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APPROXIMATE CALCULATION OF THE CAPUTO FRACTIONAL-ORDER DERIVATIVE

^{1,2}Hayotov A.R., ^{1,3}Pirnapasov S.O.

sardor.pirnapasov@gmail.com

¹V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, 9, University str., Tashkent, 100174 Uzbekistan;

²Central Asian University, 264, Milliy bog str., Barkamol MFY, M. Ulugbek district, Tashkent, 111221 Uzbekistan;

³Webster University in Tashkent, 13, A. Navoi Avenue, Tashkent, 100011 Uzbekistan

Background: The Caputo fractional-order derivative of order $\alpha \in (0, 1)$ is represented by a weakly singular integral of the classical first-order derivative against the kernel $(x - t)^{-\alpha}$, and its accurate numerical evaluation is central to the modeling of processes with memory. Existing methods (the $L1$ scheme, Grünwald–Letnikov-type sums, product-integration schemes) fix an interpolation rule for the derivative first and only afterwards estimate the resulting error, so their coefficients are not chosen to directly minimize that error.

Purpose: The purpose of this work is to construct, for an arbitrary partition of the interval of integration, a quadrature formula for the Caputo derivative whose coefficients are optimal from the outset, in the sense of Sard, in a suitable Sobolev-type Hilbert space.

Method: We pose the approximation problem directly in the space $L_2^{(1)}(a, x)$ and use the φ -function method: on each subinterval of the partition we construct an explicit function φ_k related to the singular kernel, determine its free constant by minimizing the corresponding quadratic error functional, and thereby obtain the coefficients of the quadrature formula in closed form.

Results: Explicit closed-form expressions are obtained for the coefficients $A_0, A_1, \dots, A_n$ of the optimal quadrature formula for an arbitrary node partition, and these are specialized to the case of equally spaced nodes. As an independent check, the constructed formula is shown to reduce, in a particular case, to a known optimal quadrature formula obtained earlier by a different method for the generalized Abel integral equation.

Implication: The result extends the φ -function construction of Sard-optimal quadrature formulas, previously used mainly for regular and oscillatory integrals, to the weakly singular kernels that arise in fractional calculus, and it provides explicit, ready-to-use coefficients for the accurate numerical evaluation of the Caputo derivative on arbitrary meshes.

Keywords: Caputo derivative, fractional calculus, optimal quadrature formula, φ -function method, Sard's sense, Sobolev space, Abel integral equation

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1 Introduction

1.1 Problems leading to Caputo-type derivatives

Fractional-order differentiation and integration operators arise naturally whenever a process depends not only on its present state but also, in a weighted way, on its entire

history. Because of this memory property, fractional calculus has become a standard modeling tool in a wide range of applied problems: anomalous diffusion in porous and heterogeneous media, viscoelastic and rheological models, fractional-order control systems, and fractional relaxation–oscillation equations, to name a few. Comprehensive discussions of the current state, computational pitfalls, and open problems of the field can be found in the survey papers [1, 2].

Among the several nonequivalent definitions of a fractional derivative (Riemann–Liouville, Grünwald–Letnikov, Caputo, and others), the Caputo derivative

$$(D_{a+}^{\alpha} y)(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{y'(t)}{(x-t)^{\alpha}} dt, \quad \alpha \in (0, 1),$$

plays a distinguished role in applications. Unlike the Riemann–Liouville derivative, the Caputo derivative of a constant is zero, and initial-value problems formulated with it require initial data of the classical form $y(a) = y_0$, exactly as for integer-order differential equations. This is one of the main reasons fractional relaxation and diffusion equations, fractional oscillators, and many other applied models are usually written with the Caputo derivative rather than another fractional derivative. At the same time, as the formula above shows, the Caputo derivative of y is nothing but a weakly singular, weighted integral of the classical derivative $y'(t)$, so computing $D_{a+}^{\alpha} y$ accurately comes down to computing an integral with an integrable singularity at $t = x$.

Closely related to the Caputo derivative is the Abel integral equation and its generalizations, since the Abel equation of the first kind is solved precisely through an integral of the same weakly singular type. This connection is well known and has motivated a fairly large body of work on constructing accurate quadrature rules for integrals with a $(x-t)^{-\alpha}$ -type kernel, some of which we use below when we compare our result with the existing literature.

1.2 Existing numerical methods for the Caputo derivative

A number of methods for the approximate calculation of the Caputo derivative already exist.

Probably the most popular one is the $L1$ scheme: the function $y(t)$ is replaced by a piecewise-linear interpolant on the mesh $t_0, t_1, \dots, t_n$, and then the integral against the singular kernel $(x-t)^{-\alpha}$ is computed exactly for this interpolant [3, 4]. It is simple to program and is used very often, although it only gives first-order accuracy, and this accuracy gets worse if y is not smooth enough near $t = a$.

Another group of methods comes from the Grünwald–Letnikov definition of the fractional derivative: the derivative is approximated directly by a (shifted) Grünwald–Letnikov sum, or, more generally, by a fractional linear multistep method built from the power series of $(1-z)^{\alpha}$, as was proposed by Lubich [5]. These give convolution-type formulas, and their coefficients can be found by a recursion.

A third approach, product integration, again keeps the singular part $(x-t)^{-\alpha}$ exact and only approximates $y'(t)$ – usually by a piecewise linear or quadratic polynomial. The fractional Adams–Bashforth–Moulton predictor–corrector method of Diethelm, Ford and Freed [6] is probably the best-known method of this type. There are also spline-based and collocation methods, which can give higher accuracy but need more smoothness of y and are more complicated to implement.

What all of these methods have in common is that the interpolation (or collocation) rule for $f(t) = y'(t)$ is chosen first, more or less independently of the error, and only

afterwards is the error of the resulting formula estimated. So the coefficients of these quadrature formulas are not really chosen to minimize the error on a given class of functions – they simply come from whatever interpolation scheme was used.

1.3 Optimal-in-the-sense-of-Sard quadrature formulas

A different line of work, closer in spirit to what we do here, constructs quadrature formulas that are optimal from the start, in the sense of Sard: for a fixed set of nodes, the coefficients are chosen so that the norm of the error functional over a given Hilbert (or Sobolev-type) space of functions is as small as possible. The foundational construction of such formulas in the space $W_2^{(m,m-1)}$ was given in [7].

Formulas of this kind for weakly singular kernels of the $(x - t)^{-\alpha}$ type have been obtained, for instance, for the numerical solution of the generalized Abel integral equation in the space $L_2^{(1)}$ [8], with further developments and refinements of this construction given in [9, 10]; and, more recently, in the space $W_2^{(2,1)}$, where the coefficients also involve the derivative φ' of the unknown function [10]. A related optimal quadrature formula with a derivative term, for the right-sided Riemann–Liouville integral, was constructed in [11], extending earlier work on the right Riemann–Liouville fractional integral and on fractional integrals in a Hilbert space [12, 13]. Analogous optimal quadrature and weighted quadrature formulas with derivative terms have also been constructed for other classes of singular kernels, including Hadamard-type and Cauchy-type singular integrals [14–16], for periodic functions [17], and, quite recently, in a fully weighted and consecutively optimized setting in general Sobolev spaces [18–21].

The φ -function method that we use below – building an explicit representer of the kernel on each mesh subinterval and choosing its free constants to minimize the quadratic error functional – has likewise been used to construct optimal quadrature formulas for Fourier-type and exponentially weighted integrals [22–24].

1.4 Contribution of the present work

Here we follow this second line of work rather than the first. Instead of choosing an interpolation rule for $f(t) = y'(t)$ first, we pose the approximation problem for the integral representation (1) of the Caputo derivative directly in the Sobolev-type Hilbert space $L_2^{(1)}(a, x)$, and look for the quadrature formula

$$\int_a^x \frac{f(t)}{(x-t)^\alpha} dt = \sum_{k=0}^n A_k f(t_k) + R_n(f)$$

whose coefficients A_k minimize the error functional R_n over the unit ball of $L_2^{(1)}(a, x)$, for fixed nodes $T = (t_0, \dots, t_n)$ – that is, a formula that is optimal in the sense of Sard. To build it for the weakly singular kernel $(x - t)^{-\alpha}$ coming from the Caputo derivative, we use the φ -function method: on every subinterval of the partition we construct an explicit function φ_k related to the kernel, find its free constant by minimizing the corresponding norm, and in this way obtain closed-form expressions for $A_0, A_1, \dots, A_n$. As we show at the end of the paper, our result also lets us recover, as a special case, an optimal quadrature formula obtained earlier by a different route for the related generalized Abel integral equation [8], which is a useful check on the derivation.

The Caputo derivative of order α ($\alpha \in (0, 1)$) of a function $y(t)$ is defined as follows [25]:

$$(D_{a+}^\alpha y)(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{y'(t)}{(x-t)^\alpha} dt. \quad (1)$$

Thus, the Caputo derivative of fractional order α is expressed through a weakly singular weighted integral of the classical first-order derivative of the function. To approximately compute the derivative of the form (1), we construct an optimal quadrature formula in a certain Hilbert space in the present work. To construct such a quadrature formula, we use the φ -function method.

2 The φ -function method for the quadrature formula

We partition the segment $[a, x]$ by the nodes

$$a = t_0 < t_1 < \cdots < t_{k-1} < t_k < \cdots < t_n = x, \quad (2)$$

and on each subinterval $t \in [t_{k-1}, t_k]$ we consider a function $\varphi_k(t)$ satisfying

$$\varphi'_k(t) = \frac{1}{(x-t)^\alpha}, \quad k = 1, 2, \dots, n, \quad (3)$$

and we set

$$\varphi(t)|_{t \in [t_{k-1}, t_k]} = \varphi_k(t).$$

Consider the integral $I(f)$, which is precisely the left-hand side of the quadrature formula (1). Splitting it over the subintervals of the partition and using (3), we obtain

$$\begin{aligned} I(f) &:= \int_a^x \frac{1}{(x-t)^\alpha} f(t) dt = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \frac{1}{(x-t)^\alpha} f(t) dt \\ &= \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \varphi'_k(t) \cdot f(t) dt = \left| \begin{array}{l} u = f, \quad du = f' dt \\ \varphi'_k dt = dv, \quad v = \varphi_k(t) \end{array} \right| \\ &= \sum_{k=1}^n \left(\varphi_k(t) f(t) \right) \Big|_{t_{k-1}}^{t_k} - \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \varphi_k(t) \cdot f'(t) dt \\ &= \sum_{k=1}^n \left(\varphi_k(t_k) f(t_k) - \varphi_k(t_{k-1}) f(t_{k-1}) \right) - \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \varphi_k(t) \cdot f'(t) dt. \end{aligned}$$

Continuing,

$$\begin{aligned} I(f) &= \sum_{k=1}^n \varphi_k(t_k) f(t_k) - \sum_{k=1}^n \varphi_k(t_{k-1}) f(t_{k-1}) - \int_a^x \varphi(t) \cdot f'(t) dt \\ &= \sum_{k=1}^n \varphi_k(t_k) \cdot f(t_k) - \sum_{k=0}^{n-1} \varphi_{k+1}(t_k) f(t_k) + R_n(f), \end{aligned} \quad (4)$$

where

$$R_n(f) = - \int_a^x \varphi(t) \cdot f'(t) dt. \quad (5)$$

By the Cauchy–Schwarz inequality,

$$|R_n(f)| = \left| \int_a^x \varphi(t) \cdot f'(t) dt \right| \leq \sqrt{\int_a^x \varphi^2(t) dt} \cdot \sqrt{\int_a^x (f'(t))^2 dt}. \quad (6)$$

Rewriting (4) by separating the boundary terms ($k = 0$ and $k = n$) from the interior sum, we get

$$\begin{aligned} I(f) &= \sum_{k=1}^{n-1} \varphi_k(t_k) \cdot f(t_k) + \varphi_n(t_n) f(t_n) - \sum_{k=1}^{n-1} \varphi_{k+1}(t_k) \cdot f(t_k) - \varphi_1(t_0) \cdot f(t_0) + R_n(f) \\ &= -\varphi_1(t_0) \cdot f(t_0) + \sum_{k=1}^{n-1} (\varphi_k(t_k) - \varphi_{k+1}(t_k)) f(t_k) + \varphi_n(t_n) f(t_n) + R_n(f). \end{aligned} \quad (7)$$

Comparing (7) with the quadrature formula (1), we identify the coefficients

$$\begin{aligned} A_0 &= -\varphi_1(t_0), \\ A_k &= \varphi_k(t_k) - \varphi_{k+1}(t_k), \quad k = \overline{1, n-1}, \\ A_n &= \varphi_n(t_n). \end{aligned} \quad (8)$$

So once the functions φ_k are known, formulas (8) give the coefficients of the quadrature formula directly.

2.1 Determining the function φ

Integrating (3) once gives, on each subinterval $[t_{k-1}, t_k]$,

$$\varphi_k(t) = \frac{(x-t)^{1-\alpha}}{\alpha-1} + C_k = -\frac{(x-t)^{1-\alpha}}{1-\alpha} + C_k, \quad (9)$$

with a free constant C_k on every subinterval. By (7), the error is controlled by $\|\varphi\|_{L_2}$, so we still have to choose the constants C_k so as to make

$$F_n(A) = \int_a^x \varphi^2(t) dt = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \varphi_k^2(t) dt \quad (10)$$

as small as possible. Because the subintervals do not overlap, this amounts to minimizing, separately for each k ,

$$F_k(C_k) = \int_{t_{k-1}}^{t_k} \varphi_k^2(t) dt \rightarrow \min, \quad C_k - ? \quad (11)$$

$$\begin{aligned} F_k(C_k) &= \int_{t_{k-1}}^{t_k} \left(\frac{(x-t)^{1-\alpha}}{\alpha-1} + C_k \right)^2 dt \\ &= \int_{t_{k-1}}^{t_k} \left(\frac{(x-t)^{2-2\alpha}}{(\alpha-1)^2} + 2 \cdot \frac{(x-t)^{1-\alpha}}{\alpha-1} \cdot C_k + C_k^2 \right) dt. \end{aligned} \quad (12)$$

Setting the derivative with respect to C_k to zero,

$$F'_k(C_k) = 2C_k(t_k - t_{k-1}) + \frac{2}{\alpha-1} \int_{t_{k-1}}^{t_k} (x-t)^{1-\alpha} dt = 0, \quad (13)$$

i.e.

$$C_k(t_k - t_{k-1}) = -\frac{1}{\alpha-1} \int_{t_{k-1}}^{t_k} (x-t)^{1-\alpha} dt. \quad (14)$$

Using $\int (x-t)^{1-\alpha} dt = -\frac{(x-t)^{2-\alpha}}{2-\alpha} + \text{const}$, the right-hand side of (14) evaluates to

$$-\frac{1}{\alpha-1} \cdot \frac{(x-t_{k-1})^{2-\alpha} - (x-t_k)^{2-\alpha}}{2-\alpha} = \frac{(x-t_k)^{2-\alpha} - (x-t_{k-1})^{2-\alpha}}{(\alpha-1)(2-\alpha)},$$

so that

$$C_k = -\frac{(x-t_k)^{2-\alpha} - (x-t_{k-1})^{2-\alpha}}{(1-\alpha)(2-\alpha)(t_k - t_{k-1})}, \quad (15)$$

where we used $(\alpha-1)(2-\alpha) = -(1-\alpha)(2-\alpha)$. It is easy to check directly (by substituting (15) back into (13)) that this is indeed the minimizer, since F_k is a convex quadratic function of C_k .

Substituting (15) into (9), we obtain the explicit form of φ_k :

$$\varphi_k(t) = -\frac{(x-t)^{1-\alpha}}{1-\alpha} - \frac{(x-t_k)^{2-\alpha} - (x-t_{k-1})^{2-\alpha}}{(1-\alpha)(2-\alpha)(t_k - t_{k-1})}. \quad (16)$$

2.2 The coefficients A_0, A_k, A_n

For $k=1$, (16) gives

$$\varphi_1(t) = -\frac{(x-t)^{1-\alpha}}{1-\alpha} - \frac{(x-t_1)^{2-\alpha} - (x-t_0)^{2-\alpha}}{(1-\alpha)(2-\alpha)(t_1 - t_0)}. \quad (17)$$

Since $A_0 = -\varphi_1(t_0)$ and $t_0 = a$, formula (17) evaluated at $t = t_0$ gives

$$A_0 = \frac{(x-a)^{1-\alpha}}{1-\alpha} + \frac{(x-t_1)^{2-\alpha} - (x-a)^{2-\alpha}}{(1-\alpha)(2-\alpha)(t_1 - a)}. \quad (18)$$

Next, evaluating (16) at $t = t_k$ and the analogous expression for φ_{k+1} at $t = t_k$,

$$\varphi_k(t_k) = -\frac{(x-t_k)^{1-\alpha}}{1-\alpha} - \frac{(x-t_k)^{2-\alpha} - (x-t_{k-1})^{2-\alpha}}{(1-\alpha)(2-\alpha)(t_k - t_{k-1})}, \quad (19)$$

$$\varphi_{k+1}(t_k) = -\frac{(x-t_k)^{1-\alpha}}{1-\alpha} - \frac{(x-t_{k+1})^{2-\alpha} - (x-t_k)^{2-\alpha}}{(1-\alpha)(2-\alpha)(t_{k+1} - t_k)}. \quad (20)$$

The $(x-t_k)^{1-\alpha}/(1-\alpha)$ terms cancel in $A_k = \varphi_k(t_k) - \varphi_{k+1}(t_k)$, and we obtain

$$A_k = \frac{1}{(1-\alpha)(2-\alpha)} \left(\frac{(x-t_{k+1})^{2-\alpha} - (x-t_k)^{2-\alpha}}{t_{k+1} - t_k} - \frac{(x-t_k)^{2-\alpha} - (x-t_{k-1})^{2-\alpha}}{t_k - t_{k-1}} \right), \quad (21)$$

$k = \overline{1, n-1}$.

Finally, since $t_n = x$, formula (16) for $k = n$ gives $\varphi_n(t)$ with $(x-t_n)^{1-\alpha} = (x-t_n)^{2-\alpha} = 0$ at $t = t_n$, so that

$$A_n = \varphi_n(t_n) = \frac{(x-t_{n-1})^{1-\alpha}}{(1-\alpha)(2-\alpha)}. \quad (22)$$

We collect these results in the following theorem.

Theorem 1. Among all quadrature formulas of the form (7) with fixed nodes $T = (t_0, t_1, \dots, t_n)$, $a = t_0 < t_1 < \dots < t_n = x$, the quadrature formula

$$\int_a^x \frac{f(t)}{(x-t)^\alpha} dt \cong \sum_{k=0}^n A_k f(t_k) + R_n(f) \quad (23)$$

that is optimal in the sense of Sard in the space $L_2^{(1)}(a, x)$ has coefficients given by

$$A_0 = \frac{(x-a)^{1-\alpha}}{1-\alpha} + \frac{(x-t_1)^{2-\alpha} - (x-a)^{2-\alpha}}{(1-\alpha)(2-\alpha)(t_1-a)}, \quad (24)$$

$$A_k = \frac{1}{(1-\alpha)(2-\alpha)} \left(\frac{(x-t_{k+1})^{2-\alpha} - (x-t_k)^{2-\alpha}}{t_{k+1} - t_k} - \frac{(x-t_k)^{2-\alpha} - (x-t_{k-1})^{2-\alpha}}{t_k - t_{k-1}} \right), \quad (25)$$

$$k = \overline{1, n-1},$$

$$A_n = \frac{(x-t_{n-1})^{1-\alpha}}{(1-\alpha)(2-\alpha)}. \quad (26)$$

Corollary 1. When the nodes of (2) are equally spaced, $h = t_k - t_{k-1} = \frac{x-a}{n}$ for every k , the coefficients of the optimal quadrature formula (23) take the form

$$A_0 = \frac{(x-a)^{1-\alpha}}{1-\alpha} + \frac{(x-a-h)^{2-\alpha} - (x-a)^{2-\alpha}}{(1-\alpha)(2-\alpha)h}, \quad (27)$$

$$A_k = \frac{h^{1-\alpha}}{(1-\alpha)(2-\alpha)} \left((n-k+1)^{2-\alpha} - 2(n-k)^{2-\alpha} + (n-k-1)^{2-\alpha} \right), \quad k = \overline{1, n-1}, \quad (28)$$

$$A_n = \frac{h^{1-\alpha}}{(1-\alpha)(2-\alpha)}. \quad (29)$$

Proof. With $a = t_0$ and equal spacing, $x - t_k = x - a - kh = (n-k)h$; substituting this into (24)–(26) and factoring out $h^{2-\alpha}$ from the differences of powers in (25) gives (27)–(29) after simplification.

Remark 1. It is worth checking this result against a formula obtained by an entirely different method for a closely related problem. Setting $a = 0$ in Corollary 1 and replacing α by $1 - \alpha$ throughout (so that $2 - \alpha$ becomes $1 + \alpha$, etc.), formulas (27)–(29) turn into

$$A_0 = \frac{x^\alpha}{\alpha} + \frac{h^{-1}}{\alpha(\alpha+1)} \left((x-h)^{\alpha+1} - x^{\alpha+1} \right),$$

$$A_k = \frac{h^{-1}}{\alpha(\alpha+1)} \left((x-(k+1)h)^{\alpha+1} - 2(x-kh)^{\alpha+1} + (x-(k-1)h)^{\alpha+1} \right), \quad k = \overline{1, n-1},$$

$$A_n = \frac{h^\alpha}{\alpha(\alpha+1)},$$

which is exactly the optimal quadrature formula obtained by Shadimetov and Daliev in the space $L_2^{(1)}(0, x)$ for the generalized Abel integral equation (see [8, Theorem 1]). In other words, that earlier result is recovered as the particular case $a = 0$, $\alpha \mapsto 1 - \alpha$ of Corollary 1 above, which gives a useful independent check on our derivation.

3 Conclusion

In this paper, we constructed an optimal quadrature formula, in the sense of Sard, for the weakly singular integral that represents the Caputo fractional-order derivative of order $\alpha \in (0, 1)$ in the Sobolev-type Hilbert space $L_2^{(1)}(a, x)$. Using the φ -function method, we obtained explicit closed-form expressions for the coefficients $A_0, A_1, \dots, A_n$ of the formula for an arbitrary partition of the interval of integration, and we specialized these expressions to the practically important case of equally spaced nodes. As a further check on the derivation, we showed that, as a particular case, the constructed formula reduces exactly to a known optimal quadrature formula obtained earlier by a different method for the generalized Abel integral equation, which confirms the correctness of our result. In future work, we plan to extend the φ -function approach used here to Caputo derivatives of order $\alpha \in (m - 1, m)$, $m > 1$, and to Sobolev spaces of higher smoothness $L_2^{(m)}(a, x)$, as well as to study the corresponding error estimates and carry out numerical experiments illustrating the accuracy of the constructed formulas.

Declarations

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Author contributions. Hayotov A.R.: problem statement, supervision. Pirnapasov S.O.: derivation of the optimal quadrature formula, writing of the manuscript.

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ПРИБЛИЖЁННОЕ ВЫЧИСЛЕНИЕ ДРОБНОЙ ПРОИЗВОДНОЙ КАПУТО

^{1,2}Хаётов А.Р., ^{1,3}Пирнапасов С.О.

sardor.pirnapasov@gmail.com

¹Институт математики имени В. И. Романовского Академии наук Республики Узбекистан, 100174, Узбекистан, г. Ташкент, ул. Университетская, 9;²Центрально-Азиатский университет, 111221, Узбекистан, г. Ташкент, ул. Миллий бог, 264;³Вебстерский университет в Ташкенте, 100011, Узбекистан, г. Ташкент, просп. А. Навои, 13

Предпосылки: Дробная производная Капуто порядка $\alpha \in (0, 1)$ представляется в виде слабо сингулярного интеграла от классической производной первого порядка с ядром $(x - t)^{-\alpha}$, и её точное численное вычисление играет центральную роль при моделировании процессов с памятью. Известные методы ($L1$ -схема, суммы типа Грюнвальда–Летникова, схемы продукт-интегрирования) сначала фиксируют правило интерполяции производной и лишь затем оценивают возникающую погрешность, поэтому их коэффициенты не выбираются с целью прямой минимизации этой погрешности.

Цель: Цель данной работы — построить для произвольного разбиения интервала интегрирования квадратурную формулу для производной Капуто, коэффициенты которой оптимальны с самого начала, в смысле Сарда, в подходящем гильбертовом пространстве типа Соболева.

Метод: Задача приближения ставится непосредственно в пространстве $L_2^{(1)}(a, x)$, и используется метод φ -функций: на каждом подынтервале разбиения строится явная функция φ_k , связанная с сингулярным ядром, её свободная константа определяется минимизацией соответствующего квадратичного функционала погрешности, и таким образом коэффициенты квадратурной формулы получаются в замкнутом виде.

Результаты: Получены явные замкнутые выражения для коэффициентов $A_0, A_1, \dots, A_n$ оптимальной квадратурной формулы при произвольном разбиении узлов, которые затем специализируются на случай равноотстоящих узлов. В качестве независимой проверки показано, что построенная формула в частном случае совпадает с известной оптимальной квадратурной формулой, ранее полученной другим методом для обобщённого интегрального уравнения Абеля.

Значимость: Результат расширяет метод φ -функций построения оптимальных в смысле Сарда квадратурных формул, ранее применявшийся в основном для регулярных и осциллирующих интегралов, на слабо сингулярные ядра, возникающие в дробном исчислении, и даёт явные, готовые к использованию коэффициенты для точного численного вычисления производной Капуто на произвольных сетках.

Ключевые слова: производная Капуто, дробное исчисление, оптимальная квадратурная формула, метод φ -функций, в смысле Сарда, пространство Соболева, интегральное уравнение Абеля

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Содержание

Хаётов А.Р., Мухторова М.М.

Функции гистосплайна при восстановлении функций по интегральным значениям 5

Олимов Н.Н.

Численное решение линейных интегральных уравнений Фредгольма второго рода на основе оптимальных интерполяционных формул с производными . . 16

Бабаев С.С., Элмуродова С.С.

Численное приближение дробных интегралов: оптимальный квадратурный подход, основанный на фи-функциях 29

Кулдошев Х.М.

Выражения для коэффициентов оптимальных квадратурных формул с параметром 41

Хаётов А.Р., Хусаинов А.

Альтернативный вывод и численный анализ оптимальной интерполяционной формулы в гильбертовом пространстве 61

Хаитов Т.О.

Оптимальные методы численного интегрирования, основанные на алгебраических и тригонометрических функциях 72

Шадиметов Х.М., Нафасов А.Ю.

Квадрат нормы функционала погрешности интерполяционной формулы с третьей производной в гильбертовом пространстве 85

Болтаев А.К., Пардаева О.Ф.

Интерполяционные сплайны, минимизирующие полунорму в гильбертовом пространстве Соболева 98

Бердимуродова У.А.

Явные коэффициенты и оценка погрешности оптимальной квадратурной формулы 116

Полвонов С.З.

Оптимальная квадратурная формула с кратными узлами в гильбертовом пространстве 127

Барраева С.Ш.

Оптимальные квадратурные формулы с тригонометрическими весами в пространстве Соболева 141

Нуралиев Ф.А., Абдуллаева Г.Ш.

Сравнительный анализ алгебраико-гиперболических натуральных сплайнов шестого порядка и интегральных квазиинтерполяционных сплайнов 153

Ахмедов Д.М., Шомаликова М.Ш.

Оптимальная квадратурная формула типа Эрмита со слабосингулярным весом методом РН-функций 167

Хаётов А.Р., Пирнапасов С.О.

Приближённое вычисление дробной производной Капуто 180

Буваширов Д.С., Алиев Ш.Е.

Оптимальные квадратурные формулы для интегралов типа Коши с тригонометрическими весами 190

Насруллаев П.А., Джурабоева О.С., Саманова Г.Б.

Консервативная модель переноса примесей в атмосфере над сложным рельефом 201

Contents

<i>Hayotov A.R., Mukhtorova M.M.</i>	
Histospline functions in reconstructing functions using integral values	5
<i>Olimov N.N.</i>	
Numerical solution of linear Fredholm integral equations of the second kind based on optimal interpolation formulas with derivatives	16
<i>Babaev S.S., Elmurodova S.S.</i>	
Numerical approximation of fractional integrals: an optimal quadrature approach based on phi-functions	29
<i>Kuldoshev Kh.M.</i>	
Expressions for the coefficients of optimal quadrature formulas with a parameter	41
<i>Hayotov A.R., Husainov A.</i>	
Alternative derivation and numerical analysis of an optimal interpolation formula in the Hilbert space	61
<i>Khaitov T.O.</i>	
Optimal numerical integration rules based on algebraic and trigonometric functions	72
<i>Shadimetov Kh.M., Nafasov A.Y.</i>	
The squared norm of the error functional for a third-derivative interpolation formula in a Hilbert space	85
<i>Boltaev A.K., Pardaeva O.F.</i>	
Interpolation splines minimizing a semi-norm in a Sobolev Hilbert space	98
<i>Berdimuradova U.A.</i>	
Explicit coefficients and error estimate for an optimal quadrature formula	116
<i>Polvonov S.Z.</i>	
Optimal quadrature formula with multiple nodes in a Hilbert space	127
<i>Barraeva S.Sh.</i>	
Optimal quadrature formulas with trigonometric weights in a Sobolev space	141
<i>Nuraliev F.A., Abdullaeva G.Sh.</i>	
A comparative analysis of sixth-order algebraic-hyperbolic natural splines and integral quasi-interpolating splines	153
<i>Akhmedov D.M., Shomalikova M.Sh.</i>	
Optimal Hermite-type quadrature with weakly singular weight via PHI-functions	167
<i>Hayotov A.R., Pirnapasov S.O.</i>	
Approximate calculation of the Caputo fractional-order derivative	180
<i>Buvasharov D.S., Aliev Sh.E.</i>	
Optimal quadrature formulas for highly oscillatory Cauchy-type integrals	190
<i>Nasrullayev P.A., Juraboyeva O.S., Samanova G.B.</i>	
A mass-conserving model of pollutant transport in the atmosphere over complex terrain	201