

UDC 519.644.3

OPTIMAL QUADRATURE FORMULAS FOR HIGHLY OSCILLATORY CAUCHY-TYPE INTEGRALS*

^{1,2}*Buvashero D.S.*, ¹*Aliev Sh.E.*

dilshodbuvashero@gmail.com

¹V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, 9, University str., Tashkent, 100174 Uzbekistan;

²Tashkent State Technical University named after Islam Karimov, 2, University str., Tashkent, 100095 Uzbekistan

Background: Singular integrals with oscillatory kernels arise in various areas of computational mathematics and applied science. Their numerical evaluation is challenging because singular behavior and rapid oscillations occur simultaneously, which may significantly reduce the accuracy and efficiency of conventional quadrature methods. Existing approaches often treat these features separately and do not fully address the construction of optimal quadrature formulas within a functional-analytic framework.

Purpose: The purpose of this study is to construct optimal quadrature formulas for the numerical evaluation of singular integrals with oscillatory kernels and to establish their optimality in an appropriate Sobolev space. Particular emphasis is placed on obtaining explicit quadrature coefficients that are suitable for practical computation.

Method: The quadrature problem is formulated as an extremal problem for the corresponding error functional in a Sobolev space. Following the Sobolev method, the quadrature coefficients are determined by minimizing the norm of the error functional over a prescribed class of quadrature formulas. The analytical properties of both the singular integrand and the oscillatory kernel are incorporated directly into the construction. This formulation provides a unified treatment of singularities and rapid oscillations and leads to explicit analytical expressions for the quadrature coefficients.

Results: The constructed formulas possess the minimum-norm property of the associated error functional in the underlying Sobolev space. The explicit coefficient representations permit direct numerical implementation and provide accurate evaluation of singular oscillatory integrals without excessive mesh refinement. The numerical performance demonstrates that the proposed approach remains effective in the presence of both singular behavior and rapid oscillations.

Implication: The proposed methodology provides a rigorous and computationally efficient framework for constructing optimal quadrature formulas for singular oscillatory integrals. It extends the functional-analytic approach to numerical integration and offers a useful basis for the treatment of related singular integral operators.

Keywords: singular integral, high oscillation, optimal quadrature formula, optimal coefficients, Sobolev space, error functional, numerical integration

Citation: Buvashero D.S., Aliev Sh.E. 2026. Optimal quadrature formulas for highly oscillatory Cauchy-type integrals. *Problems of Computational and Applied Mathematics*. 4/1(75): 190-200.

DOI: 10.71310/pcam.4-1_75.2026.15

*I express my gratitude to the V.I. Romanovsky Institute of Mathematics for providing comfortable working conditions, and to Professor Abdullo Hayotov and Dilshod Akhmedov for their kind assistance in improving this paper.

1 Introduction

A broad range of numerical techniques has been developed for the accurate and efficient approximation of singular integrals. These problems have received considerable attention because the presence of singularities creates significant difficulties in both theoretical analysis and numerical implementation. In particular, optimal quadrature formulas for singular and high oscillation integrals have been extensively studied by Kh. M. Shadimetov, A. R. Hayotov, D. M. Akhmedov and others [1–13]. Their studies provide important theoretical foundations for the construction of optimal quadrature formulas and enable accurate numerical evaluation of these types of integrals.

In addition to these developments, considerable attention has been devoted to the numerical treatment of singular integrals involving exponential weights and oscillatory behavior. Such problems are particularly challenging because the singularity of the integrand is accompanied by rapid variations, which may substantially affect the accuracy and stability of conventional numerical integration methods. To address these difficulties, several specialized quadrature techniques have been proposed. G. E. Okecha [14] developed effective quadrature rules based on Lagrange and Hermite interpolation. These approaches provide suitable numerical approximations while taking into account the specific properties of weighted singular integrals.

M. R. Capobianco and G. Criscuolo [15] introduced a stable numerical method based on Jacobi orthogonal polynomials. The use of orthogonal polynomial techniques provides favorable approximation properties and contributes to the stability of the resulting numerical procedure. Their approach demonstrates the effectiveness of specially constructed polynomial approximations in the treatment of singular integrals with nonstandard weight functions.

Further progress has been achieved through interpolation-based methods with improved convergence properties. H. Wang and S. Xiang [16] developed a uniformly convergent scheme based on Chebyshev interpolation at Clenshaw–Curtis points. The proposed approach takes advantage of the approximation properties of Chebyshev polynomials and the favorable distribution of Clenshaw–Curtis nodes, resulting in an efficient and accurate numerical procedure.

Similarly, Guo He and Shuhuang Xiang [17] constructed interpolatory quadrature formulas using Clenshaw–Curtis nodes and efficient techniques for the computation of modified moments. Their method provides an effective framework for the numerical evaluation of integrals for which standard quadrature techniques may encounter difficulties. The efficient treatment of modified moments also contributes to reducing the computational cost of the numerical procedure.

The aforementioned studies demonstrate the continuous development of numerical methods for singular, hypersingular, weighted, and oscillatory integrals. The proposed approaches differ in their approximation techniques, stability characteristics, convergence properties, and computational complexity. Nevertheless, they share a common objective: to overcome the numerical difficulties associated with singularities and rapidly varying integrands while maintaining high accuracy.

These results provide a solid theoretical and computational foundation for further research in the field of optimal quadrature formulas. In particular, they motivate the development of numerical schemes capable of simultaneously treating singular behavior and strong oscillations. Such methods are of significant interest in computational mathematics and in applications where singular and highly oscillatory integrals naturally arise.

2 Statement of the problem

We consider the problem of numerically evaluating Cauchy principal value integrals of oscillatory functions

$$I = \text{p.v.} \int_{-1}^1 \frac{\varphi(x)}{x-t} \sin(\omega x) dx = \lim_{\varepsilon \rightarrow 0^+} \int_{|x-t| \geq \varepsilon} \frac{\varphi(x)}{x-t} \sin(\omega x) dx, \quad -1 < t < 1, \quad (1)$$

where $\omega \in \mathbb{R} \setminus \{0\}$. The integral (1) exists if the function $\varphi(x)$ satisfies Hölder's condition on $[-1, 1]$. For large ω , the highly oscillatory behavior of the integrand poses significant challenges for accurate numerical evaluation, while conventional numerical integration methods may become ineffective.

Consider the quadrature formula for (1)

$$\int_{-1}^1 \frac{\sin(\omega x) \varphi(x)}{x-t} dx \cong \sum_{\beta=0}^N A_{\beta} \varphi(x_{\beta}), \quad (2)$$

where $-1 < t < 1$, $x_{\beta} = h\beta - 1$ are the nodes of the quadrature formula, $h = \frac{2}{N}$, N is a natural number, $\beta = \overline{0, N}$, $\varphi(x) \in L_2^{(1)}(-1, 1)$, and $\omega \neq 0$. A_{β} are the coefficients of the quadrature formula (2).

The difference between the integral and the corresponding quadrature sum

$$(R, \varphi) = \int_{-1}^1 \frac{\sin(\omega x) \varphi(x)}{x-t} dx - \sum_{\beta=0}^N A_{\beta} \varphi(x_{\beta}) \quad (3)$$

is called the error of the quadrature formula (2).

The space $L_2^{(1)}(-1, 1)$ is a Hilbert space, and the inner product of two functions is defined as follows:

$$\langle \varphi, \psi \rangle = \int_{-1}^1 \varphi'(x) \psi'(x) dx.$$

The norm of the function is defined by the following formula:

$$\|\varphi\|_{L_2^{(1)}}^2 = \langle \varphi, \varphi \rangle = \int_{-1}^1 (\varphi'(x))^2 dx.$$

The error functional corresponding to the error is given by

$$R(x) = \frac{\varepsilon_{[-1,1]}(x) \sin(\omega x)}{x-t} - \sum_{\beta=0}^N A_{\beta} \delta(x - x_{\beta}), \quad (4)$$

where $\varepsilon_{[-1,1]}(x)$ is the indicator of the interval $[-1, 1]$, and $\delta(x)$ is Dirac's delta-function.

The error functional $R(x)$ is defined in the space $L_2^{(1)}(-1, 1)$, the following condition must be satisfied:

$$(R, 1) = 0. \quad (5)$$

This equality ensures that the quadrature formula (2) is exact for constants.

3 Form of the norm of the error functional

We find the form of the norm of the error functional by using Riesz's theorem [18]. That is, for every continuous linear functional defined on a Hilbert space, there exists a unique element ψ_R in this space such that the following identity holds for all φ :

$$(R, \varphi) = \langle \psi_R, \varphi \rangle$$

and

$$\|R(\psi)\|_{L_2^{(1)*}} = \|\psi_R\|_{L_2^{(1)}}.$$

To determine the norm of the error functional, it is necessary to determine the function ψ_R . For this purpose, we employ the concept of the extremal function introduced by S. L. Sobolev [19].

This yields the following equality:

$$(R, \psi_R) = \|R\|_{L_2^{(1)*}(-1,1)} \cdot \|\psi_R\|_{L_2^{(1)}(-1,1)} = \|\psi_R\|_{L_2^{(1)}(-1,1)}^2 = \|R\|_{L_2^{(1)*}(-1,1)}^2.$$

Consequently, the norm of the error functional is obtained as

$$\|R\|_{L_2^{(1)*}(-1,1)}^2 = (R, \psi_R).$$

The extremal function ψ_R corresponding to the error functional R in the space $L_2^{(1)}(-1, 1)$, as established by S. L. Sobolev [19], is given by:

$$\psi_R(x) = -(R(x) * G_1(x) + P_0), \quad (6)$$

where

$$G_1(x) = \frac{|x|}{2} = \frac{x \cdot \text{sign}(x)}{2}$$

is a fundamental solution of the second order differential equation, satisfying:

$$\frac{d^2}{dx^2} G_1(x) = \delta(x)$$

and P_0 is constant.

Now, we can compute the norm of the error functional using the extremal function (6):

$$\begin{aligned} \|R\|_{L_2^{(1)*}(-1,1)}^2 &= (R(x), \psi_R(x)) = \int_{-1}^1 R(x) \cdot (R(x) * G_1(x) + P_0) dx \\ &= \int_{-1}^1 R(x) \cdot (R(x) * G_1(x)) dx + \int_{-1}^1 R(x) \cdot P_0 dx \\ &= \int_{-1}^1 R(x) \cdot (R(x) * G_1(x)) dx. \end{aligned}$$

We evaluate the convolution

$$\begin{aligned} R(x) * G_1(x) &= \int_{-\infty}^{\infty} R(y) G_1(x - y) dy \\ &= \int_{-1}^1 \frac{\sin(\omega y) \cdot |x - y|}{2(y - t)} dy - \sum_{\beta=0}^N A_{\beta} \cdot \frac{|x - x_{\beta}|}{2}. \end{aligned}$$

3.1 Compute the norm

We obtained

$$\|R\|_{L_2^{(1)*}(-1,1)}^2 = (R, R * G_1) = \int_{-1}^1 R(x)(-R(x) * G_1(x)) dx.$$

Using the convolution, we get the following form of the norm

$$\begin{aligned} \|R\|^2 &= - \int_{-1}^1 \left[\frac{\sin(\omega x)}{x-t} \varepsilon_{[-1,1]}(x) - \sum_{\beta=0}^N A_\beta \delta(x-x_\beta) \right] \\ &\quad \times \left(\int_{-1}^1 \frac{\sin(\omega y) \cdot |x-y|}{2(y-t)} dy - \sum_{\beta=0}^N A_\beta \frac{|x-x_\beta|}{2} \right) dx \\ &= - \left[\int_{-1}^1 \int_{-1}^1 \frac{\sin(\omega y) \cdot \sin(\omega x) \cdot |x-y|}{2(x-t)(y-t)} dx dy - \sum_{\gamma=0}^N A_\gamma \int_{-1}^1 \frac{\sin(\omega x) |x-x_\gamma|}{2(x-t)} dx \right. \\ &\quad \left. - \sum_{\beta=0}^N A_\beta \int_{-1}^1 \frac{\sin(\omega y) |y-y_\beta|}{2(y-t)} dy + \sum_{\beta=0}^N \sum_{\gamma=0}^N A_\beta A_\gamma \frac{|h\beta-h\gamma|}{2} \right]. \end{aligned}$$

After we expand all terms, we obtain the following as one of the main results.

Theorem 1. *The squared norm of the error functional (4) for the quadrature formula (2) is given by*

$$\begin{aligned} \|R\|^2 &= - \left[\int_{-1}^1 \int_{-1}^1 \frac{\sin(\omega x) \sin(\omega y) |x-y|}{2(x-t)(y-t)} dx dy \right. \\ &\quad \left. - \sum_{\beta=0}^N \int_{-1}^1 \frac{A_\beta \sin(\omega x) |x-h\beta+1|}{(x-t)} dx + \sum_{\beta=0}^N \sum_{\gamma=0}^N A_\beta A_\gamma \frac{|h\beta-h\gamma|}{2} \right]. \end{aligned} \quad (7)$$

4 Optimal coefficients of the quadrature formula

Using the expressions obtained above, the squared norm of the error functional $\|R\|^2$ is expressed as a functional of the coefficients A_β . Subject to the condition (5), its minimum with respect to A_β is determined by the method of Lagrange multipliers. Accordingly, we introduce the following Lagrange function:

$$\Psi(A_\beta, \lambda) = \|R(x)\|^2 - 2\lambda \left(\int_{-1}^1 \frac{\sin(\omega x)}{x-t} dx - \sum_{\beta=0}^N A_\beta \right). \quad (8)$$

Differentiating the Lagrange function with respect to the unknowns A_β , $\beta = \overline{0, N}$, and λ and setting the resulting derivatives equal to zero, we obtain the following system of linear equations:

$$\begin{cases} \sum_{\beta=0}^N A_\beta \frac{|h\beta-h\gamma|}{2} + \lambda = f_1(h\gamma), & \gamma = 0, 1, \dots, N, \\ \sum_{\beta=0}^N A_\beta = g_1, \end{cases} \quad (9)$$

where

$$\begin{aligned}
 g_1 &= \operatorname{Im} \left[e^{2\pi i \omega t} \left(\operatorname{Ci}(2\pi \omega(1-t)) - \operatorname{Ci}(2\pi \omega(1+t)) \right) \right. \\
 &\quad \left. + i \left(\operatorname{Si}(2\pi \omega(1-t)) + \operatorname{Si}(2\pi \omega(1+t)) \right) \right], \\
 f_1(h\beta) &= \frac{1}{2} \operatorname{Im} \left[\frac{e^{2\pi i \omega} + e^{-2\pi i \omega} - 2e^{2\pi i \omega(\beta h - 1)}}{2\pi i \omega} + (-\beta h + t + 1)e^{2\pi i \omega t} \right. \\
 &\quad \times \left(\operatorname{Ci}(2\pi \omega(1-t)) - 2 \operatorname{Ci}(2\pi \omega|\beta h - t - 1|) + \operatorname{Ci}(2\pi \omega(1+t)) \right) \\
 &\quad + i \left[\operatorname{Si}(2\pi \omega(1-t)) - 2 \operatorname{sign}(\beta h - t - 1) \operatorname{Si}(2\pi \omega|\beta h - t - 1|) \right. \\
 &\quad \left. \left. - \operatorname{Si}(2\pi \omega(1+t)) \right] \right].
 \end{aligned}$$

Here

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t} dt$$

and

$$\operatorname{Ci}(x) = C + \ln x + \int_0^x \frac{\cos t - 1}{t} dt$$

are respectively the cosine and sine integrals, with C being the Euler constant [20].

The solution of the system (9) yields the optimal coefficients A_β of the quadrature formula, which minimize the error in the Sobolev space $L_2^{(1)}(-1, 1)$. The system (9) is a solvable linear system with $(N+2)$ unknowns, and its solution determines the conditional minimum of $\|R\|^2$. However, solving this system by standard methods is impractical. Therefore, we employ the Sobolev method to determine the optimal coefficients of the quadrature formula from system (9).

We rewrite the system (9) in the following convolution form:

$$\begin{cases} A_\beta * \frac{|h\beta|}{2} + \lambda = f_1(h\beta), & \beta = 0, \dots, N, \\ A_\beta = 0, & \text{for } \beta < 0, \beta > N, \\ \sum_{\beta=0}^N A_\beta = g_1. \end{cases}$$

We simplify the resulting system further:

$$\begin{aligned}
 A_\beta &= hD_1(h\beta) * u(h\beta), \\
 u(h\beta) &= A_\beta \cdot \frac{|h\beta|}{2} + \lambda.
 \end{aligned}$$

Since the convolution with the discrete operator $D_1(h\beta)$ is determined [4], we need to find the form of $u(h\beta)$ for $\beta \in (-\infty, \infty)$, given

$$\beta = 0, 1, 2, \dots, N, \quad u(h\beta) = f_1(h\beta).$$

Definition 1. We define the convolution of two discrete argument functions $\varphi(h\beta)$ and $\psi(h\beta)$ as the inner product as follows:

$$\varphi(h\beta) * \psi(h\beta) = [\varphi(h\gamma), \psi(h\beta - h\gamma)] = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma) \psi(h\beta - h\gamma).$$

In addition, to carry out calculations, we need the discrete analogue of the differential operator $\frac{d^2}{dx^2}$, denoted by $D_1(h\beta)$, which is defined by the following formula [4]:

$$D_1(h\beta) = \begin{cases} 0, & |\beta| \geq 2, \\ h^{-2}, & |\beta| = 1, \\ -2h^{-2}, & \beta = 0. \end{cases}$$

The discrete analogue $D_1(h\beta)$ of the differential operator $\frac{d^2}{dx^2}$ satisfies the following properties:

$$hD_1(h\beta) * \left| \frac{h\beta}{2} \right| = \delta(h\beta),$$

$$D_1(h\beta) * (h\beta)^k = 0, \quad k = 0, 1.$$

Here, $\delta(h\beta)$ is the discrete delta function and

$$\delta(h\beta) = \begin{cases} 0, & \beta \neq 0, \\ 1, & \beta = 0. \end{cases}$$

We now proceed to solve system (9).

For $\beta \leq 0$, we get

$$\begin{aligned} u(h\beta) &= A_\beta * \frac{|h\beta|}{2} + \lambda = \sum_{\gamma=-\infty}^{\infty} A_\gamma \cdot \frac{|h\beta - h\gamma|}{2} + \lambda = \sum_{\gamma=0}^{\infty} A_\gamma \cdot \frac{|h\beta - h\gamma|}{2} + \lambda \\ &= \sum_{\gamma=0}^{\infty} A_\gamma \frac{-(h\beta - h\gamma)}{2} + \lambda = -\frac{h\beta}{2} \underbrace{\sum_{\gamma=0}^N A_\gamma}_{g_1} + \frac{1}{2} \underbrace{\sum_{\gamma=0}^N A_\gamma(h\gamma)}_{\mu} + \lambda. \end{aligned}$$

Then we have:

$$\beta \leq 0 \Rightarrow u(h\beta) = -\frac{h\beta}{2} \cdot g_1 + \mu + \lambda, \quad a_0^- = \mu + \lambda,$$

$$\beta \geq N \Rightarrow u(h\beta) = \frac{h\beta}{2} \cdot g_1 - \mu + \lambda, \quad a_0^+ = -\mu + \lambda.$$

Therefore,

$$u(h\beta) = \begin{cases} -\frac{h\beta}{2} \cdot g_1 + a_0^-, & \beta < 0, \\ f_1(h\beta), & 0 \leq \beta \leq N, \\ \frac{h\beta}{2} \cdot g_1 + a_0^+, & \beta > N, \end{cases}$$

$$\beta = 0 \Rightarrow a_0^- = f_1(0),$$

$$\beta = N \Rightarrow a_0^+ = f_1(hN) - \frac{g_1}{2}.$$

Now, we can find the coefficients

$$\begin{aligned}
 A_\beta &= hD_1(h\beta) * u(h\beta) = h \sum_{\gamma=-\infty}^{\infty} D_1(h\beta - h\gamma) \cdot u(h\gamma) \\
 &= h \left[\sum_{\gamma=-\infty}^{-1} D_1(h\beta - h\gamma)u(h\gamma) + \sum_{\gamma=0}^N D_1(h\beta - h\gamma)f_1(h\gamma) \right. \\
 &\quad \left. + \sum_{\gamma=N+1}^{\infty} D_1(h\beta - h\gamma)u(h\gamma) \right] \\
 &= h \left[\sum_{\gamma=1}^{\infty} D_1(h\beta + h\gamma)u(-h\gamma) + \sum_{\gamma=0}^N D_1(h\beta - h\gamma)f_1(h\gamma) \right. \\
 &\quad \left. + \sum_{\gamma=1}^{\infty} D_1(h\beta - h(N + \gamma))u(h(N + \gamma)) \right].
 \end{aligned}$$

For $\beta = 0$:

$$A_0 = h^{-1} \left(\frac{h}{2}g_1 + f_1(h) - f_1(0) \right),$$

for $1 \leq \beta \leq N - 1$:

$$A_\beta = h^{-1} (f_1(h(\beta - 1)) - 2f_1(h\beta) + f_1(h(\beta + 1))),$$

for $\beta = N$:

$$A_N = h^{-1} \left(f_1(h(N - 1)) - f_1(hN) + \frac{h}{2}g_1 \right).$$

The preceding analysis leads to the following result.

Theorem 2. *The coefficients A_β of the optimal quadrature formula (2) with the error functional (4) are determined by the following expressions:*

$$A_\beta = \begin{cases} h^{-1} \left(\frac{h}{2}g_1 + f_1(h) - f_1(0) \right), & \beta = 0, \\ h^{-1} (f_1(h(\beta - 1)) - 2f_1(h\beta) + f_1(h(\beta + 1))), & 1 \leq \beta \leq N - 1, \\ h^{-1} (f_1(h(N - 1)) - f_1(hN) + \frac{h}{2}g_1), & \beta = N, \end{cases}$$

where g_1 and $f_1(h\beta)$ are known.

5 Conclusions

In this work, an optimal quadrature formula in the sense of Sard has been constructed for the numerical evaluation of Cauchy-type singular integrals with highly oscillatory kernels in a first-order Sobolev space. Using the Sobolev method, explicit analytical expressions for the coefficients have been obtained through the minimization of the norm of the corresponding error functional. This provides a rigorous characterization of the optimal quadrature formula within the considered functional framework.

The proposed construction is specifically adapted to the combined presence of a Cauchy-type singularity and rapid oscillations, which are among the main sources of numerical difficulty in this class of integration problems. The explicit form of the coefficients facilitates the implementation of the resulting quadrature rule and provides a convenient basis for further error analysis.

The obtained results contribute to the numerical treatment of singular oscillatory integrals and may be useful in applications involving boundary integral equations and related computational models. The established approach also opens possibilities for further investigations, including extensions to higher-order Sobolev spaces, more general singular kernels, and broader classes of oscillatory integral operators. These directions may lead to more accurate and efficient numerical schemes for singular integration problems arising in computational mathematics and mathematical physics.

Declarations

Funding. This research received no external funding.

Conflicts of interest. The authors declare no conflict of interest.

Author contributions. The author prepared, read, and approved the final manuscript.

Data availability. The datasets and code generated during the current study are available from the author on reasonable request.

References

- [1] Akhmedov D. M., Hayotov A. R., Shadimetov Kh. M. 2018. Optimal quadrature formulas with derivatives for Cauchy type singular integrals. *Applied Mathematics and Computation*. 317: 150–159. doi: <https://doi.org/10.1016/j.amc.2017.09.009>.
- [2] Akhmedov D. M., Shadimetov Kh. M. 2022. Optimal quadrature formulas for approximate solution of the first kind singular integral equation with Cauchy kernel. *Studia Universitatis Babeş-Bolyai Mathematica*. doi: <https://doi.org/10.24193/subbmth.2022.3.15>.
- [3] Shadimetov Kh. M., Akhmedov D. M. 2024. Numerical integration formulas for hypersingular integrals. *Numerical Mathematics: Theory, Methods and Applications*. 17(3): 805–826. doi: <https://doi.org/10.4208/nmtma.OA-2024-0028>.
- [4] Shadimetov Kh. M. 2019. *Optimal lattice quadrature and cubature formulas in Sobolev spaces*. Tashkent: Science and Technology.
- [5] Shadimetov Kh. M., Akhmedov D. M. 2022. Approximate solution of a singular integral equation using the Sobolev method. *Lobachevskii Journal of Mathematics*. 43(2): 496–505.
- [6] Boltaev A., Hayotov A., Shadimetov Kh. 2021. Construction of optimal quadrature formulas exact for exponential-trigonometric functions by Sobolev's method. *Acta Mathematica Sinica, English Series*. 37(7): 1066–1088. doi: <https://doi.org/10.1007/s10114-021-9506-6>.
- [7] Shadimetov Kh. M., Hayotov A. R., Akhmedov D. M. 2013. Optimal quadrature formulas for the Cauchy type singular integral in the Sobolev space L_2 . *American Journal of Numerical Analysis*. 1(1): 22–31. doi: <https://doi.org/10.12691/ajna-1-1-4>.
- [8] Shadimetov Kh., Hayotov A., Bozarov B. 2022. Optimal quadrature formulas for oscillatory integrals in the Sobolev space. *Journal of Inequalities and Applications*. 2022: Art. 103. doi: <https://doi.org/10.1186/s13660-022-02839-4>.
- [9] Shadimetov Kh. M., Gulomov O. Kh. 2025. Optimal quadrature formulas for calculating integrals of rapidly oscillating functions. *Problems of Computational and Applied Mathematics*.
- [10] Shadimetov Kh. M., Adilkhodjaev A., Gulomov O. 2025. Optimal quadrature formulas for approximate calculation of rapidly oscillating integrals. *Results in Applied Mathematics*. 27: Art. 100627. doi: <https://doi.org/10.1016/j.rinam.2025.100627>.
- [11] Shadimetov Kh., Hayotov A., Khayriev U. 2025. Optimal quadrature formulas for approximating strongly oscillating integrals in the Hilbert space $\widetilde{W}_2^{(m,m-1)}$ of periodic func-

- tions. *Journal of Computational and Applied Mathematics*. 453: Art. 116133. doi: <https://doi.org/10.1016/j.cam.2024.116133>.
- [12] Boltaev A. K., Hayotov A. R., Shadimetov Kh. M. 2014. About coefficients and order of convergence of the optimal quadrature formula. *American Journal of Numerical Analysis*. 2(2): 35–48. doi: <https://doi.org/10.12691/ajna-2-2-2>.
- [13] Shadimetov Kh. M., Nuraliyev F. A., Mirkomilov D. M. 2025. Optimal quadrature formulas for approximate calculation of fast oscillating integral. *Problems of Computational and Applied Mathematics*. 4(68). doi: https://doi.org/10.71310/pcam.4_68.2025.08.
- [14] Okecha G. E. 1987. Quadrature formulae for Cauchy principal value integrals of oscillatory kind. *Mathematics of Computation*. 49: 259–268.
- [15] Capobianco M. R., Criscuolo G. 2003. On quadrature for Cauchy principal value integrals of oscillatory functions. *Journal of Computational and Applied Mathematics*. 156: 471–486.
- [16] Wang H., Xiang S. 2009. Uniform approximations to Cauchy principal value integrals of oscillatory functions. *Applied Mathematics and Computation*. 215: 1886–1894.
- [17] He G., Xiang S. H. 2015. An improved algorithm for the evaluation of Cauchy principal value integrals of oscillatory functions and its application. *Journal of Computational and Applied Mathematics*. 280: 1–13. doi: <https://doi.org/10.1016/j.cam.2014.11.023>.
- [18] Han W., Atkinson K. E. 2011. *Theoretical numerical analysis: a functional analysis framework*. New York: Springer.
- [19] Sobolev S. L. 1974. *Introduction to the theory of cubature formulas*. Moscow: Nauka. (In Russian)
- [20] Abramowitz M., Stegun I. A. 1964. *Handbook of mathematical functions*. Washington, D.C.: National Bureau of Standards.

УДК 519.644.3

ОПТИМАЛЬНЫЕ КВАДРАТУРНЫЕ ФОРМУЛЫ ДЛЯ ИНТЕГРАЛОВ ТИПА КОШИ С ТРИГОНОМЕТРИЧЕСКИМИ ВЕСАМИ*

^{1,2}Буваширов Д.С., ¹Алиев Ш.Е.

dilshodbuvashero@gmail.com

¹Институт математики имени В. И. Романовского Академии наук Республики Узбекистан, 100174, Узбекистан, г. Ташкент, ул. Университетская, 9;

²Ташкентский государственный технический университет имени Ислама Каримова, 100095, Узбекистан, г. Ташкент, ул. Университетская, 2

Актуальность: Сингулярные интегралы с осциллирующими ядрами встречаются в различных областях вычислительной математики и прикладных наук. Их численное вычисление представляет значительную сложность, поскольку сингулярное поведение и быстрые осцилляции проявляются одновременно, что может существенно снижать точность и эффективность стандартных квадратурных методов. Существующие подходы часто рассматривают эти особенности раздельно и не в полной мере решают задачу построения оптимальных квадратурных формул в функционально-аналитической постановке.

*Выражаю благодарность Институту математики имени В. И. Романовского за предоставленные комфортные условия для работы, а профессору Абдулло Хаётову и Дилшоду Ахмедову — за оказанную помощь в улучшении данной работы.

Цель: Целью настоящего исследования является построение оптимальных квадратурных формул для численного вычисления сингулярных интегралов с осциллирующими ядрами и установление их оптимальности в соответствующем пространстве Соболева. Особое внимание уделяется получению явных аналитических выражений для коэффициентов квадратурных формул, пригодных для практической реализации.

Метод: Квадратурная задача формулируется как экстремальная задача для соответствующего функционала погрешности в пространстве Соболева. В рамках метода Соболева коэффициенты квадратурной формулы определяются путем минимизации нормы функционала погрешности на заданном классе квадратурных формул. Аналитические свойства как сингулярного подынтегрального выражения, так и осциллирующего ядра непосредственно учитываются при построении формулы. Такой подход обеспечивает единое рассмотрение сингулярности и быстрых осцилляций и позволяет получить явные аналитические выражения для коэффициентов квадратурной формулы.

Результаты: Построенные квадратурные формулы обладают свойством минимальности нормы соответствующего функционала погрешности в рассматриваемом пространстве Соболева. Полученные явные выражения для коэффициентов позволяют непосредственно реализовать предложенный метод численно и обеспечивают точное вычисление сингулярных осциллирующих интегралов без чрезмерного сгущения сетки. Численные результаты демонстрируют эффективность предложенного подхода при одновременном наличии сингулярного поведения и быстрых осцилляций.

Практическая значимость: Предложенная методология представляет собой строгий с математической точки зрения и вычислительно эффективный подход к построению оптимальных квадратурных формул для сингулярных осциллирующих интегралов. Она расширяет функционально-аналитический подход к численному интегрированию и создает основу для исследования связанных сингулярных интегральных операторов.

Ключевые слова: сингулярный интеграл, высокая осцилляция, оптимальная квадратурная формула, оптимальные коэффициенты, пространство Соболева, функционал ошибок, численное интегрирование

Цитирование: Буваширов Д.С., Алиев Ш.Е. Оптимальные квадратурные формулы для интегралов типа Коши с тригонометрическими весами // Проблемы вычислительной и прикладной математики. – 2026. – № 4/1(75). – С. 190-200.

DOI: 10.71310/psam.4-1_75.2026.15

HISOBLASH VA AMALIY MATEMATIKA MUAMMOLARI

ПРОБЛЕМЫ ВЫЧИСЛИТЕЛЬНОЙ
И ПРИКЛАДНОЙ МАТЕМАТИКИ
PROBLEMS OF COMPUTATIONAL
AND APPLIED MATHEMATICS

ПРОБЛЕМЫ ВЫЧИСЛИТЕЛЬНОЙ И ПРИКЛАДНОЙ МАТЕМАТИКИ

№ 4/1(75) 2026

Журнал основан в 2015 году.

Издается 6 раз в год.

Учредитель:

Научно-исследовательский институт развития цифровых технологий и
искусственного интеллекта.

Главный редактор:

Равшанов Н.

Заместители главного редактора:

Арипов М.М., Шадиметов Х.М., Ахмедов Д.Д.

Ответственный секретарь:

Убайдуллаев М.Ш.

Редакционный совет:

Азамов А.А., Алоев Р.Д., Амиргалиев Е.Н. (Казахстан), Арушанов М.Л.,
Боборахимов Б.И., Бурнашев В.Ф., Джумаёзов У.З., Загребина С.А. (Россия),
Задорин А.И. (Россия), Игнатъев Н.А., Ильин В.П. (Россия),
Иманкулов Т.С. (Казахстан), Исмагилов И.И. (Россия), Кабанихин С.И. (Россия),
Курбонов Н.М., Маматов Н.С., Мирзаев Н.М., Мурадов Ф.А., Назирова Э.Ш.,
Нормуродов Ч.Б., Нуралиев Ф.А., Нуралиев Ф.М., Опанасенко В.Н. (Украина),
Расулмухамедов М.М., Садуллаева Ш.А., Старовойтов В.В. (Беларусь),
Хаётов А.Р., Халджигитов А., Хамдамов Р.Х., Хужаев И.К., Хужаеров Б.Х.,
Эшмаматова Д.Б., Дустмуродова Ш.Ж., Чье Ен Ун (Россия),
Шабозов М.Ш. (Таджикистан), Dimov I. (Болгария), Li Y. (США),
Mascagni M. (США), Min A. (Германия), Singh M. (Южная Корея).

Журнал зарегистрирован в Агентстве информации и массовых коммуникаций при
Администрации Президента Республики Узбекистан.

Свидетельство №0856 от 5 августа 2015 года.

ISSN 2181-8460, eISSN 2181-046X

При перепечатке материалов ссылка на журнал обязательна.

За точность фактов и достоверность информации ответственность несут авторы.

Адрес редакции:

100125, г. Ташкент, м-в. Буз-2, 17А.

Тел.: +(998) 71 263-41-98.

Э-почта: journals@airi.uz.

Веб-сайт: <https://journals.airi.uz>.

Дизайн и вёрстка:

Шарипов Х.Д.

Отпечатано в типографии НИИ РЦТИИ.

Подписано в печать 05.10.2026 г.

Формат 60x84 1/8. Заказ № 5. Тираж 100 экз.

PROBLEMS OF COMPUTATIONAL AND APPLIED MATHEMATICS

No. 4/1(75) 2026

The journal was established in 2015.
6 issues are published per year.

Founder:

Digital Technologies and Artificial Intelligence Development Research Institute.

Editor-in-Chief:

Ravshanov N.

Deputy Editors:

Aripov M.M., Shadimetov Kh.M., Akhmedov D.D.

Executive Secretary:

Ubaydullaev M.Sh.

Editorial Council:

Azamov A.A., Aloev R.D., Amirgaliev E.N. (Kazakhstan), Arushanov M.L.,
Boborakhimov B.I., Burnashev V.F., Djumayozov U.Z., Zagrebina S.A. (Russia),
Zadorin A.I. (Russia), Ignatiev N.A., Ilyin V.P. (Russia), Imankulov T.S. (Kazakhstan),
Ismagilov I.I. (Russia), Kabanikhin S.I. (Russia), Kurbonov N.M., Mamatov N.S.,
Mirzaev N.M., Muradov F.A., Nazirova E.Sh., Normurodov Ch.B., Nuraliev F.A.,
Nuraliev F.M., Opanasenko V.N. (Ukraine), Sadullaeva Sh.A.,
Starovoitov V.V. (Belarus), Khayotov A.R., Khaldjigitov A., Khamdamov R.Kh.,
Khujaev I.K., Khujayorov B.Kh., Eshmamatova D.B., Dustmurodova Sh.J.,
Chye En Un (Russia), Shabozov M.Sh. (Tajikistan), Dimov I. (Bulgaria), Li Y. (USA),
Mascagni M. (USA), Min A. (Germany), Singh M. (South Korea).

The journal is registered by Agency of Information and Mass Communications under the
Administration of the President of the Republic of Uzbekistan.

Certificate of Registration No. 0856 of 5 August 2015.

ISSN 2181-8460, eISSN 2181-046X

At a reprint of materials the reference to the journal is obligatory.
Authors are responsible for the accuracy of the facts and reliability of the information.

Address:

100125, Tashkent, Buz-2, 17A.

Tel.: +(998) 71 263-41-98.

E-mail: journals@airi.uz.

Web-site: <https://journals.airi.uz>.

Layout design:

Sharipov Kh.D.

DTAIRI printing office.

Signed for print 05.10.2026

Format 60x84 1/8. Order No. 5. Print run of 100 copies.

Содержание

Хаётов А.Р., Мухторова М.М.

Функции гистосплайна при восстановлении функций по интегральным значениям 5

Олимов Н.Н.

Численное решение линейных интегральных уравнений Фредгольма второго рода на основе оптимальных интерполяционных формул с производными . . 16

Бабаев С.С., Элмуродова С.С.

Численное приближение дробных интегралов: оптимальный квадратурный подход, основанный на фи-функциях 29

Кулдошев Х.М.

Выражения для коэффициентов оптимальных квадратурных формул с параметром 41

Хаётов А.Р., Хусаинов А.

Альтернативный вывод и численный анализ оптимальной интерполяционной формулы в гильбертовом пространстве 61

Хаитов Т.О.

Оптимальные методы численного интегрирования, основанные на алгебраических и тригонометрических функциях 72

Шадиметов Х.М., Нафасов А.Ю.

Квадрат нормы функционала погрешности интерполяционной формулы с третьей производной в гильбертовом пространстве 85

Болтаев А.К., Пардаева О.Ф.

Интерполяционные сплайны, минимизирующие полунорму в гильбертовом пространстве Соболева 98

Бердимуродова У.А.

Явные коэффициенты и оценка погрешности оптимальной квадратурной формулы 116

Полвонов С.З.

Оптимальная квадратурная формула с кратными узлами в гильбертовом пространстве 127

Барраева С.Ш.

Оптимальные квадратурные формулы с тригонометрическими весами в пространстве Соболева 141

Нуралиев Ф.А., Абдуллаева Г.Ш.

Сравнительный анализ алгебраико-гиперболических натуральных сплайнов шестого порядка и интегральных квазиинтерполяционных сплайнов 153

Ахмедов Д.М., Шомаликова М.Ш.

Оптимальная квадратурная формула типа Эрмита со слабосингулярным весом методом РН-функций 167

Хаётов А.Р., Пирнапасов С.О.

Приближённое вычисление дробной производной Капуто 180

Буваширов Д.С., Алиев Ш.Е.

Оптимальные квадратурные формулы для интегралов типа Коши с тригонометрическими весами 190

Насруллаев П.А., Джурабоева О.С., Саманова Г.Б.

Консервативная модель переноса примесей в атмосфере над сложным рельефом 201

Contents

<i>Hayotov A.R., Mukhtorova M.M.</i>	
Histospline functions in reconstructing functions using integral values	5
<i>Olimov N.N.</i>	
Numerical solution of linear Fredholm integral equations of the second kind based on optimal interpolation formulas with derivatives	16
<i>Babaev S.S., Elmurodova S.S.</i>	
Numerical approximation of fractional integrals: an optimal quadrature approach based on phi-functions	29
<i>Kuldoshev Kh.M.</i>	
Expressions for the coefficients of optimal quadrature formulas with a parameter	41
<i>Hayotov A.R., Husainov A.</i>	
Alternative derivation and numerical analysis of an optimal interpolation formula in the Hilbert space	61
<i>Khaitov T.O.</i>	
Optimal numerical integration rules based on algebraic and trigonometric functions	72
<i>Shadimetov Kh.M., Nafasov A.Y.</i>	
The squared norm of the error functional for a third-derivative interpolation formula in a Hilbert space	85
<i>Boltaev A.K., Pardaeva O.F.</i>	
Interpolation splines minimizing a semi-norm in a Sobolev Hilbert space	98
<i>Berdimuradova U.A.</i>	
Explicit coefficients and error estimate for an optimal quadrature formula	116
<i>Polvonov S.Z.</i>	
Optimal quadrature formula with multiple nodes in a Hilbert space	127
<i>Barraeva S.Sh.</i>	
Optimal quadrature formulas with trigonometric weights in a Sobolev space	141
<i>Nuraliev F.A., Abdullaeva G.Sh.</i>	
A comparative analysis of sixth-order algebraic-hyperbolic natural splines and integral quasi-interpolating splines	153
<i>Akhmedov D.M., Shomalikova M.Sh.</i>	
Optimal Hermite-type quadrature with weakly singular weight via PHI-functions	167
<i>Hayotov A.R., Pirnapasov S.O.</i>	
Approximate calculation of the Caputo fractional-order derivative	180
<i>Buvasharov D.S., Aliev Sh.E.</i>	
Optimal quadrature formulas for highly oscillatory Cauchy-type integrals	190
<i>Nasrullayev P.A., Juraboyeva O.S., Samanova G.B.</i>	
A mass-conserving model of pollutant transport in the atmosphere over complex terrain	201