reads

$$\frac{\partial(Du)}{\partial x} + \frac{\partial(Dv)}{\partial y} + \frac{\partial W}{\partial \sigma} = 0, \quad W(x, y, \sigma) = - \int_0^\sigma \left[\frac{\partial(Du)}{\partial x} + \frac{\partial(Dv)}{\partial y} \right] d\sigma', \quad (5)$$

where the integration starts from the ground value $W = 0$. Equation (5) determines W , and hence the physical vertical velocity, from the horizontal wind. We stress that the frequently used simplification $w = 0$ in (3) gives $W|_{\sigma=0} = -(u \partial h / \partial x + v \partial h / \partial y) \neq 0$ on

every slope: the ground then behaves as a sink on windward slopes and as a source of clean air on lee slopes. The consequences are quantified in Section 4.

The horizontal wind is constructed from single-point data. The meteorological input is a time series of the 10-m wind speed V_{10} and direction φ (direction from which the wind blows), air temperature T_a , and global radiation R_g at one point. The horizontal components are $u_{10} = -V_{10} \sin \varphi$, $v_{10} = -V_{10} \cos \varphi$. The vertical profile is logarithmic in the height above ground and is multiplied by a column factor $\Gamma(x, y)$:

$$(u, v) = (u_{10}, v_{10}) f(\sigma D) \Gamma(x, y), \quad f(z) = \frac{\ln(z/z_0)}{\ln(10/z_0)} \quad (z > z_0),$$

$$\Gamma = \frac{H_{top} \int_0^1 f(\sigma H_{top}) d\sigma}{D \int_0^1 f(\sigma D) d\sigma}, \quad (6)$$

where z_0 is the roughness length. The factor Γ makes the column-integrated horizontal mass flux $\int_0^1 D(u, v) d\sigma = (u_{10}, v_{10}) H_{top} \int_0^1 f(\sigma H_{top}) d\sigma$ independent of (x, y) . Its horizontal divergence therefore vanishes, and (5) gives $W = 0$ also at the upper boundary $\sigma = 1$ (rigid lid). Over elevated terrain $\Gamma > 1$, which represents the acceleration of the flow over hills [8]. Because the lid is rigid, the speed-up is overestimated when the obstacle height is not small compared with H_{top} ; with $H_{top} = 800$ m and a 200 m ridge the maximum speed-up is 1.34 (Fig. 1).

Turbulent exchange is parameterised through the friction velocity and the Obukhov length [12],

$$u_* = \max\left(\frac{\kappa V_{10}}{\ln(10/z_0)}, 0.05\right), \quad L_M = -\frac{u_*^3 \rho c_p T_a}{\kappa g H},$$

$$H = \begin{cases} 0.3 R_g, & R_g > 5 \text{ W m}^{-2}, \\ -30 \text{ W m}^{-2}, & R_g \leq 5 \text{ W m}^{-2}, \end{cases} \quad (7)$$

where the sensible heat flux H is a simple radiation-based estimate [16], and $L_M = 10^6$ m (neutral) is taken for $|H| \leq 1 \text{ W m}^{-2}$. The eddy diffusivities combine surface-layer similarity with a parabolic decay towards the upper boundary [15]:

$$K_x = K_y = \frac{\kappa u_* z_{ag}}{Sc_t \varphi_m(z_{ag}/L_M)} (1 - \sigma)^2, \quad K_z = \frac{\kappa u_* z_{ag}}{Sc_t \varphi_h(z_{ag}/L_M)} (1 - \sigma)^2, \quad (8)$$

with the Businger–Dyer functions [13, 14] $\varphi_m = (1 - 16\zeta)^{-1/4}$, $\varphi_h = (1 - 16\zeta)^{-1/2}$ for $\zeta < 0$ and $\varphi_m = \varphi_h = 1 + 5\zeta$ for $\zeta \geq 0$. The functions are limited to $0.1 \leq \varphi \leq 10$ and the diffusivities to $K_{\min} \leq K \leq K_{\max}$.

A point source of strength q located at height z_s above the ground at (x_s, y_s) is $S = q \delta(x - x_s) \delta(y - y_s) \delta(z_{ag} - z_s)$, so its height above the terrain is preserved wherever it is placed. Initially $C = C_b$ (background). On a lateral boundary with outward normal $\mathbf{n}$, $C = C_b$ where the flow enters ($\mathbf{V} \cdot \mathbf{n} < 0$) and $\partial C / \partial n = 0$ where it leaves. At the upper boundary the diffusive flux vanishes and $W = 0$. At the terrain surface the vertical flux equals dry deposition minus surface emission,

$$\left(-w_g C - \frac{K_z}{D} \frac{\partial C}{\partial \sigma} \right) \Big|_{\sigma=0} = -(v_d + w_g) C + F_s, \quad (9)$$

which is the σ -form of the Robin condition $K_z \partial C / \partial z = v_d C - F_s$; v_d is the dry deposition velocity and F_s the surface emission flux.

3 Numerical method

The computational box is divided into $N_x \times N_y \times N_\sigma$ cells with centres $x_i = (i - \frac{1}{2})\Delta x$, $y_j = (j - \frac{1}{2})\Delta y$, $\sigma_k = (k - \frac{1}{2})\Delta\sigma$. The physical volume of cell (i, j, k) is $\mathcal{V}_{ij} = \Delta x \Delta y D_{ij} \Delta\sigma$. The terrain is evaluated at cell centres and at face centres, so that D_{ij} , $D_{i+1/2,j}$ and $D_{i,j+1/2}$ are available. Integrating (4) over a cell gives the semi-discrete balance

$$D_{ij} \frac{dC_{ijk}}{dt} = -\frac{F_{i+1/2} - F_{i-1/2}}{\Delta x} - \frac{P_{j+1/2} - P_{j-1/2}}{\Delta y} - \frac{G_{k+1/2} - G_{k-1/2}}{\Delta\sigma} + D_{ij} S_{ijk} - \lambda D_{ij} C_{ijk}, \quad (10)$$

where only the index that changes is written. Each face flux is computed once and used by both adjacent cells. The horizontal fluxes combine first-order upwinding with central diffusion,

$$F_{i+1/2} = U_{i+1/2}^+ C_i - U_{i+1/2}^- C_{i+1} - \frac{(DK_x)_{i+1/2}}{\Delta x} (C_{i+1} - C_i), \quad U_{i+1/2} = D_{i+1/2} u_{i+1/2}, \quad (11)$$

with $a^+ = \max(a, 0)$, $a^- = \max(-a, 0)$; $P_{j+1/2}$ is analogous, with $V_{j+1/2} = D_{j+1/2} v_{j+1/2}$. The vertical flux is

$$G_{k+1/2} = W_{k+1/2}^{*+} C_k - W_{k+1/2}^{*-} C_{k+1} - \left(\frac{K_z}{D}\right)_{k+1/2} \frac{C_{k+1} - C_k}{\Delta\sigma}, \quad W^* = W - w_g, \quad (12)$$

and at the ground $G_{1/2} = -(v_d + w_g) C_{ij1} + F_s$ according to (9). At the lateral and upper boundaries the diffusive flux is zero and the advective flux takes the upwind value, C_b at inflow faces. The source of a point emission is distributed over the neighbouring cells with trilinear (cloud-in-cell) weights in (x, y, z_{ag}) , which conserves q exactly and removes the dependence of the solution on the position of the source within a cell.

The contravariant velocity at the horizontal faces of each column is obtained from the discrete analogue of (5), integrated upwards from the ground:

$$W_{1/2} = 0, \quad W_{k+1/2} = W_{k-1/2} - \Delta\sigma \left[\frac{U_{i+1/2} - U_{i-1/2}}{\Delta x} + \frac{V_{j+1/2} - V_{j-1/2}}{\Delta y} \right]_k. \quad (13)$$

The factor Γ in (6) is evaluated with the same discrete column sums, $\Delta\sigma \sum_k U_{i+1/2,k} = \text{const}$, so that $W_{N_\sigma+1/2}$ vanishes to round-off (in the computations below $|W_{N_\sigma+1/2}| < 10^{-13} \text{ m s}^{-1}$).

The system (10) is advanced by the explicit Euler method, $C^{n+1} = C^n + \Delta t^n R(C^n)$. Writing the update of each cell as $C_{ijk}^{n+1} = (1 - \Delta t a_{ijk}) C_{ijk}^n + \Delta t \sum_m b_m C_m^n + \Delta t S_{ijk}$ with non-negative neighbour coefficients b_m , the diagonal coefficient is

$$a_{ijk} = \frac{1}{D_{ij}} \left[\frac{U_{i+1/2}^+ + U_{i-1/2}^-}{\Delta x} + \frac{V_{j+1/2}^+ + V_{j-1/2}^-}{\Delta y} + \frac{W_{k+1/2}^{*+} + W_{k-1/2}^{*-}}{\Delta\sigma} \right. \\ \left. + \frac{(DK_x)_{i+1/2} + (DK_x)_{i-1/2}}{\Delta x^2} + \frac{(DK_y)_{j+1/2} + (DK_y)_{j-1/2}}{\Delta y^2} \right. \\ \left. + \frac{(K_z/D)_{k+1/2} + (K_z/D)_{k-1/2}}{\Delta\sigma^2} \right] + \lambda, \quad (14)$$

where $W_{1/2}^* = -(v_d + w_g)$ at the ground. The time step is chosen as

$$\Delta t^n = \frac{\alpha}{\max_{ijk} a_{ijk}}, \quad 0 < \alpha \leq 1, \quad (15)$$

and is recomputed whenever the meteorological fields are updated. Expression (14) contains the outgoing advective rates and the diffusive rates of each cell. It is a local, sharper form of the usual combined Courant and diffusion restriction, and in the σ -coordinate it automatically accounts for the thin cells over elevated terrain through the factors $1/D_{ij}$ and $1/D^2$. We use $\alpha = 0.9$.

The scheme has three discrete properties, referred to below as (P1)–(P3), which are used as verification targets in Section 4. The first of them (P1) is exact conservation of mass. Multiplying (10) by $\Delta x \Delta y \Delta \sigma$ and summing over all cells, the interior face fluxes cancel in pairs, which gives the exact discrete budget

$$M^{n+1} = M^n + \Delta t^n (E - \Phi_{lat} - \Phi_{top} - \Phi_{dep} - \lambda M^n), \quad M = \sum_{ijk} \mathcal{V}_{ij} C_{ijk}, \quad (16)$$

where E is the emission rate and $\Phi_{lat}, \Phi_{top}, \Phi_{dep}$ are the net outflows through the lateral boundaries, the upper boundary and the ground, all evaluated from the same face fluxes. Accumulating these terms gives the budget residual $\varepsilon_M = (M - M_0 - \sum (E - \Phi_{lat} - \Phi_{top} - \Phi_{dep} - \lambda M) \Delta t) / \sum E \Delta t$, which must vanish to round-off. With the ground flux restricted to (9), no mass crosses the terrain surface other than by deposition.

The second property (P2) is positivity: if $C^n \geq 0$, $C_b \geq 0$, $S \geq 0$ and $\Delta t^n \leq 1 / \max a_{ijk}$, then $C^{n+1} \geq 0$, since C^{n+1} is a combination of non-negative values with non-negative coefficients. No clipping is required, so (P1) is not disturbed.

The third property (P3) is free-stream preservation: if $C^n \equiv C_b$, $S = 0$, $\lambda = 0$ and $v_d = w_g = 0$, then all diffusive fluxes vanish and the advective part of the update of every cell is C_b times the left-hand side of the discrete continuity equation (13), which is zero. Hence $C^{n+1} \equiv C_b$: the terrain-following grid does not generate spurious sources in a uniform field.

The input data are prepared as follows. For a domain centred at (φ_0, λ_0) the terrain is taken from the Copernicus DEM [18] at 31×31 regularly spaced points retrieved through the Open-Meteo interface [19], referred to the lowest point, and interpolated bilinearly to cell and face centres; it is limited to $h \leq H_{top} - 10$ m. Hourly ERA5 values of V_{10} , φ , T_a and R_g [17] are interpolated linearly in time, the wind by its vector components (u_{10}, v_{10}) to avoid the discontinuity of the direction at $0^\circ/360^\circ$. In the computational algorithm the terrain, D and the source weights are computed once. Every 300 s of model time the meteorological parameters are interpolated, and u, v (6), K (8), W (13) and Δt (15) are recomputed. The face fluxes and the cell update (10) are then evaluated, and the budget terms are accumulated. Near-ground concentrations are reported in the first cell, $C(\sigma_1)$, whose centre lies at $z_{ag} = D \Delta \sigma / 2$. The solver is written in Python with the flux and update loops compiled by Numba [23]; a step on 3×10^5 cells takes about 3 ms on two processor cores.

4 Verification

All quantities in this section are computed with the scheme of Section 3. The synthetic terrain is a two-dimensional Gaussian ridge $h = 200 \exp(-(x - 3000)^2 / (2 \cdot 500^2))$ m (maximum slope 0.24) in a domain $L_x \times L_y \times H_{top} = 6000 \times 3000 \times 800$ m. The wind is westerly, $V_{10} = 3 \text{ m s}^{-1}$, $z_0 = 0.1$ m, $Sc_t = 0.7$, $K_{\min} = 0.1$ and $K_{\max} = 200 \text{ m}^2 \text{ s}^{-1}$. Stable conditions correspond to $R_g = 0$, $T_a = 288$ K ($L_M = 56$ m) and unstable conditions to $R_g = 600 \text{ W m}^{-2}$, $T_a = 298$ K ($L_M = -9.7$ m). Figure 1 shows the grid, the speed-up factor of the mass-consistent wind, and the diagnosed contravariant velocity: W is positive on the windward slope and negative on the lee slope, is zero at the ground and at the lid, and does not exceed 9 mm s^{-1} .

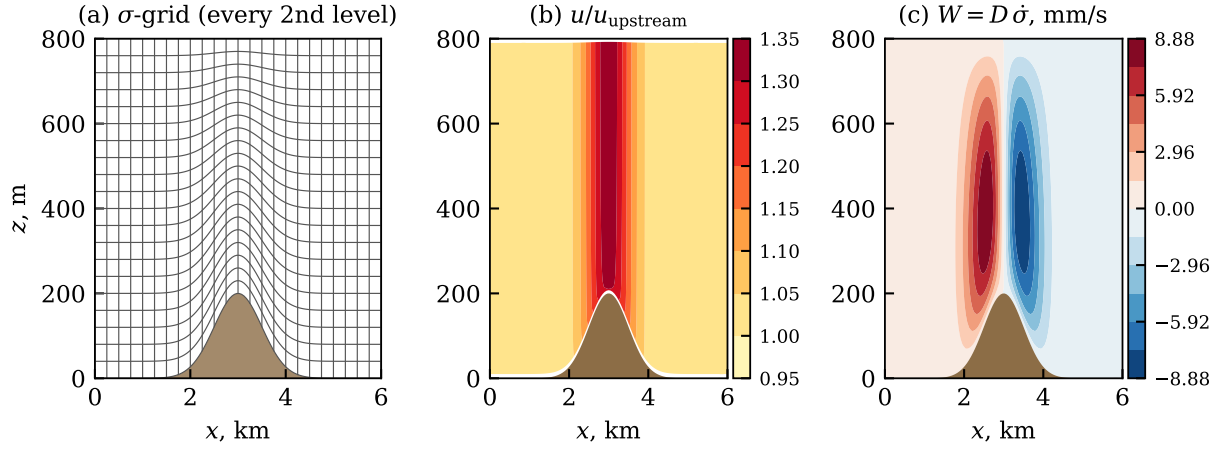


Figure 1 Two-dimensional ridge: (a) terrain-following grid (every second σ -level); (b) speed-up of the mass-consistent wind (6) relative to the upstream profile; (c) contravariant vertical velocity $W = D\dot{\sigma}$ diagnosed from (13).

The first two tests check the discrete properties (P3) and (P1). Starting from $C \equiv C_b = 1$ with $S = 0$, $v_d = 0$ and inflow value C_b , the solution over the ridge remained equal to 1 exactly (maximum deviation 0 in double precision) after 1800 s, in both stable and unstable conditions, as required by (P3). In all computations reported in this paper, including the time-dependent Tashkent runs with deposition, the budget residual satisfied $|\varepsilon_M| < 2 \times 10^{-12}$, and no negative concentrations occurred.

The accuracy of the scheme was then assessed against an analytical solution. For flat terrain, constant wind U and constant isotropic diffusivity K , the steady concentration from a point source in an unbounded medium is [1]

$$C_a(\mathbf{x}) = \frac{q}{4\pi K r} \exp\left(-\frac{U(r - (x - x_s))}{2K}\right), \quad r = |\mathbf{x} - \mathbf{x}_s|. \quad (17)$$

The ground and the lid are included by image sources at $-z_s$, $2H_{top} - z_s$, $-2H_{top} + z_s$ and $2H_{top} + z_s$. With $U = 2 \text{ m s}^{-1}$, $K = 20 \text{ m}^2 \text{ s}^{-1}$, $q = 1$, a $1600 \times 800 \times 400 \text{ m}$ domain and the source at $(300, 400, 200) \text{ m}$, the numerical solution at $t = 1400 \text{ s}$ was compared with (17) in the plane of the source for $100 \leq x - x_s \leq 1000 \text{ m}$, $|y - y_s| \leq 150 \text{ m}$, using the fractional bias FB , normalised mean-square error $NMSE$, fraction within a factor of two $FAC2$, and correlation R [24], together with the relative L_2 error (Table 1, Fig. 2a). All measures are far inside the acceptance ranges commonly used for dispersion models ($|FB| < 0.3$, $NMSE < 1.5$, $FAC2 > 0.5$) [24], and the L_2 error decreases with an observed order of 1.5 and 1.2 over the two refinements. The order is between one and two because the cell Péclet number $U\Delta x/K$ decreases from 4 to 1 on these grids and the discretisation error changes from upwind-dominated to diffusion-dominated.

Table 1 Comparison with the analytical solution (17) (flat terrain, cubic cells).

$\Delta x = \Delta z, \text{ m}$	Points	FB	$NMSE$	$FAC2$	R	Rel. L_2 error
40	184	0.032	0.0083	1.00	0.9978	0.072
20	720	0.011	0.0010	1.00	0.9995	0.026
10	2700	0.005	0.0002	1.00	0.9999	0.011

Convergence over terrain was examined with a uniform line source along y ($q = 5 \times 10^{-5} \text{ kg s}^{-1} \text{ m}^{-1}$, $x_s = 1000 \text{ m}$, $z_s = 50 \text{ m}$), which reduces the problem to the x - σ plane over the ridge; it was solved on four grids $N_x \times N_\sigma = 60 \times 20$, 120×40 , 240×80 and 480×160 . The near-ground concentration at $t = 3600 \text{ s}$ (Fig. 2b) converges: the relative L_2 difference from the finest grid over $1.5 \leq x \leq 5.9 \text{ km}$ is 25%, 5.4% and 3.0% in stable conditions and 3.7%, 1.8% and 0.6% in unstable conditions. The stable case converges more slowly because the thin plume is resolved by only a few cells on the coarsest grid. Halving α from 0.9 to 0.45 and 0.225 changes the transient solution at $t = 900 \text{ s}$ by 1.4%, 0.55% and 0.18% relative to $\alpha = 0.1125$, consistent with first-order accuracy in time; the value $\alpha = 0.9$ is therefore adequate for the purposes of the paper.

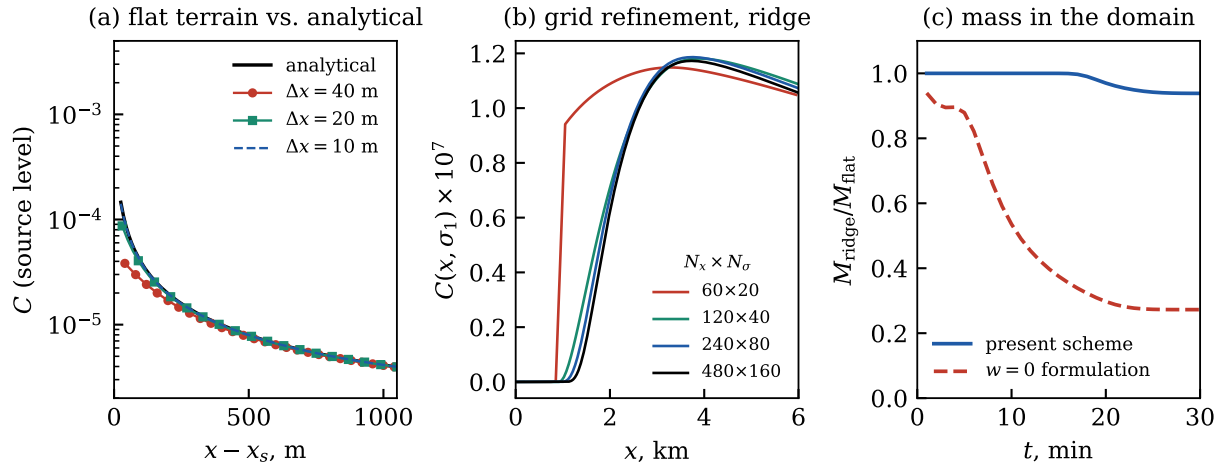


Figure 2 Verification: (a) centreline concentration from the analytical solution (17) and from three grids; (b) near-ground concentration over the ridge on four grids (stable stratification, line source); (c) ratio of the mass in the domain with and without the ridge for the present scheme and for the formulation with $w = 0$ (stable stratification, $v_d = 0$).

To quantify the effect discussed after (5), the same point-source problem (stable stratification, $z_s = 50 \text{ m}$, $q = 0.05 \text{ kg s}^{-1}$, $v_d = 0$) was solved with a node-based upwind scheme that uses $w = 0$ in (3), i.e. $W = -(1 - \sigma)(u \partial h / \partial x + v \partial h / \partial y)$, and the non-weighted form of the horizontal fluxes. Since deposition is switched off, before the plume reaches the outflow boundary the mass in the domain must be the same with and without the ridge. For the present scheme the ratio $M_{\text{ridge}}/M_{\text{flat}}$ equals 1.000 until $t \approx 16 \text{ min}$ and then decreases slightly as the accelerated plume starts to leave the domain. For the $w = 0$ formulation it falls to 0.88, 0.54 and 0.38 at 5, 10 and 15 min, and to 0.27 at 30 min (Fig. 2c). The lost mass leaves through the windward slope. On the lee side the same formulation predicts near-ground concentrations lower than over flat terrain by a factor of 600 at $x = 4$ – 5 km (Fig. 4a). This attenuation is an artefact of the missing kinematic condition, not a physical blocking effect.

5 Results and discussion

A point source ($q = 0.05 \text{ kg s}^{-1}$) at $(1000, 1500) \text{ m}$ emits for one hour under the conditions of Section 4, with $v_d = 5 \times 10^{-3} \text{ m s}^{-1}$ and a $120 \times 60 \times 40$ grid. Figure 3 shows the concentration in the vertical plane through the source. Under stable stratification the plume from a 50 m source stays in a shallow near-ground layer and follows the terrain over the ridge. Under unstable stratification it is mixed through the whole layer within

about 1 km of the source. Figure 4 compares the near-ground concentrations with and without the ridge, and Table 2 lists the ratios for three source heights.

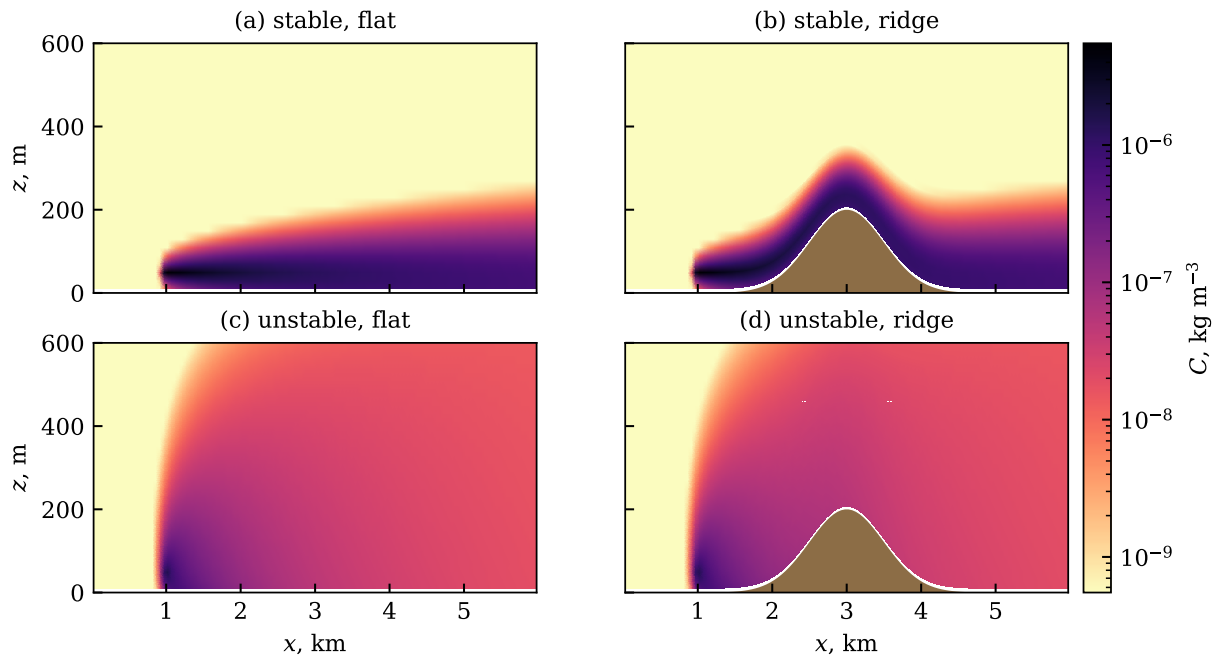


Figure 3 Concentration in the vertical plane $y = 1500$ m through the source ($z_s = 50$ m, $t = 3600$ s): (a, b) stable stratification over flat terrain and over the ridge; (c, d) unstable stratification. Logarithmic colour scale.

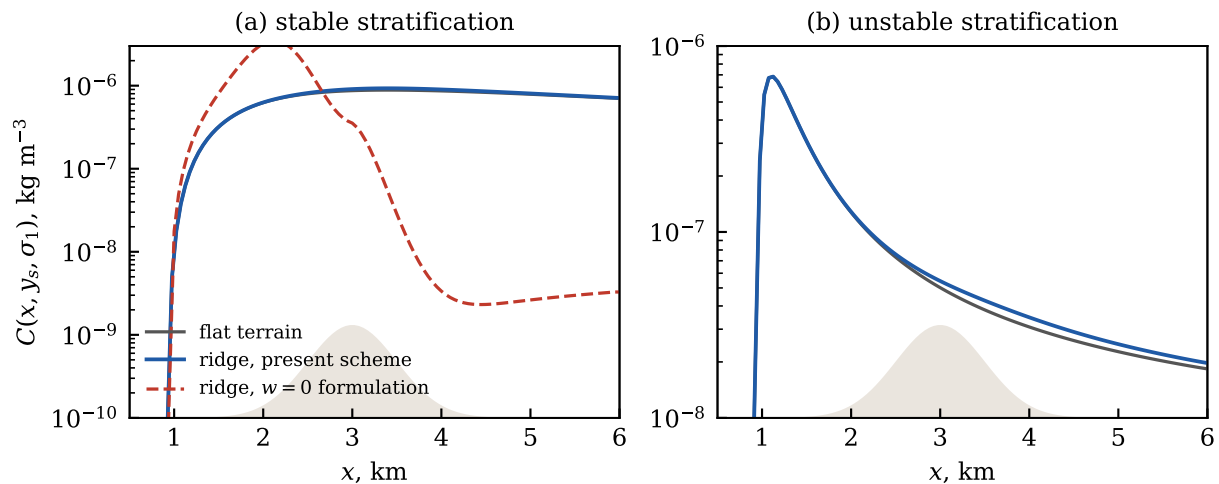


Figure 4 Near-ground concentration along the plume axis ($z_s = 50$ m, $t = 3600$ s): (a) stable and (b) unstable stratification. The shaded profile shows the ridge. The dashed line in (a) is the $w = 0$ formulation (without deposition).

Two conclusions follow. First, with a consistent kinematic condition the ridge does not screen the lee side. Instead, it raises the near-ground concentration on the crest and lee slope by 2–13% for a 50 m source and by up to 51% for a 150 m source in stable conditions. The mechanism is the convergence of the σ -surfaces over the ridge: a plume that travels at a fixed fraction of the column depth comes closer to the ground where the column is thinner. The effect grows with the source height because, in stable conditions, an elevated

Table 2 Near-ground concentration over the ridge relative to flat terrain, C_{ridge}/C_{flat} , at $t = 3600$ s; the ridge crest is at $x = 3$ km. The last column gives the flat-terrain value at $x = 3$ km.

Stratification	z_s , m	$x = 2.5$ km	$x = 3.0$ km	$x = 3.5$ km	$x = 4.0$ km	C_{flat} , kg m^{-3}
stable	50	1.02	1.04	1.05	1.04	8.7×10^{-7}
stable	100	1.07	1.17	1.22	1.20	7.7×10^{-8}
stable	150	1.15	1.37	1.51	1.48	1.8×10^{-9}
unstable	50	1.04	1.09	1.12	1.13	5.0×10^{-8}
unstable	100	1.03	1.08	1.12	1.12	4.6×10^{-8}
unstable	150	1.03	1.07	1.11	1.12	4.3×10^{-8}

plume reaches the ground mainly through this geometric approach rather than through weak vertical mixing. The acceleration $\Gamma > 1$ acts in the opposite direction, which is why the net effect for a low source is only a few per cent. Second, stability dominates. At $x = 3$ km the near-ground concentration from the 50 m source is 17 times larger in stable than in unstable conditions, and for the 150 m source it is 24 times smaller, because the elevated stable plume has not yet reached the ground. These findings are deliberately conservative. The kinematic wind (6) cannot represent flow splitting or upstream blocking in strongly stratified flow [7]; such effects require a dynamic or diagnostic wind model and are outside the present scope. The model does ensure that any terrain effect it reports comes from the transport itself and not from a loss of mass.

The model was also applied to the real terrain and meteorology of Tashkent. The domain is 10×10 km centred at 41.351° N, 69.336° E, with $H_{top} = 600$ m, a $100 \times 100 \times 30$ grid ($\Delta x = \Delta y = 100$ m) and $v_d = 5 \times 10^{-3} \text{ m s}^{-1}$. A point source of an inert gas ($q = 0.2 \text{ kg s}^{-1}$, $z_s = 40$ m) is placed at the centre, and the calmest 24-hour period of January 2025 (lowest mean V_{10} in the ERA5 series) is simulated from 00:00 local time, with and without the terrain.

The model has several limitations that define its scope. The wind is kinematic: it follows the terrain, is mass-consistent under a rigid lid, and is horizontally uniform apart from the terrain factor Γ . Upstream blocking and flow splitting in strongly stable flow, slope and valley winds, and the urban canopy are not represented. The single-point meteorological input and the radiation-based heat flux (7) are appropriate for domains of a few tens of kilometres with moderate relief. The upwind scheme is first-order accurate in space and time; it was chosen for its monotonicity and positivity, and the verification shows that its numerical diffusion is acceptable at the grid spacings used. A higher-order limited scheme can replace (11) without changing properties (P1) and (P3).

6 Conclusions

A terrain-following model of pollutant dispersion in the atmospheric boundary layer has been formulated in Jacobian-weighted conservative form. The contravariant vertical velocity is diagnosed from continuity, so the terrain surface is exactly impermeable, and the horizontal wind is obtained from single-point data and made mass-consistent by a column-flux normalisation. The finite-volume scheme conserves mass to round-off (budget residual below 2×10^{-12}), preserves a uniform field exactly, and remains positive under a local time-step bound. It reproduces the analytical point-source solution with $FB = 0.005$, $NMSE = 2 \times 10^{-4}$, $R = 0.9999$ and $FAC2 = 1$, and converges under grid and time-step refinement over terrain. The widely used assumption of a horizontal wind in the σ -equation violates the kinematic surface condition: over a 200 m ridge it loses more than 60% of the emitted mass in 15 min and produces a spurious lee-side attenuation of

the order of 10^2 – 10^3 . With the conservative formulation, a moderate ridge changes near-ground concentrations by 2–51%, most strongly for elevated sources in stable conditions, whereas stability changes them by more than an order of magnitude. Coupled with the Copernicus DEM and ERA5 reanalysis, the model provides a reproducible basis for rapid assessment of pollution over real terrain. Planned extensions are a diagnostic wind model with stability-dependent terrain blocking and a higher-order monotone advection scheme.

Nomenclature

C concentration (kg m^{-3}); C_b background concentration; x, y, z Cartesian coordinates (m); h terrain height; H_{top} height of the upper boundary; $D = H_{top} - h$ column depth (Jacobian); σ terrain-following coordinate; $z_{ag} = \sigma D$ height above ground; u, v, w wind components (m s^{-1}); $\dot{\sigma}$, $W = D\dot{\sigma}$ contravariant vertical velocity; Γ column factor of the wind; V_{10}, φ 10-m wind speed and direction; z_0 roughness length; w_g settling velocity; v_d dry deposition velocity; F_s surface emission flux; K_x, K_y, K_z eddy diffusivities ($\text{m}^2 \text{s}^{-1}$); u_* friction velocity; L_M Obukhov length; H sensible heat flux; R_g global radiation (W m^{-2}); T_a air temperature; φ_m, φ_h universal functions; $\kappa = 0.41$ von Kármán constant; Sc_t turbulent Schmidt number; S, q source term and source strength; λ decay rate; M mass in the domain; α time-step safety factor; $FB, NMSE, FAC2, R$ statistical measures.

References

- [1] Seinfeld J. H., Pandis S. N. 2016. *Atmospheric chemistry and physics: from air pollution to climate change*. 3rd ed. Hoboken: John Wiley & Sons. 1152 p.
- [2] Stull R. B. 1988. *An introduction to boundary layer meteorology*. Dordrecht: Kluwer Academic Publishers. 666 p. doi: <https://doi.org/10.1007/978-94-009-3027-8>.
- [3] Cimorelli A. J., Perry S. G., Venkatram A. [et al.] 2005. AERMOD: a dispersion model for industrial source applications. Part I: general model formulation and boundary layer characterization. *Journal of Applied Meteorology*. 44(5): 682–693. doi: <https://doi.org/10.1175/JAM2227.1>.
- [4] Scire J. S., Strimaitis D. G., Yamartino R. J. 2000. *A user's guide for the CALPUFF dispersion model (version 5)*. Concord, MA: Earth Tech, Inc. 521 p.
- [5] Stohl A., Forster C., Frank A., Seibert P., Wotawa G. 2005. Technical note: the Lagrangian particle dispersion model FLEXPART version 6.2. *Atmospheric Chemistry and Physics*. 5: 2461–2474. doi: <https://doi.org/10.5194/acp-5-2461-2005>.
- [6] Byun D., Schere K. L. 2006. Review of the governing equations, computational algorithms, and other components of the Models-3 Community Multiscale Air Quality (CMAQ) modeling system. *Applied Mechanics Reviews*. 59(2): 51–77. doi: <https://doi.org/10.1115/1.2128636>.
- [7] Snyder W. H., Thompson R. S., Eskridge R. E. [et al.] 1985. The structure of strongly stratified flow over hills: dividing-streamline concept. *Journal of Fluid Mechanics*. 152: 249–288. doi: <https://doi.org/10.1017/S0022112085000684>.
- [8] Jackson P. S., Hunt J. C. R. 1975. Turbulent wind flow over a low hill. *Quarterly Journal of the Royal Meteorological Society*. 101(430): 929–955. doi: <https://doi.org/10.1002/qj.49710143015>.
- [9] Phillips N. A. 1957. A coordinate system having some special advantages for numerical forecasting. *Journal of Meteorology*. 14(2): 184–185. doi: [https://doi.org/10.1175/1520-0469\(1957\)014<0184:ACSHSS>2.0.CO;2](https://doi.org/10.1175/1520-0469(1957)014<0184:ACSHSS>2.0.CO;2).
- [10] Gal-Chen T., Somerville R. C. J. 1975. On the use of a coordinate transformation for the solution of the Navier–Stokes equations. *Journal of Computational Physics*. 17(2): 209–228. doi: [https://doi.org/10.1016/0021-9991\(75\)90037-6](https://doi.org/10.1016/0021-9991(75)90037-6).

- [11] Schär C., Leuenberger D., Fuhrer O., Lüthi D., Girard C. 2002. A new terrain-following vertical coordinate formulation for atmospheric prediction models. *Monthly Weather Review*. 130(10): 2459–2480. doi: [https://doi.org/10.1175/1520-0493\(2002\)130<2459:ANTFVC>2.0.CO;2](https://doi.org/10.1175/1520-0493(2002)130<2459:ANTFVC>2.0.CO;2).
- [12] Monin A.S., Obukhov A.M. 1954. Osnovnye zakonomernosti turbulentnogo peremeshivaniya v prizemnom sloe atmosfery [Basic laws of turbulent mixing in the surface layer of the atmosphere]. *Trudy Geofizicheskogo instituta AN SSSR*. 24(151): 163–187. (In Russian)
- [13] Businger J.A., Wyngaard J.C., Izumi Y., Bradley E.F. 1971. Flux-profile relationships in the atmospheric surface layer. *Journal of the Atmospheric Sciences*. 28(2): 181–189. doi: [https://doi.org/10.1175/1520-0469\(1971\)028<0181:FPRITA>2.0.CO;2](https://doi.org/10.1175/1520-0469(1971)028<0181:FPRITA>2.0.CO;2).
- [14] Dyer A.J. 1974. A review of flux-profile relationships. *Boundary-Layer Meteorology*. 7(3): 363–372. doi: <https://doi.org/10.1007/BF00240838>.
- [15] O'Brien J.J. 1970. A note on the vertical structure of the eddy exchange coefficient in the planetary boundary layer. *Journal of the Atmospheric Sciences*. 27(8): 1213–1215. doi: [https://doi.org/10.1175/1520-0469\(1970\)027<1213:ANOTVS>2.0.CO;2](https://doi.org/10.1175/1520-0469(1970)027<1213:ANOTVS>2.0.CO;2).
- [16] Holtslag A.A.M., van Ulden A.P. 1983. A simple scheme for daytime estimates of the surface fluxes from routine weather data. *Journal of Climate and Applied Meteorology*. 22(4): 517–529. doi: [https://doi.org/10.1175/1520-0450\(1983\)022<0517:ASSFDE>2.0.CO;2](https://doi.org/10.1175/1520-0450(1983)022<0517:ASSFDE>2.0.CO;2).
- [17] Hersbach H., Bell B., Berrisford P. [et al.] 2020. The ERA5 global reanalysis. *Quarterly Journal of the Royal Meteorological Society*. 146(730): 1999–2049. doi: <https://doi.org/10.1002/qj.3803>.
- [18] European Space Agency 2021. *Copernicus global digital elevation model (COP-DEM)*. Distributed by OpenTopography. doi: <https://doi.org/10.5270/ESA-c5d3d65>.
- [19] Zippenfenig P. 2023. *Open-Meteo.com Weather API*. Computer software. Zenodo. doi: <https://doi.org/10.5281/zenodo.7970649>.
- [20] Ravshanov N., Muradov F., Akhmedov D. 2020. Operator splitting method for numerical solving the atmospheric pollutant dispersion problem. *Journal of Physics: Conference Series*. 1441: Art. 012164. doi: <https://doi.org/10.1088/1742-6596/1441/1/012164>.
- [21] Ravshanov N., Nazarov Sh.E., Boborakhimov B. 2024. Modeling the process of pollutant spread in the atmosphere with account for the capture of particles by vegetation elements. *Lobachevskii Journal of Mathematics*. 45(3): 1213–1226. doi: <https://doi.org/10.1134/S1995080224600638>.
- [22] Ravshanov N., Boborahimov B.I., Berdiyrov Sh.Sh. 2026. Numerical modeling of filtration and transport processes in a cylindrical porous filter using the finite volume method. *Problems of Computational and Applied Mathematics*. 1(71): 28–42. doi: https://doi.org/10.71310/pcam.1_71.2026.03.
- [23] Lam S.K., Pitrou A., Seibert S. 2015. Numba: a LLVM-based Python JIT compiler. *Proceedings of the Second Workshop on the LLVM Compiler Infrastructure in HPC*. Art. 7. doi: <https://doi.org/10.1145/2833157.2833162>.
- [24] Chang J.C., Hanna S.R. 2004. Air quality model performance evaluation. *Meteorology and Atmospheric Physics*. 87(1–3): 167–196. doi: <https://doi.org/10.1007/s00703-003-0070-7>.

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КОНСЕРВАТИВНАЯ МОДЕЛЬ ПЕРЕНОСА ПРИМЕСЕЙ В АТМОСФЕРЕ НАД СЛОЖНЫМ РЕЛЬЕФОМ

^{1*} *Насруллаев П.А., ¹Джурабоева О.С., ²Саманова Г.Б.***uzbekpy@gmail.com*¹ Научно-исследовательский институт развития цифровых технологий и искусственного интеллекта, 100125, Узбекистан, г. Ташкент, м-в Буз-2, 17А;² Ташкентский государственный медицинский университет, 100109, Узбекистан, г. Ташкент, ул. Фароби, 2

Разработаны математическая модель и верифицированный численный метод для расчёта переноса пассивных и оседающих примесей в пограничном слое атмосферы над сложным рельефом. Уравнение адвекции–диффузии записано в координате, следующей рельефу, в полностью консервативной форме с якобианом преобразования. Контравариантная вертикальная скорость определяется из дискретного уравнения неразрывности, поэтому кинематическое условие на поверхности рельефа выполняется точно и поток вещества через земную поверхность отсутствует. Горизонтальный ветер, задаваемый по точечному прогнозу или реанализу логарифмическим профилем по высоте над поверхностью, согласуется с уравнением неразрывности нормировкой потока в столбе при условии жёсткой крышки. Коэффициенты турбулентной диффузии определяются по теории подобия Монина — Обухова с параболическим затуханием к верхней границе слоя, а параметр устойчивости вычисляется по часовым значениям солнечной радиации и температуры воздуха. Конечно-объёмная схема с общими потоками через грани, противопоточной аппроксимацией и явным шагом по времени, выбираемым из локального условия положительности, сохраняет массу с точностью до ошибок округления, точно сохраняет однородное поле и не порождает отрицательных концентраций. Верификация включает сравнение с аналитическим решением для точечного источника, исследование сходимости по сетке и шагу по времени и замкнутый баланс массы с невязкой менее $2 \cdot 10^{-12}$. Показано, что распространённое упрощение с горизонтальным ветром ($w = 0$) нарушает кинематическое условие на поверхности: над хребтом высотой 200 м за 15 мин теряется более 60% выброшенной массы, а за хребтом возникает ложное ослабление концентрации на два–три порядка. В консервативной постановке хребет изменяет приземные концентрации на 2–51%, тогда как устойчивость атмосферы изменяет их более чем на порядок. Модель связана с цифровой моделью рельефа Copernicus и реанализом ERA5 и применена к территории г. Ташкента.

Ключевые слова: пограничный слой атмосферы, перенос примесей, координата, следующая рельефу, метод конечных объёмов, сохранение массы, теория подобия Монина — Обухова

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