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A COMPARATIVE ANALYSIS OF SIXTH-ORDER ALGEBRAIC-HYPERBOLIC NATURAL SPLINES AND INTEGRAL QUASI-INTERPOLATING SPLINES

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Algebraic-hyperbolic splines are important in numerical analysis because they combine smoothness and approximation accuracy. In a number of applications, pointwise function values may be noisy or unavailable, while subinterval integral values constitute the available data. Existing integro-spline constructions address this situation, but comparison with sixth-order algebraic-hyperbolic natural splines constructed from point values remains relevant.

The purpose of this work is to construct a sixth-order algebraic-hyperbolic natural interpolation spline and to compare its approximation accuracy with several spline methods constructed from integral values.

The interpolating natural spline is constructed in a functional space associated with a third-order differential operator containing a hyperbolic parameter. A discrete analogue of the corresponding operator is used to derive the computational algorithm and explicit systems for the spline coefficients. An important reproduction property of the proposed spline is established: it exactly reproduces constant and hyperbolic functions.

Numerical comparisons are carried out for two test functions and several numbers of knots. The approximation errors of the proposed algebraic-hyperbolic natural spline are compared with those of an algebraic-hyperbolic integro quasi-interpolant, local integro-splines, an integro-quadratic spline, and an algebraic-trigonometric integro-spline. For the reported tests, the proposed method yields smaller maximum approximation errors.

The results indicate that the constructed algebraic-hyperbolic natural spline is a useful high-accuracy approximation tool and provides a basis for comparison between point-interpolation and integral-data spline techniques.

Keywords: algebraic-hyperbolic spline, natural spline, integro-interpolation, differential operator, convergence rate, error estimation

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1 Introduction

Initially, the concept of a spline function was introduced as a mathematical apparatus by I. J. Schoenberg in 1946 [1]. The theory of splines can be approached from two distinct perspectives: the first is an algebraic approach, which relies on examining the system of equations that defines the spline; the second is a variational approach, which utilizes the fundamental integral relation between functions belonging to the spaces $K^m(a, b)$ and $K^{2m}(a, b)$ and the approximating splines. Here,

$$K^m(a, b) = \{ \varphi : [a, b] \rightarrow \mathbb{R}, \varphi^{(m-1)}(x) \text{ is absolutely continuous, } \varphi^{(m)} \in L_2(a, b) \}, \quad (1)$$

and this class of functions $K^m(a, b)$ forms a Hilbert space factorized with respect to the following inner product:

$$\langle f, g \rangle_{K^m(a,b)} = \int_a^b Lf(x)Lg(x) dx.$$

In this space, the norm of a function is defined via the above inner product as follows:

$$\|f\|_{K^m(a,b)} = \sqrt{\int_a^b (Lf(x))^2 dx}. \quad (2)$$

Here, L is a differential operator of the form

$$L \equiv a_m(x) \frac{d^m}{dx^m} + a_{m-1}(x) \frac{d^{m-1}}{dx^{m-1}} + \cdots + a_1(x) \frac{d}{dx} + a_0(x),$$

where $a_m(x) \neq 0$ and $a_j(x) \in C^j[a, b]$ for $j = 0, 1, \dots, m$.

The extensive development of splines constructed via the variational method is linked to the scientific work of J.C. Holladay published in 1957 [2]. In that paper, it was shown that a piecewise cubic spline belonging to the class $C^2[a, b] = \{f : [a, b] \rightarrow \mathbb{R} \mid f'' \text{ is continuous}\}$ minimizes the following functional:

$$\int_a^b [f''(x)]^2 dx.$$

Later, J.C. Holladay's work was generalized by J.H. Ahlberg, E.N. Nilson, and J.L. Walsh, resulting in the first systematic monograph on spline theory [3].

According to [3], let a grid $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ and a set of values $Y = \{y_i \mid i = 0, 1, \dots, N\}$ be given. Among all functions $f(x) \in K^m(a, b)$ satisfying the interpolation conditions $f(x_i) = y_i$ for $i = 0, 1, \dots, N$, the generalized spline $S_\Delta(Y; x)$, when it exists, minimizes the quantity:

$$\int_a^b (Lf(x))^2 dx. \quad (3)$$

In solving these variational problems in practice, particularly when high-order derivatives are involved, systems of linear equations arise. To solve such systems, the method proposed by S.L. Sobolev is widely used. Using Sobolev's method, interpolation splines minimizing semi-norms have been constructed in various Sobolev-type functional spaces, including $L_2^{(m)}(0, 1)$, $K_2(P_3)$, and $K^7(a, b)$ [4–6].

In recent years, Sobolev's method has been further developed and applied to a broad class of approximation and numerical integration problems. In particular, optimal quadrature formulas for oscillatory integrals in Sobolev spaces have been constructed, with explicit expressions obtained for their optimal coefficients [7]. Optimal interpolation formulas have also been derived by using discrete analogues of differential operators, demonstrating the effectiveness of this approach for determining optimal interpolation coefficients in Sobolev spaces [8]. Related developments include optimal quadrature formulas for the numerical integration of fractional integrals in Hilbert spaces [9] and algorithms based on discrete analogues of higher-order differential operators for constructing optimal integration formulas [10].

The scope of the Sobolev approach has also been extended to the numerical solution of integral equations. Optimal quadrature formulas with derivatives have been applied to singular integral equations of the first kind with Cauchy kernels [11, 12]. For generalized Abel integral equations, optimal formulas in Sobolev spaces have been used to develop accurate approximate-analytical solution algorithms [13]. More recently, optimal quadrature methods have been applied to Fredholm integral equations of the second kind, illustrating the effectiveness of Sobolev-based optimization techniques in the numerical treatment of integral equations [14].

Further progress has been made in the construction of optimal integration formulas in Hilbert and Sobolev spaces. An algorithm based on discrete analogues of differential operators has been developed for constructing optimal quadrature formulas that are exact for hyperbolic functions and polynomials [15]. Optimal quadrature formulas with equally spaced nodes have also been obtained by Sobolev's method [16]. In addition, Sard's problem for optimal quadrature formulas in a Sobolev-type space has been investigated using the extremal function, Lagrange multipliers, and a discrete analogue of a high-order differential operator [17]. Weighted optimal quadrature formulas with derivatives have subsequently been constructed in Sobolev spaces and applied to the approximate evaluation of regular, singular, fractional, and oscillatory integrals [18]. Recent results have further extended these ideas to spaces of differentiable functions with mixed derivative data [19].

Of particular relevance to the present study are recent applications of Sobolev's method to spline approximation. Natural spline interpolation has been investigated in a Hilbert space, where Sobolev's method is used to obtain an algorithm for solving the system of linear algebraic equations that determines the coefficients of the natural spline [20]. These developments show that Sobolev's method provides a unified computational framework for optimal interpolation, quadrature formulas, numerical solution of integral equations, and the construction of natural splines, and therefore provides a natural methodological basis for the high-order algebraic-hyperbolic spline construction considered in the present work.

While the aforementioned variational approaches and Sobolev space formulations effectively address interpolation based on discrete point values $f(x_i)$, many real-world problems present a different constraint structure. In various physical and mathematical models, individual node values are inaccessible, and the available data consists solely of subinterval integrals. This limitation necessitates the transition from point-wise spline interpolation to integro-spline reconstruction techniques. In many practical applications across mechanics, climatology, oceanography, and numerical analysis, node values of an unknown function $f(x)$ are unavailable, and only its subinterval integrals

$$I_i = \int_{x_i}^{x_{i+1}} f(x)dx, \quad i = 0, 1, \dots, n-1, \quad (4)$$

on a partition $a = x_0 < x_1 < \dots < x_N = b$ are known. The goal of integro-interpolation is to reconstruct an approximating function $S(x)$ satisfying $I_i = \int_{x_i}^{x_{i+1}} S(x)dx$.

Spline functions offer a natural framework for this task. H. Behforooz [21, 22] first introduced integro-cubic and integro-quintic splines based on Hermite and Hermite-Birkhoff interpolation, although these constructions required additional boundary conditions and did not directly address the approximation of derivatives. To simplify the construction, T. Zhanlav and R. Mijiddorj [23] proposed local integro-cubic splines and investigated their approximation properties. Subsequently, higher-order schemes were developed to

improve the approximation accuracy. In particular, J.M. Wu and X.L. Zhang [24] constructed an integro-sextic spline interpolation scheme and established its superconvergence properties.

Now, we construct the interpolating natural spline that minimizes the integral (3) for $L_3 = \frac{d^3}{dx^3} - v^2 \frac{d}{dx}$, where $v \in \mathbb{R} \setminus \{0\}$. Then we compare the proposed spline with several existing splines constructed from integral values.

2 An interpolating natural spline minimizing the norm in the space $K_{2,v}^{(3)}$

According to the general theory presented in [3], the natural spline minimizing integral (3) when $L_3 = \frac{d^3}{dx^3} - v^2 \frac{d}{dx}$ is a sixth-order algebraic-hyperbolic natural interpolation spline defined as follows, where the corresponding Hilbert space is denoted by $K_{2,v}^{(3)}$.

Definition 1. A sixth-order algebraic-hyperbolic natural spline is a uniquely determined function satisfying the following conditions:

1. $S_{\Delta,3}(x)$ is a linear combination of the functions $\sinh(vx)$, $\cosh(vx)$, 1 , $x \sinh(vx)$, $x \cosh(vx)$, and x on each interval $(x_\beta, x_{\beta+1})$ for $\beta = 0, 1, \dots, N-1$;
2. $S_{\Delta,3}(x)$ is a linear combination of the functions $\sinh(vx)$, $\cosh(vx)$, and 1 on the intervals $(-\infty, a)$ and (b, ∞) ;
3. $S_{\Delta,3}(x)$ satisfies the following continuity and natural boundary conditions:

$$S_{\Delta,3}^{(\alpha)}(x_\beta - 0) = S_{\Delta,3}^{(\alpha)}(x_\beta + 0), \quad \alpha = 0, 1, 2, 3, 4, \quad \beta = 1, 2, \dots, N-1,$$

$$L_3 S_{\Delta,3}(a) = 0, \quad L_3 S_{\Delta,3}(b) = 0, \quad (L_3 S_{\Delta,3}(a))' = 0, \quad (L_3 S_{\Delta,3}(b))' = 0;$$

4. $S_{\Delta,3}(x)$ satisfies the interpolation conditions:

$$S_{\Delta,3}(x_\beta) = f(x_\beta), \quad \beta = 0, 1, \dots, N.$$

Based on the definition above, we obtain the following theorem regarding the explicit form and coefficients of the sixth-order algebraic-hyperbolic natural spline.

Theorem 1. The function minimizing the norm in the space $K_{2,v}^{(3)}$ is an algebraic-hyperbolic natural spline, which takes the following form:

$$S_{\Delta,3}(x) = \sum_{\gamma=0}^N C_\gamma G_3(x - x_\gamma) + d_1 \sinh(vx) + d_2 \cosh(vx) + d_3, \quad (5)$$

where C_γ ($\gamma = 0, 1, \dots, N$), d_1 , d_2 , and d_3 are the coefficients of spline (5) satisfying the following system of linear equations:

$$\sum_{\gamma=0}^N C_\gamma G_3(x_\beta - x_\gamma) + d_1 \sinh(vx_\beta) + d_2 \cosh(vx_\beta) + d_3 = f(x_\beta), \quad \beta = 0, 1, \dots, N, \quad (6)$$

$$\sum_{\gamma=0}^N C_\gamma \sinh(vx_\gamma) = 0, \quad (7)$$

$$\sum_{\gamma=0}^N C_\gamma \cosh(vx_\gamma) = 0, \quad (8)$$

$$\sum_{\gamma=0}^N C_{\gamma} = 0, \quad (9)$$

and $G_3(x)$ is the fundamental solution of the operator $L_3^* L_3$, given by [25]:

$$G_3(x) = \frac{\text{sign}(x)}{4v^5} (vx \cosh(vx) - 3 \sinh(vx) - 2vx). \quad (10)$$

In the next section, we give an algorithm to solve (6)–(9) equation system.

3 An algorithm for solving the system of equations

Using the discrete analogue of the differential operator $L_3^* L_3$, we present an algorithm for solving the system of equations (6)–(9). We assume that the nodes x_{β} are equally spaced. **Algorithmic Scheme:**

Step 1: Initialization and convolution formulation:

Set the node coordinates $x_{\beta} = h\beta$ ($h = 1/N$) and zero-pad the coefficient sequence by assuming $C_{\beta} = 0$ for $\beta < 0$ and $\beta > N$. Using the discrete convolution operation, the original system (6)–(9) is rewritten as follows:

$$G_3(h\beta) * C_{\beta} + d_1 \sinh(vx_{\beta}) + d_2 \cosh(vx_{\beta}) + d_3 = f(x_{\beta}), \quad \beta = 0, 1, \dots, N, \quad (11)$$

$$\sum_{\gamma=0}^N C_{\gamma} \sinh(vx_{\gamma}) = 0, \quad (12)$$

$$\sum_{\gamma=0}^N C_{\gamma} \cosh(vx_{\gamma}) = 0, \quad (13)$$

$$\sum_{\gamma=0}^N C_{\gamma} = 0, \quad (14)$$

where $G_3(h\beta)$ is the function with discrete argument corresponding to (10).

Step 2: Introduction of additional notations and discrete operator:

In (11), for arbitrary values of β , we introduce the following notation.

$$\mu_3(h\beta) = G_3(h\beta) * C_{\beta}, \quad (15)$$

$$u_3(h\beta) = \mu_3(h\beta) + d_1 \sinh(vh\beta) + d_2 \cosh(vh\beta) + d_3, \quad (16)$$

where $u_3(h\beta) = f(h\beta)$ for $0 \leq \beta \leq N$. Convoluting both sides of (16) with the discrete analogue $D_3(h\beta)$ of the differential operator $\frac{d^6}{dx^6} - 2v^2 \frac{d^4}{dx^4} + v^4 \frac{d^2}{dx^2}$, we obtain the inversion formula:

$$C_{\beta} = D_3(h\beta) * u_3(h\beta). \quad (17)$$

Step 3: Evaluation of outer regions and parameter definition:

To determine $u_3(h\beta)$ outside $[0, 1]$ (for $\beta < 0$ and $\beta > N$), compute the convolution $\mu_3(h\beta) = G_3(h\beta) * C_{\beta}$. Introducing the coefficients

$$b_1 = \frac{1}{4} \sum_{\gamma=-\infty}^{\infty} C_{\gamma} (vh\gamma) \sinh(vh\gamma), \quad b_2 = \frac{1}{4} \sum_{\gamma=-\infty}^{\infty} C_{\gamma} (vh\gamma) \cosh(vh\gamma),$$

$$b_3 = \frac{1}{2} \sum_{\gamma=-\infty}^{\infty} C_{\gamma} (hv\gamma),$$

we obtain $\mu_3(h\beta) = -b_1 \sinh(vh\beta) + b_2 \cosh(vh\beta) + b_3$ for $\beta \leq 0$, and $\mu_3(h\beta) = b_1 \sinh(vh\beta) - b_2 \cosh(vh\beta) - b_3$ for $\beta \geq N$. Defining $d_1^\pm = d_1 \mp b_1$, $d_2^\pm = d_2 \pm b_2$, $d_3^\pm = d_3 \pm b_3$, the complete function $u_3(h\beta)$ is constructed as:

$$u_3(h\beta) = \begin{cases} d_1^- \sinh(vh\beta) + d_2^- \cosh(vh\beta) + d_3^-, & \beta \leq 0, \\ f(h\beta), & 0 \leq \beta \leq N, \\ d_1^+ \sinh(vh\beta) + d_2^+ \cosh(vh\beta) + d_3^+, & \beta \geq N. \end{cases} \quad (18)$$

Step 4: System solution and spline reconstruction:

Utilize the exterior condition:

$$C_\beta = D_3(h\beta) * u_3(h\beta) = 0, \quad h\beta \notin [0, 1], \quad (19)$$

to solve for the unknown parameters $d_1^-, d_2^-, d_3^-, d_1^+, d_2^+, d_3^+$. Finally, using the calculated profile of $u_3(h\beta)$, evaluate (17) to obtain the explicit values of the spline coefficients C_β ($\beta = 0, 1, \dots, N$) and d_1, d_2, d_3 .

Using the above algorithm, in the next section we present a theorem on the coefficients C_β ($\beta = 0, 1, \dots, N$), d_1, d_2, d_3 of the interpolating spline.

4 The coefficients of an interpolating spline

When $L_3 = \frac{d^3}{dx^3} - v^2 \frac{d}{dx}$, we obtained the following theorem for the coefficients of a sixth-order algebraic-hyperbolic interpolating natural spline that minimizes integral (3).

Theorem 2. *The coefficients of the interpolation spline (5) minimizing the semi-norm in the space $K_{2,v}^{(3)}$ with equally spaced knots are given by the following formulas:*

$$\begin{aligned} C_0 &= \frac{2v^5}{p_4^{(4)}} \{ -d_1^- \sinh vh + d_2^- \cosh vh + d_3^- + C_3 f(0) + f(h) \} + \\ &\quad + \frac{2v^5}{p_4^{(4)}} \cdot \sum_{k=1}^2 \frac{A_k}{\lambda_k} [M_k + \sum_{\gamma=0}^N \lambda_k^\gamma f(h\gamma) + \lambda_k^N N_k], \\ C_\beta &= \frac{2v^5}{p_4^{(4)}} \cdot \{ f(h(\beta-1)) + C_3 f(h\beta) + f(h(\beta+1)) \} + \\ &\quad + \frac{2v^5}{p_4^{(4)}} \cdot \sum_{k=1}^2 \frac{A_k}{\lambda_k} [\lambda_k^\beta M_k + \sum_{\gamma=0}^N \lambda_k^{|\beta-\gamma|} f(h\gamma) + \lambda_k^{N-\beta} N_k], \\ C_N &= \frac{2v^5}{p_4^{(4)}} \cdot \{ d_1^+ \sinh(v(h+1)) + d_2^+ \cosh(v(h+1)) + d_3^+ + f(1-h) + C_3 f(1) \} + \\ &\quad + \frac{2v^5}{p_4^{(4)}} \cdot \sum_{k=1}^2 \frac{A_k}{\lambda_k} [\lambda_k^N M_k + \sum_{\gamma=0}^N \lambda_k^{N-\gamma} f(h\gamma) + N_k], \\ d_i &= \frac{1}{2} (d_i^+ + d_i^-), \quad i = 1, 2, 3, \end{aligned}$$

where

$$\begin{aligned} M_k &= -d_1^- \frac{\lambda_k \sinh vh}{1 + \lambda_k^2 - 2\lambda_k \cosh(vh)} + d_2^- \frac{\lambda_k (\cosh(vh) - \lambda_k)}{1 + \lambda_k^2 - 2\lambda_k \cosh(vh)} + d_3^- \frac{\lambda_k}{1 - \lambda_k}, \\ N_k &= d_1^+ \frac{\lambda_k (\sinh(v(h+1)) - \lambda_k \sinh v)}{1 + \lambda_k^2 - 2\lambda_k \cosh(vh)} + d_2^+ \frac{\lambda_k (\cosh(v(h+1)) - \lambda_k \cosh v)}{1 + \lambda_k^2 - 2\lambda_k \cosh(vh)} + d_3^+ \frac{\lambda_k}{1 - \lambda_k}, \end{aligned}$$

and

$$A_k = \frac{(1 - \lambda_k)^2 (\lambda_k^2 - 2\lambda_k \cosh hv + 1)^2 p_4^{(4)}}{\lambda_k P'_4(\lambda_k)}, \quad k = 1, 2, \quad C_3 = -4 \cosh hv - 2 - \frac{p_3^{(4)}}{p_4^{(4)}},$$

$$p_4^{(4)} = hv \cosh hv - 3 \sinh hv + 2(hv),$$

$$p_3^{(4)} = -2(hv \cosh hv - 3 \sinh(hv)) + (-2vh + 3 \sinh(2vh)) - 8(hv) \cosh(vh),$$

and λ_k are the roots of the following polynomial satisfying $|\lambda_k| < 1$:

$$P_4(x) = (1 - x)^2 [hv \cosh hv - 3 \sinh hv] x^2 + [3 \sinh(2hv) - 2hv] x + [hv \cosh hv - 3 \sinh hv] + 2(hv)(x^2 - 2x \cosh hv + 1)^2,$$

$$d_2^- = f(0) - d_3^-, \quad d_2^+ = \frac{f(1) - d_1^+ \sinh(v) - d_3^+}{\cosh(v)},$$

and $d_1^-, d_3^-, d_1^+, d_3^+$ are found by the following equation system

$$\begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{pmatrix} \times \begin{pmatrix} d_1^- \\ d_3^- \\ d_1^+ \\ d_3^+ \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{pmatrix},$$

where

$$B_{i1} = \frac{2v^5}{p_4^{(4)}} [\sinh(vh(i-1)) + C_3 \sinh(vhi) + \sinh(vh(i+1)) +$$

$$+ \sum_{k=1}^2 \frac{A_k}{\lambda_k} \sum_{\gamma=1}^{\infty} \lambda_k^{|\gamma-i|} \sinh(vh\gamma)],$$

$$B_{i2} = \frac{2v^5}{p_4^{(4)}} [2 + C_3 + \sum_{k=1}^2 \frac{A_k}{\lambda_k} \sum_{\gamma=1}^{\infty} \lambda_k^{|\gamma-i|} -$$

$$- (\cosh(vh(i-1)) + C_3 \cosh(vhi) + \cosh(vh(i+1)) +$$

$$+ \sum_{k=1}^2 \frac{A_k}{\lambda_k} \sum_{\gamma=1}^{\infty} \lambda_k^{|\gamma-i|} \cosh(vh\gamma)],$$

$$B_{i3} = \frac{2}{p_4^{(4)} \cosh v} \sum_{k=1}^2 A_k \lambda_k^{N+i-1} \sum_{\gamma=1}^{\infty} \lambda_k^{\gamma} \sinh(vh\gamma),$$

$$B_{i4} = \frac{2}{p_4^{(4)}} [\sum_{k=1}^2 A_k \lambda_k^{N+i-1} \sum_{\gamma=1}^{\infty} \lambda_k^{\gamma} -$$

$$- \frac{1}{\cosh v} \sum_{k=1}^2 A_k \lambda_k^{N+i-1} \sum_{\gamma=1}^{\infty} \lambda_k^{\gamma} \cosh(v(h\gamma+1))], \quad i = 1, 2,$$

and

$$\begin{aligned}
B_{(i+2)1} &= \frac{2v^5}{p_4^{(4)}} \sum_{k=1}^2 A_k \lambda_k^{N+i-1} \sum_{\gamma=1}^{\infty} \lambda_k^{\gamma} \sinh(vh\gamma), \\
B_{(i+2)2} &= \frac{2v^5}{p_4^{(4)}} \left[\sum_{k=1}^2 A_k \lambda_k^{N+i-1} \sum_{\gamma=1}^{\infty} \lambda_k^{\gamma} - \frac{1}{\cosh v} \sum_{k=1}^2 A_k \lambda_k^{N+i-1} \sum_{\gamma=1}^{\infty} \lambda_k^{\gamma} \cosh(vh\gamma) \right], \\
B_{(i+2)3} &= \frac{2v^5}{p_4^{(4)} \cosh v} [\sinh(vh(i-1)) + C_3 \sinh(vhi) + \sinh(vh(i+1)) + \\
&\quad + \sum_{k=1}^2 \frac{A_k}{\lambda_k} \sum_{\gamma=1}^{\infty} \lambda_k^{|\gamma-i|} \sinh(vh\gamma)], \\
B_{(i+2)4} &= \frac{2v^5}{p_4^{(4)}} [C_3 + 2 + \sum_{k=1}^2 \frac{A_k}{\lambda_k} \sum_{\gamma=1}^{\infty} \lambda_k^{|\gamma-i|} - \\
&\quad - \frac{1}{\cosh v} \{ \cosh(vh(i-1) + 1) + C_3 \cosh(vhi + 1) + \cosh(vh(i+1) + 1) + \\
&\quad + \sum_{k=1}^2 \frac{A_k}{\lambda_k} \sum_{\gamma=1}^{\infty} \lambda_k^{|\gamma-i|} \cosh(v(h\gamma + 1)) \}], \quad i = 1, 2.
\end{aligned}$$

Our next aim is to compare the constructed algebraic-hyperbolic spline with splines constructed from integral values. Therefore, in the following section, we present some important types of integro-interpolating splines.

5 Some important types of integro-interpolation splines

In this section, we give information about integro-quadratic spline, algebraic-hyperbolic quasi-interpolant, local integro-quadratic and algebraic-trigonometric quadratic B -splines.

a) In [26], integro-quadratic splines were constructed. Let us assume that $S(x) \in C^1[a, b]$ and that $S(x)$ is a quadratic polynomial on each subinterval $[x_i, x_{i+1}]$, where

$$a = x_0 < x_1 < \dots < x_N = b.$$

Then $S(x)$ is called a quadratic spline function with respect to the knots $x_0, x_1, \dots, x_N$.

If the integral values I_i of the function $f(x)$ are given on each interval $[x_i, x_{i+1}]$ and the equality

$$\int_{x_i}^{x_{i+1}} S(x) dx = I_i = \int_{x_i}^{x_{i+1}} f(x) dx, \quad i = 0, 1, \dots, N-1,$$

is satisfied, then $S(x)$ is called an integro-interpolating spline. In this case, the integral values I_i are used to construct such an integro-quadratic spline $S(x)$.

The spline $S(x)$ is defined in the following form:

$$S(x) = S(x_i)H_i(x) + I_i H_{i,i+1}(x) + S(x_{i+1})H_{i+1}(x), \quad x \in [x_i, x_{i+1}],$$

where

$$H_i(x) = \left(1 - 3 \frac{x - x_i}{x_{i+1} - x_i}\right) \left(\frac{x - x_{i+1}}{x_i - x_{i+1}}\right),$$

$$H_{i,i+1}(x) = \frac{6}{x_{i+1} - x_i} \left(\frac{x - x_{i+1}}{x_i - x_{i+1}} \right) \left(\frac{x - x_i}{x_{i+1} - x_i} \right),$$

$$H_{i+1}(x) = \left(1 - 3 \frac{x - x_{i+1}}{x_i - x_{i+1}} \right) \left(\frac{x - x_i}{x_{i+1} - x_i} \right).$$

b) In [27], an algebraic-hyperbolic quasi-interpolant spline was constructed. Let f be a real-valued function defined on the interval $I = [a, b]$, and let

$$\chi_N := \{t_i = a + i\alpha, \ i = 0, \dots, N\}$$

be a uniform partition of the interval I with step size α . Assume that the following values are known:

$$A_i := \int_{x_i}^{x_{i+1}} f(x) dx, \ i = 0, 1, \dots, N-1.$$

A non-polynomial spline $q(x)$ satisfying the following conditions is considered

$$A_i \simeq \int_{x_i}^{x_{i+1}} q(x) dx, \ i = 0, 1, \dots, N-1,$$

here, the spline q belongs to the algebraic-hyperbolic uniform space defined by the following expression:

$$\Omega_{\alpha, I} := \left\{ S \in C^1(I) : S|_{[t_i, t_{i+1}]} \in \text{span}\{1, \sinh, \cosh\}, \ 0 \leq i \leq N-1 \right\}.$$

Initially, we consider the AH quasi-interpolant defined in the following form:

$$Qf := \sum_{i=1}^{N+2} \mu_i(f) N_i$$

Here, N_i , $1 \leq i \leq N+2$, are algebraic-hyperbolic (AH) B -splines, and $\mu_i(f)$ depend on the values of the function f at the nodal points. The values of the function f at the knots are approximated using the given integral values A_i , $i = 0, 1, \dots, N-1$. Using the Taylor formula, the following representation is obtained:

$$f(x) := f(x_i) + G_2(t, x_i) f'(x_i) + G_3(t, x_i) f''(x_i) + \sum_{j=4}^{\infty} G_j(t, x_i) L_{j-1} f(t_i).$$

Here, $G_k(t, x)$ is the fundamental solution of the differential operator $L_k := D^{k-2}(D^2 - 1)$.

c) In [23], the problem of constructing quasi-interpolation of local integro-quadratic splines was considered.

d) In [28], algebraic-trigonometric quadratic B -splines were considered.

Let us consider subintervals $[x_i, x_{i+1}]$, $i = 0, 1, \dots, N-1$, on the interval $[a, b]$. In this work, the integro-spline interpolation problem is defined as follows. Let $S(x) \in C^1[a, b]$ be a function such that on each subinterval $[x_i, x_{i+1}]$ it is a quadratic algebraic-trigonometric function. If the integral values I_i of the function $f(x)$ on each interval $[x_i, x_{i+1}]$ are known and the following relations hold:

$$\int_{x_i}^{x_{i+1}} S(x) dx = \int_{x_i}^{x_{i+1}} f(x) dx = I_i, \quad i = 0, 1, \dots, N-1,$$

then $S(x)$ is called a quadratic algebraic-trigonometric integro-spline interpolant. The spline $S(x)$ is represented in the following form:

$$S(x) = S_i T_i(x) + I_i T_{i,i+1}(x) + S_{i+1} T_{i+1}(x)$$

where

$$T_i(x) = \frac{-h \sin(x - x_{i+1}) - \cos(x - x_i) + \cos(x - x_{i+1}) + \cos(h) - 1}{h \sin(h) + 2(\cos(h) - 1)}$$

$$T_{i+1}(x) = \frac{h \sin(x - x_i) + \cos(x - x_i) + \cos(x - x_{i+1}) + \cos(h) - 1}{h \sin(h) + 2(\cos(h) - 1)}$$

$$T_{i,i+1}(x) = \frac{\cos(-\frac{x_i}{2} - \frac{x_{i+1}}{2} + x) \sec(\frac{h}{2}) - 1}{2 \tan(\frac{h}{2})}$$

where $h = x_{i+1} - x_i$ denotes the step size. In the following, we compare the approximation accuracy of the constructed algebraic-hyperbolic natural spline with that of splines constructed using integral values through numerical results.

6 Numerical results

In what follows, using the values at the knots and the boundary conditions, we compare the approximation errors of the interpolating algebraic-hyperbolic natural spline constructed in the form (5) for the parameter value $v = 0.1$ with those of the integro-interpolation splines considered in cases (a)–(d), where the integral values of the function are assumed to be known.

For the function $f(x) = \cos(\pi x)$, we present a table of the maximum approximation errors obtained by the sixth-order algebraic-hyperbolic interpolating natural spline developed in this work, the algebraic-hyperbolic integro quasi-interpolant [27], the local integro-cubic spline [23], and the integro-quadratic spline interpolation [26] for different numbers of knots, namely $N = 10$, $N = 20$, and $N = 40$.

Table 1 Maximum approximation errors $R_N(f)$ for $f(x) = \cos(\pi x)$.

N	Our method	Method in [27]	Method in [23]	Method in [26]
10	1.18×10^{-4}	1.83×10^{-4}	6.21×10^{-4}	2.50×10^{-4}
20	7.37×10^{-6}	1.07×10^{-5}	4.05×10^{-5}	3.11×10^{-5}
40	4.59×10^{-7}	6.59×10^{-7}	2.55×10^{-6}	3.86×10^{-6}

For the function $g(x) = \frac{\exp(\frac{1}{1+x^2}) \tanh(\frac{x}{10\pi})}{1+16x^3}$, we present a table of the maximum approximation errors obtained by the constructed algebraic-hyperbolic natural spline in the space $K_{2,v}^{(3,2)}(0,1)$, as well as by the algebraic-hyperbolic integro quasi-interpolant [27], integro-quadratic spline [26], and algebraic-trigonometric integro-spline [28] methods for different numbers of knots $N = 8, 16, 32, 64, 128$.

The comparative analysis of the numerical results presented in Tables 1 and 2 shows that the proposed algebraic-hyperbolic natural spline demonstrates higher approximation accuracy compared with the traditional integro-splines based on integral mean values. The obtained errors in approximating the test functions confirm that the constructed model provides a more effective tool for the reconstruction of function characteristics.

Table 2 Maximum approximation errors $R_N(g)$ for the function $g(x) = \frac{\exp(\frac{1}{1+x^2}) \tanh(\frac{x}{10\pi})}{1+16x^3}$.

N	Our method	Method in [27]	Method in [26]	Method in [28]
8	1.64×10^{-4}	1.24×10^{-3}	1.64×10^{-2}	1.69×10^{-2}
16	9.22×10^{-6}	7.81×10^{-5}	1.00×10^{-3}	1.01×10^{-3}
32	7.91×10^{-7}	4.88×10^{-6}	6.26×10^{-5}	6.28×10^{-5}
64	7.7×10^{-8}	3.05×10^{-7}	3.91×10^{-6}	3.92×10^{-6}
128	8.27×10^{-9}	1.90×10^{-8}	2.44×10^{-7}	2.45×10^{-7}

7 Conclusion

In this work, the problem of constructing a sixth-order algebraic-hyperbolic natural spline using a discrete analogue of the differential operator was considered.

To evaluate the approximation accuracy of the proposed algebraic-hyperbolic natural spline, numerical experiments were performed for two test functions. The obtained results were compared with various types of integro-splines constructed based on integral values, including the algebraic-hyperbolic quasi-interpolant, integro quadratic spline, local integro cubic spline and algebraic-trigonometric integro-spline.

The computational experiments showed that the proposed algebraic-hyperbolic natural spline produced smaller errors than the considered integro-spline methods in approximating the test functions. This demonstrates the potential of the constructed spline model as an effective tool for high-accuracy approximation problems.

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СРАВНИТЕЛЬНЫЙ АНАЛИЗ АЛГЕБРАИКО-ГИПЕРБОЛИЧЕСКИХ НАТУРАЛЬНЫХ СПЛАЙНОВ ШЕСТОГО ПОРЯДКА И ИНТЕГРАЛЬНЫХ КВАЗИИНТЕРПОЛЯЦИОННЫХ СПЛАЙНОВ

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Предпосылки: Алгебраико-гиперболические сплайны важны в численном анализе благодаря сочетанию гладкости и высокой точности аппроксимации. Во многих задачах точечные значения функции могут быть зашумлены или недоступны, тогда как исходными данными служат интегральные значения на подотрезках. Поэтому актуально сравнение методов, использующих точечные и интегральные данные.

Цель: Цель работы состоит в построении алгебраико-гиперболического натурального интерполяционного сплайна шестого порядка и сравнении его точности с рядом сплайновых методов, построенных по интегральным значениям.

Метод: Натуральный интерполяционный сплайн строится в функциональном пространстве, связанном с дифференциальным оператором третьего порядка, содержащим гиперболический параметр. Дискретный аналог соответствующего дифференциального оператора используется для построения вычислительного алгоритма

и получения явных систем для определения коэффициентов сплайна. Установлено важное воспроизводящее свойство построенного сплайна: он точно воспроизводит постоянные и гиперболические функции.

Результаты: Для двух тестовых функций и различных чисел узлов проведено численное сравнение с алгебраико-гиперболическим интегральным квазиинтерполянт, локальными интегро-сплайнами, интегро-квадратичным и алгебраико-тригонометрическим интегро-сплайнами. В приведённых вычислительных экспериментах предложенный метод даёт меньшие максимальные погрешности аппроксимации.

Значимость: Полученные результаты показывают возможность применения построенного натурального сплайна в задачах высокоточной аппроксимации и позволяют сопоставлять методы по точечным и интегральным данным.

Ключевые слова: алгебраико-гиперболический сплайн, натуральный сплайн, интегро-интерполяция, дифференциальный оператор, скорость сходимости, оценка погрешности

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