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OPTIMAL QUADRATURE FORMULAS WITH TRIGONOMETRIC WEIGHTS IN A SOBOLEV SPACE

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Background: The accurate and efficient numerical evaluation of highly oscillatory integrals remains a challenging problem in computational mathematics and applications such as mathematical physics, wave dynamics, optics, and signal processing. Classical approaches, including Filon-, Levin-, and asymptotic-type methods, are effective for highly oscillatory integrals but do not directly address the construction of optimal quadrature formulas in Sobolev spaces or the minimization of the associated error functional.

Purpose: This study aims to construct an optimal quadrature formula for highly oscillatory integrals with trigonometric weights in Sobolev spaces, determine its quadrature coefficients, and investigate the behavior of the corresponding error.

Method: The integral is decomposed into sine and cosine components and transformed through piecewise integration to obtain a formulation suitable for the Sobolev-space framework. The quadrature coefficients are derived by minimizing the norm of the associated error functional. A system of conditions characterizing the optimal coefficients is established, and explicit expressions are obtained for uniformly distributed nodes. This approach incorporates both the oscillatory structure of the integrand and the functional properties of the underlying Sobolev space into the quadrature construction.

Results: Explicit analytical expressions for the optimal quadrature coefficients are obtained. The derived formulas satisfy the required exactness conditions for constant and linear functions. Numerical experiments demonstrate that the quadrature error exhibits a non-monotonic dependence on the oscillation parameter, with substantial error reduction observed for specific parameter values.

Implication: The proposed quadrature formula provides a Sobolev-space-based optimal approach for highly oscillatory integrals and extends existing quadrature techniques by incorporating oscillatory behavior directly into the optimization of the approximation error.

Keywords: highly oscillatory integrals, optimal quadrature formula, optimal coefficients, Sobolev space, error functional, numerical integration

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1 Introduction

Highly oscillatory integrals arise in many areas of applied mathematics and mathematical physics, including wave propagation, optics, electrodynamics, quantum mechanics, and signal processing [1, 2]. Their accurate and efficient numerical evaluation becomes particularly challenging when the oscillation frequency is high. Such integrals also occur in Fourier and Bessel transforms, tomographic reconstruction, and other computational problems involving rapidly varying functions [3, 4]. Quadrature formulas provide an important class of numerical methods for their approximation [5, 6].

The numerical treatment of highly oscillatory integrals has a long history. Filon introduced one of the first specialized quadrature methods for trigonometric integrals [5]. Levin proposed a differential-equation-based approach for oscillatory integrals, while Iserles and Nørsett developed methods incorporating derivative information [4, 6]. Olver investigated asymptotic techniques for the numerical evaluation of highly oscillatory functions [7]. These approaches have provided effective tools for a wide range of oscillatory integration problems.

The theory of optimal quadrature formulas is closely connected with approximation theory and functional analysis. Fundamental results by Sard, Sobolev, and Nikolsky established important foundations for optimal integration and cubature formulas in functional spaces. In this framework, the error of a quadrature formula is represented by an error functional, and the optimal formula is obtained by minimizing an appropriate norm of this functional [8, 9]. This approach provides a rigorous framework for determining optimal coefficients and estimating the corresponding approximation error.

Optimal quadrature formulas in Sobolev spaces have been studied extensively, including formulas involving derivatives and formulas with positive coefficients [10, 11]. Further developments have considered optimal formulas in the sense of Sard, Sobolev spaces of complex-valued functions, and higher-order Sobolev spaces [12–14]. In these studies, extremal functions of error functionals are used to derive explicit representations of the optimal coefficients and corresponding error estimates.

Derivative optimal quadrature and interpolation formulas in Sobolev spaces have also attracted considerable attention [10, 11, 15]. The use of equally spaced nodes is of particular practical interest because it leads to convenient composite formulas and facilitates their numerical implementation [16]. Recent studies have further investigated the minimization of error-functional norms and the construction of formulas with explicitly determined coefficients [9, 16].

Fourier integrals represent an important class of highly oscillatory problems for which Sobolev-space methods are particularly useful. An optimal quadrature formula for the approximate calculation of Fourier integrals was constructed using the φ -function method, which provides a systematic procedure for determining optimal coefficients through the minimization of the corresponding error-functional norm [17]. Related results on optimal interpolation formulas with derivatives and auxiliary methods further develop this functional approach [18–20].

Despite these developments, the construction of optimal quadrature formulas with derivatives for highly oscillatory integrals with trigonometric weights remains insufficiently studied. In particular, incorporating derivative information together with oscillatory sine and cosine weights within a Sobolev-space framework requires further investigation.

In this paper, we construct optimal quadrature formulas with derivatives for highly oscillatory integrals with trigonometric weights in the Sobolev space $L_2^{(1)}(a, b)$. The coefficients are determined using the φ -function method by minimizing the norms of the corresponding error functionals. The exactness properties of the constructed formulas are established, and their approximation errors are examined numerically.

2 Problem statement and method

In this paper, we consider the highly oscillatory integral of the following form in the space $L_2^{(1)}(a, b)$

$$I(f) = \int_a^b e^{2\pi i \omega x} f(x) dx, \quad (1)$$

where ω is the oscillation parameter, $f(x)$ is an integrable function, and $a = x_0 < x_1 < \dots < x_n = b$.

According to Euler's formula,

$$e^{2\pi i \omega x} = \cos(2\pi \omega x) + i \sin(2\pi \omega x) \quad (2)$$

holds. Therefore, we separate the integral (1) into its real and imaginary parts:

$$I(f) = \underbrace{\int_a^b \cos(2\pi \omega x) f(x) dx}_{I_c(f)} + i \underbrace{\int_a^b \sin(2\pi \omega x) f(x) dx}_{I_s(f)} \quad (3)$$

and introduce the notations $I_c(f)$ and $I_s(f)$ respectively:

$$I_c(f) = \int_a^b \cos(2\pi \omega x) f(x) dx, \quad (4)$$

$$I_s(f) = \int_a^b \sin(2\pi \omega x) f(x) dx. \quad (5)$$

We construct optimal quadrature formulas for integrals (4) and (5):

$$I_c(f) \cong \sum_{k=0}^n A_k^c f(x_k), \quad I_s(f) \cong \sum_{k=0}^n A_k^s f(x_k). \quad (6)$$

To transform the integrals in (4) and (5) into a form suitable for constructing the optimal quadrature formula, we introduce the functions $\varphi_k^c(x)$ and $\varphi_k^s(x)$ satisfying

$$(\varphi_k^c(x))' = \cos(2\pi \omega x), \quad (\varphi_k^s(x))' = \sin(2\pi \omega x), \quad k = 1, \dots, n.$$

Here, $\varphi_k^c(x)$ and $\varphi_k^s(x)$ denote the corresponding functions on the interval $[x_{k-1}, x_k]$.

Hence, their general forms are given by

$$\begin{aligned} \varphi_k^c(x) &= \frac{1}{2\pi\omega} \sin(2\pi \omega x) + C_k^c, \\ \varphi_k^s(x) &= -\frac{1}{2\pi\omega} \cos(2\pi \omega x) + C_k^s, \quad k = 1, \dots, n, \end{aligned} \quad (7)$$

where C_k^c and C_k^s are constants of integration to be determined optimally.

Integrating by parts with $u = f(x)$ and $dv = (\varphi_k^c(x))' dx$ for $I_c(f)$ and $dv = (\varphi_k^s(x))' dx$ for $I_s(f)$, we obtain $du = f'(x) dx$ and $v = \varphi_k^c(x)$ or $v = \varphi_k^s(x)$, respectively. Thus, the integrals $I_c(f)$ and $I_s(f)$ are transformed as follows:

$$\begin{aligned} I_c(f) &= \int_a^b \cos(2\pi \omega x) f(x) dx = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} (\varphi_k^c(x))' f(x) dx = \sum_{k=1}^n [\varphi_k^c(x) f(x)]_{x_{k-1}}^{x_k} - \\ &\quad - \sum_{k=1}^n \int_{x_{k-1}}^{x_k} \varphi_k^c(x) f'(x) dx = \sum_{k=1}^n \varphi_k^c(x_k) f(x_k) - \sum_{k=1}^n \varphi_k^c(x_{k-1}) f(x_{k-1}) - \end{aligned}$$

$$\begin{aligned}
& - \sum_{k=1}^n \int_{x_{k-1}}^{x_k} \varphi_k^c(x) f'(x) dx = \sum_{k=1}^n \varphi_k^c(x_k) f(x_k) - \sum_{k=0}^{n-1} \varphi_{k+1}^c(x_k) f(x_k) - \\
& - \sum_{k=1}^n \int_{x_{k-1}}^{x_k} \varphi_k^c(x) f'(x) dx = \varphi_n^c(x_n) f(x_n) + \sum_{k=1}^{n-1} (\varphi_k^c(x_k) - \varphi_{k+1}^c(x_k)) f(x_k) - \\
& - \varphi_1^c(x_0) f(x_0) - \sum_{k=1}^n \int_{x_{k-1}}^{x_k} \varphi_k^c(x) f'(x) dx.
\end{aligned} \tag{8}$$

Applying this method to expression (6) as well, we obtain the following results:

$$\begin{aligned}
I_c(f) &= \varphi_n^c(x_n) f(x_n) + \sum_{k=1}^{n-1} (\varphi_k^c(x_k) - \varphi_{k+1}^c(x_k)) f(x_k) - \\
& - \varphi_1^c(x_0) f(x_0) - \sum_{k=1}^n \int_{x_{k-1}}^{x_k} \varphi_k^c(x) f'(x) dx,
\end{aligned} \tag{9}$$

$$\begin{aligned}
I_s(f) &= \varphi_n^s(x_n) f(x_n) + \sum_{k=1}^{n-1} (\varphi_k^s(x_k) - \varphi_{k+1}^s(x_k)) f(x_k) - \\
& - \varphi_1^s(x_0) f(x_0) - \sum_{k=1}^n \int_{x_{k-1}}^{x_k} \varphi_k^s(x) f'(x) dx.
\end{aligned} \tag{10}$$

Introducing the following notations for the coefficients in (9) and (10), respectively:

$$\begin{aligned}
A_0^c &= -\varphi_1^c(x_0), \\
A_k^c &= \varphi_k^c(x_k) - \varphi_{k+1}^c(x_k), \quad k = \overline{1, n-1}, \\
A_n^c &= \varphi_n^c(x_n),
\end{aligned} \tag{11}$$

$$\begin{aligned}
A_0^s &= -\varphi_1^s(x_0), \\
A_k^s &= \varphi_k^s(x_k) - \varphi_{k+1}^s(x_k), \quad k = \overline{1, n-1}, \\
A_n^s &= \varphi_n^s(x_n)
\end{aligned} \tag{12}$$

the corresponding error functionals are given by

$$\begin{aligned}
R_{n,1}^c(f) &= - \sum_{k=1}^n \int_{x_{k-1}}^{x_k} \varphi_k^c(x) f'(x) dx, \\
R_{n,1}^s(f) &= - \sum_{k=1}^n \int_{x_{k-1}}^{x_k} \varphi_k^s(x) f'(x) dx.
\end{aligned}$$

To determine the optimal values of the constants C_k^c and C_k^s , we introduce the corresponding error functionals:

$$F_k(C_k^c) = \int_{x_{k-1}}^{x_k} (\varphi_k^c(x))^2 dx, \quad F_k(C_k^s) = \int_{x_{k-1}}^{x_k} (\varphi_k^s(x))^2 dx. \tag{13}$$

By minimizing the error functionals in (13), we determine the optimal values of the constants C_k^c and C_k^s .

$$\frac{dF_k(C_k^c)}{dC_k^c} = 0,$$

$$\frac{dF_k(C_k^s)}{dC_k^s} = 0.$$

Solving the resulting equations yields the optimal values:

$$C_k^c = -\frac{1}{(x_k - x_{k-1})(2\pi\omega)^2} (\cos(2\pi\omega x_k) - \cos(2\pi\omega x_{k-1})), \quad (14)$$

$$C_k^s = \frac{1}{(x_k - x_{k-1})(2\pi\omega)^2} (\sin(2\pi\omega x_k) - \sin(2\pi\omega x_{k-1})). \quad (15)$$

Using these coefficients in (7), we obtain the following expressions:

$$\varphi_k^c = \frac{1}{2\pi\omega} \sin(2\pi\omega x) - \frac{1}{(x_k - x_{k-1})(2\pi\omega)^2} (\cos(2\pi\omega x_k) - \cos(2\pi\omega x_{k-1})), \quad (16)$$

$$\varphi_k^s = -\frac{1}{2\pi\omega} \cos(2\pi\omega x) + \frac{1}{(x_k - x_{k-1})(2\pi\omega)^2} (\sin(2\pi\omega x_k) - \sin(2\pi\omega x_{k-1})). \quad (17)$$

Theorem 1. *The coefficients of the approximation formula of the form (6) in the space $L_2^{(1)}(a, b)$, for $\omega \in \mathbb{R} \setminus \{0\}$, are determined by (11) and (12). For the equally spaced nodes*

$$x_k = a + kh, \quad h = \frac{b-a}{n}, \quad k = 0, 1, \dots, n,$$

the optimal coefficients A_k^c and A_k^s take the following simplified forms:

$$A_0^c = -\frac{\sin(2\pi\omega x_0)}{2\pi\omega} + \frac{\cos(2\pi\omega x_1) - \cos(2\pi\omega x_0)}{h(2\pi\omega)^2},$$

$$A_k^c = -\frac{4 \cos(2\pi\omega x_k) \sin^2(\pi\omega h)}{h(2\pi\omega)^2}, \quad k = 1, \dots, n-1, \quad (18)$$

$$A_n^c = \frac{\sin(2\pi\omega x_n)}{2\pi\omega} - \frac{\cos(2\pi\omega x_n) - \cos(2\pi\omega x_{n-1})}{h(2\pi\omega)^2}.$$

$$A_0^s = \frac{\cos(2\pi\omega x_0)}{2\pi\omega} - \frac{\sin(2\pi\omega x_1) - \sin(2\pi\omega x_0)}{h(2\pi\omega)^2},$$

$$A_k^s = \frac{4 \sin(2\pi\omega x_k) \sin^2(\pi\omega h)}{h(2\pi\omega)^2}, \quad k = 1, \dots, n-1, \quad (19)$$

$$A_n^s = -\frac{\cos(2\pi\omega x_n)}{2\pi\omega} + \frac{\sin(2\pi\omega x_n) - \sin(2\pi\omega x_{n-1})}{h(2\pi\omega)^2}.$$

Thus, the quadrature formula of the form (6), with the optimal coefficients obtained by minimizing the error functional defined in (13), is referred to as an optimal quadrature formula.

2.1 Finding the norms of φ_k^c and φ_k^s

First, we determine the norm of φ_k^c . From expression (16), we obtain the square of its norm.

$$\begin{aligned} \|\varphi^c\|^2 &= \sum_{k=1}^n \int_{x_{k-1}}^{x_k} (\varphi_k^c(x))^2 dx = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} \left[\frac{\sin^2(2\pi\omega x)}{(2\pi\omega)^2} - \frac{2 \sin(2\pi\omega x) \cos(2\pi\omega x_k)}{(2\pi\omega)^3(x_k - x_{k-1})} - \right. \\ &\quad \left. - \frac{2 \sin(2\pi\omega x) \cos(2\pi\omega x_{k-1})}{(2\pi\omega)^3(x_k - x_{k-1})} + \frac{(\cos(2\pi\omega x_k) - \cos(2\pi\omega x_{k-1}))^2}{(2\pi\omega)^4(x_k - x_{k-1})^2} \right] dx = \end{aligned}$$

$$= \sum_{k=1}^n \left[\frac{1}{(2\pi\omega)^2} \left(\frac{x_k - x_{k-1}}{2} - \frac{\sin(4\pi\omega x_k) - \sin(4\pi\omega x_{k-1})}{8\pi\omega} \right) + \frac{3(\cos(2\pi\omega x_k) - \cos(2\pi\omega x_{k-1}))^2}{(2\pi\omega)^4(x_k - x_{k-1})} \right].$$

$$\|\varphi^c\|^2 = \sum_{k=1}^n \left[\frac{1}{(2\pi\omega)^2} \left(\frac{x_k - x_{k-1}}{2} - \frac{\sin(4\pi\omega x_k) - \sin(4\pi\omega x_{k-1})}{8\pi\omega} \right) + \frac{3(\cos(2\pi\omega x_k) - \cos(2\pi\omega x_{k-1}))^2}{(2\pi\omega)^4(x_k - x_{k-1})} \right].$$

Similarly, for φ_k^s we obtain

$$\begin{aligned} \|\varphi^s\|^2 &= \sum_{k=1}^n \left[\frac{1}{(2\pi\omega)^2} \left(\frac{x_k - x_{k-1}}{2} + \frac{\sin(4\pi\omega x_k) - \sin(4\pi\omega x_{k-1})}{8\pi\omega} \right) + \right. \\ &\quad \left. + \frac{(\sin(2\pi\omega x_k) - \sin(2\pi\omega x_{k-1}))(\cos(2\pi\omega x_k) - \cos(2\pi\omega x_{k-1}))}{(2\pi\omega)^4(x_k - x_{k-1})} \right]. \\ \|\varphi^s\|^2 &= \sum_{k=1}^n \int_{x_{k-1}}^{x_k} (\varphi_k^s(x))^2 dx = \sum_{k=1}^n \left[\frac{x_k - x_{k-1}}{2(2\pi\omega)^2} + \right. \\ &\quad \left. + \frac{\sin(4\pi\omega x_k) - \sin(4\pi\omega x_{k-1})}{8\pi\omega(2\pi\omega)^2} - \frac{(\sin(2\pi\omega x_k) - \sin(2\pi\omega x_{k-1}))^2}{(2\pi\omega)^4(x_k - x_{k-1})} \right]. \end{aligned}$$

3 Results and discussion

In this section, we apply the proposed optimal quadrature formula to the numerical evaluation of several integrals.

$$\int_a^b f(x) \sin(2\pi\omega x) dx \approx \sum_{k=0}^n A_k^s f(x_k), \quad x_k - x_{k-1} = h.$$

Example 1. Let $f(x) = 1$, it is straightforward to verify that the integral has the following solution:

$$\int_a^b \sin(2\pi\omega x) dx = \frac{\cos(2\pi\omega a) - \cos(2\pi\omega b)}{2\pi\omega}.$$

Then we apply optimal quadrature formula (6) to evaluate the integral.

$$\begin{aligned} \sum_{k=0}^n A_k^s &= A_0^s + \sum_{k=1}^{n-1} A_k^s + A_n^s = \frac{1}{2\pi\omega} \cos(2\pi\omega x_0) - \frac{\sin(2\pi\omega x_1) - \sin(2\pi\omega x_0)}{(x_1 - x_0)(2\pi\omega)^2} + \\ &\quad + \frac{\sin(2\pi\omega x_1) - \sin(2\pi\omega x_0)}{(x_1 - x_0)(2\pi\omega)^2} - \frac{\sin(2\pi\omega x_2) - \sin(2\pi\omega x_1)}{(x_2 - x_1)(2\pi\omega)^2} + \dots + \\ &\quad + \frac{\sin(2\pi\omega x_{n-1}) - \sin(2\pi\omega x_{n-2})}{(x_{n-1} - x_{n-2})(2\pi\omega)^2} - \frac{\sin(2\pi\omega x_n) - \sin(2\pi\omega x_{n-1})}{(x_n - x_{n-1})(2\pi\omega)^2} - \frac{1}{2\pi\omega} \cos(2\pi\omega x_n) + \\ &\quad + \frac{\sin(2\pi\omega x_n) - \sin(2\pi\omega x_{n-1})}{(x_n - x_{n-1})(2\pi\omega)^2} = \frac{1}{2\pi\omega} \cos(2\pi\omega x_0) - \frac{1}{2\pi\omega} \cos(2\pi\omega x_n). \end{aligned}$$

Since $x_0 = a$ and $x_n = b$, we conclude that our optimal quadrature formula is exact for constant functions.

First, we use the following notations.

$$x_k = a + kh, \quad k = 0, 1, \dots, n, \quad h = \frac{b-a}{n}, \quad (20)$$

$$\theta = \pi\omega, \quad s_k = \sin(2\pi\omega x_k), \quad \mu_0 = \frac{\cos(2\pi\omega x_0)}{2\pi\omega}, \quad \mu_n = -\frac{\cos(2\pi\omega x_n)}{2\pi\omega}. \quad (21)$$

$$A_0 = \frac{\cos(2\pi\omega x_0)}{2\pi\omega} - \frac{\sin(2\pi\omega x_1) - \sin(2\pi\omega x_0)}{4(x_1 - x_0)\pi^2\omega^2} = \mu_0 + \frac{s_0 - s_1}{4h\theta^2},$$

$$\begin{aligned} A_k &= \frac{\sin(2\pi\omega x_k) - \sin(2\pi\omega x_{k-1})}{4(x_k - x_{k-1})\pi^2\omega^2} - \frac{\sin(2\pi\omega x_{k+1}) - \sin(2\pi\omega x_k)}{4(x_{k+1} - x_k)\pi^2\omega^2} = \\ &= \frac{2s_k - s_{k-1} - s_{k+1}}{4h\theta^2}, \quad k = 1, 2, \dots, n-1. \end{aligned}$$

$$A_n = -\frac{\cos(2\pi\omega x_n)}{2\pi\omega} + \frac{\sin(2\pi\omega x_n) - \sin(2\pi\omega x_{n-1})}{4(x_n - x_{n-1})\pi^2\omega^2} = \mu_n + \frac{s_n - s_{n-1}}{4h\theta^2}.$$

The sum may be expressed in the following form:

$$\sum_{k=0}^n A_k \cdot x_k = \sum_{k=0}^n A_k \cdot (a + h \cdot k) = \sum_{k=0}^n A_k \cdot a + h \cdot \sum_{k=0}^n A_k \cdot k.$$

I. The two sums are evaluated separately.

$$\begin{aligned} \sum_{k=0}^n A_k \cdot a &= a \cdot \sum_{k=0}^n A_k = a \cdot (A_0 + \sum_{k=1}^{n-1} A_k + A_n) = \\ &= a \cdot \left(\mu_0 + \frac{s_0 - s_1}{4h\theta^2} + \sum_{k=1}^{n-1} \frac{2s_k - s_{k-1} - s_{k+1}}{4h\theta^2} + \mu_n + \frac{s_n - s_{n-1}}{4h\theta^2} \right) = \\ &= a \cdot \left(\mu_0 + \frac{s_0 - s_1}{4h\theta^2} + \frac{2s_1 - s_0 - s_2}{4h\theta^2} + \frac{2s_2 - s_1 - s_3}{4h\theta^2} + \frac{2s_3 - s_2 - s_4}{4h\theta^2} + \dots + \right. \\ &\quad \left. + \frac{2s_{n-2} - s_{n-3} - s_{n-1}}{4h\theta^2} + \frac{2s_{n-1} - s_{n-2} - s_n}{4h\theta^2} + \mu_n + \frac{s_n - s_{n-1}}{4h\theta^2} \right) = a \cdot (\mu_0 + \mu_n). \end{aligned}$$

Taking a as a common factor yields the result for $f(x) = 1$. Specifically, since

$$\sum_{k=0}^n A_k \cdot a = a \cdot (\mu_0 + \mu_n),$$

we obtain

$$\sum_{k=0}^n A_k = \mu_0 + \mu_n = \frac{\cos(2\pi\omega x_0)}{2\pi\omega} - \frac{\cos(2\pi\omega x_n)}{2\pi\omega} = \int_a^b \sin(2\pi\omega x) dx.$$

II.

$$\sum_{k=0}^n A_k \cdot k = A_0 \cdot 0 + \sum_{k=1}^{n-1} A_k \cdot k + A_n \cdot n =$$

$$\begin{aligned}
&= (\mu_0 + \frac{s_0 - s_1}{4h\theta^2}) \cdot 0 + \sum_{k=1}^{n-1} \frac{2s_k - s_{k-1} - s_{k+1}}{4h\theta^2} \cdot k + n \cdot \mu_n + n \cdot \frac{s_n - s_{n-1}}{4h\theta^2} = \\
&= \frac{1}{4h\theta^2} [\sum_{k=1}^{n-1} 2s_k \cdot k - \sum_{k=1}^{n-1} s_{k-1} \cdot k - \sum_{k=1}^{n-1} s_{k+1} \cdot k] + n \cdot \mu_n + n \cdot \frac{s_n - s_{n-1}}{4h\theta^2} = \\
&= \frac{1}{4h\theta^2} [\sum_{k=1}^{n-1} 2s_k \cdot k - \sum_{k=0}^{n-2} s_k \cdot (k+1) - \sum_{k=2}^n s_k \cdot (k-1)] + n \cdot \mu_n + n \cdot \frac{s_n - s_{n-1}}{4h\theta^2} = \\
&= \frac{1}{4h\theta^2} [(2s_1 + \sum_{k=2}^{n-2} 2s_k \cdot k + 2s_{n-1}(n-1)) - (s_0 + 2s_1 + \sum_{k=2}^{n-2} s_k \cdot (k+1)) - \\
&\quad - (\sum_{k=2}^{n-2} s_k \cdot (k-1) + s_{n-1}(n-2) + s_n(n-1))] + n \cdot \mu_n + n \cdot \frac{s_n - s_{n-1}}{4h\theta^2} = \\
&= \frac{1}{4h\theta^2} [2s_1 + 2s_{n-1}(n-1) - (s_0 + 2s_1) - (s_{n-1}(n-2) + s_n(n-1)) + \\
&\quad + \sum_{k=2}^{n-2} s_k [2k - (k+1) - (k-1)]] + n \cdot \mu_n + n \cdot \frac{s_n - s_{n-1}}{4h\theta^2} = \\
&= \frac{1}{4h\theta^2} [2s_1 + 2s_{n-1}(n-1) - (s_0 + 2s_1) - (s_{n-1}(n-2) + \\
&\quad + s_n(n-1)) + n(s_n - s_{n-1})] + n\mu_n = \frac{1}{4h\theta^2} (s_n - s_0) + n\mu_n.
\end{aligned}$$

The numerical performance of the proposed quadrature formula was evaluated using the test function $f(x) = x^3$ on $[-1, 1]$. Fixing the number of nodes at $n = 10$, the frequency parameter ω was varied from 10 to 20. For each ω , we computed the exact and approximate integrals to determine the absolute error

$$E_n(f) = |I(f) - Q_n(f)|,$$

where $I(f)$ and $Q_n(f)$ represent the exact integral and its quadrature approximation, respectively. The error analysis is summarized in Table 1.

Table 1 indicates a non-monotonic, oscillatory trend in the absolute error as ω increases. Notably, sharper error reductions appear at $\omega = 10, 15$, and 20 , a phenomenon driven by the integrand's frequency and the weight structure of the quadrature nodes. Figure 1 demonstrates that our quadrature formula maintains high accuracy as ω increases.

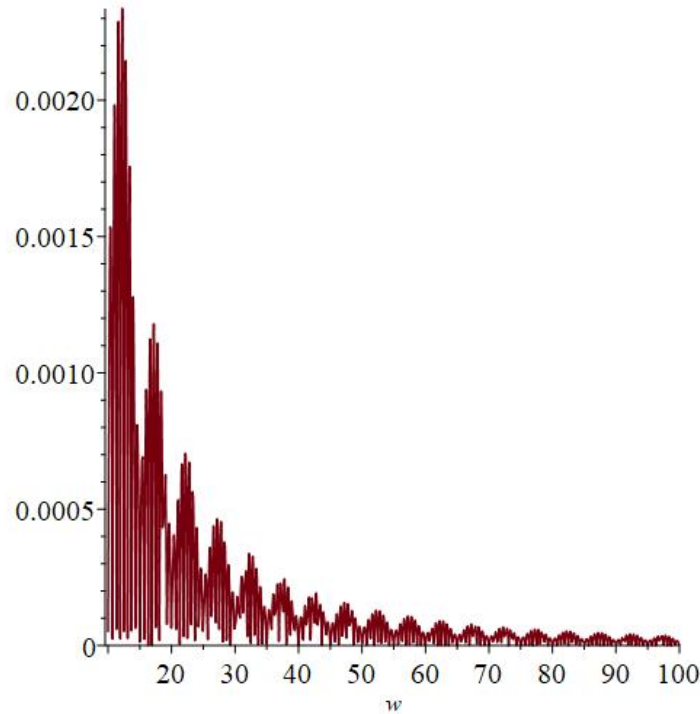
4 Conclusion

Optimal quadrature formulas incorporating derivative values have been constructed for highly oscillatory integrals with trigonometric weight functions within the Sobolev space $L_2^{(1)}(a, b)$. The methodology rests on formulating the quadrature error as a linear error functional and subsequently minimizing its norm over the specified Sobolev space.

By employing the φ -function method, the quadrature coefficients were explicitly derived from the system of optimality conditions. The resulting formulas leverage both function and first-derivative values at the nodes, directly absorbing the oscillatory behavior through the trigonometric weights.

Table 1 Absolute errors for the test function $f(x) = x^3$ with $n = 10$.

ω	$E_n(f)$
10	$4.837730 \cdot 10^{-5}$
11	$1.954607 \cdot 10^{-3}$
12	$1.005947 \cdot 10^{-3}$
13	$9.030123 \cdot 10^{-4}$
14	$1.246740 \cdot 10^{-3}$
15	$1.433399 \cdot 10^{-5}$
16	$9.292260 \cdot 10^{-4}$
17	$5.053357 \cdot 10^{-4}$
18	$4.678252 \cdot 10^{-4}$
19	$6.743811 \cdot 10^{-4}$
20	$6.047100 \cdot 10^{-6}$

**Figure 1** Absolute error for $f(x) = x^3$, $n = 10$.

Explicit expressions for the error functionals and their norms were established. Exactness analysis verified that the constructed formulas reproduce the target test functions precisely, confirming the theoretical consistency of the scheme.

Numerical experiments illustrated the error behavior across different oscillation regimes. The results indicate that the approximation error depends jointly on the node count and the oscillation parameter ω ; notably, the error does not decrease monotonically with increasing frequency, which reflects the intrinsic nature of oscillatory integration.

Overall, this Sobolev-space framework provides a rigorous foundation for integrating derivative data into weighted quadrature rules. The proposed approach can be extended to higher-order Sobolev spaces $L_2^{(m)}(a, b)$, non-uniform node distributions, and broader classes of highly oscillatory kernels.

Declarations

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ОПТИМАЛЬНЫЕ КВАДРАТУРНЫЕ ФОРМУЛЫ С ТРИГОНОМЕТРИЧЕСКИМИ ВЕСАМИ В ПРОСТРАНСТВЕ СОБОЛЕВА

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Актуальность: Точный и эффективный численный расчет сильно осциллирующих интегралов остается сложной задачей вычислительной математики и таких прикладных областей, как математическая физика, динамика волн, оптика и обработка сигналов. Классические подходы, включая методы типа Филона, Левина и асимптотические методы, эффективны для вычисления сильно осциллирующих интегралов, однако непосредственно не рассматривают построение оптимальных квадратурных формул в пространствах Соболева и минимизацию соответствующего функционала погрешности.

Цель: Целью настоящего исследования является построение оптимальной квадратурной формулы для сильно осциллирующих интегралов с тригонометрическими весами в пространствах Соболева, определение ее квадратурных коэффициентов и исследование поведения соответствующей погрешности.

Метод: Интеграл разлагается на синусоидальную и косинусоидальную составляющие, после чего с помощью поэлементного интегрирования преобразуется к форме, пригодной для применения методов теории пространств Соболева. Квадратурные коэффициенты определяются путем минимизации нормы соответствующего функционала погрешности. Устанавливается система условий, характеризующая оптимальные коэффициенты, и получаются их явные выражения для равномерно распределенных узлов. Предложенный подход учитывает как осциллирующую структуру подынтегральной функции, так и функциональные свойства соответствующего пространства Соболева при построении квадратурной формулы.

Результаты: Получены явные аналитические выражения для оптимальных квадратурных коэффициентов. Выведенные формулы удовлетворяют необходимым

условиям точности для постоянных и линейных функций. Численные эксперименты показывают немонотонную зависимость погрешности квадратурной формулы от параметра осцилляции, причем для определенных значений данного параметра наблюдается существенное уменьшение погрешности.

Практическая значимость: Предложенная квадратурная формула представляет собой оптимальный подход к вычислению сильно осциллирующих интегралов на основе пространств Соболева и расширяет существующие методы численного интегрирования за счет непосредственного учета осциллирующего характера интеграла при оптимизации погрешности приближения.

Ключевые слова: сильно осциллирующие интегралы, оптимальная квадратурная формула, оптимальные коэффициенты, пространство Соболева, функционал погрешности, численное интегрирование

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