

UDC 519.644

EXPLICIT COEFFICIENTS AND ERROR ESTIMATE FOR AN OPTIMAL QUADRATURE FORMULA

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Accurate numerical integration is essential in many computational applications, particularly when the integrands possess higher-order smoothness. Optimal quadrature formulas in Sobolev spaces provide a rigorous framework for controlling approximation errors, while classical formulas such as the Euler–Maclaurin formula may be less flexible for general distributions of quadrature nodes. This motivates the development of accurate quadrature formulas with explicit error estimates and applicability beyond uniform partitions.

This paper aims to construct an optimal quadrature formula in the Sobolev space $L_2^{(5)}(a, b)$ and to establish explicit expressions for its coefficients and error characteristics using the φ -function method.

The proposed approach represents the error of the quadrature formula through a φ -function and determines the coefficients of the quadrature formula by solving the corresponding optimality conditions. This procedure yields explicit analytical expressions for the quadrature coefficients and an explicit estimate of the approximation error. In addition, a closed-form expression for the norm of the error of the quadrature formula is obtained, together with a simplified form for equally spaced quadrature nodes. The construction is formulated for arbitrary node distributions and therefore does not require a uniform partition of the integration interval.

The resulting quadrature formula provides a rigorous error estimate and exhibits high numerical accuracy. Numerical experiments conducted for several test functions demonstrate that the proposed formula produces smaller absolute approximation errors than the classical Euler–Maclaurin quadrature formula. The numerical findings are also consistent with the theoretically established error estimates.

The obtained results extend the range of applicable optimal quadrature formulas in Sobolev spaces and provide a flexible and accurate alternative to classical numerical integration methods, particularly when non-uniform quadrature nodes are required.

Keywords: optimal quadrature formula, φ -function method, Sobolev space, error estimate, numerical integration, Euler–Maclaurin formula

Citation: Berdimuradova U.A. 2026. Explicit coefficients and error estimate for an optimal quadrature formula. *Problems of Computational and Applied Mathematics*. 4/1(75): 116-126.

DOI: 10.71310/pcam.4-1_75.2026.09

1 Introduction

Numerical integration is one of the fundamental topics in computational mathematics and approximation theory. Since exact evaluation of definite integrals is often impossible for many practical problems, the development of efficient numerical integration methods remains an important area of research. The accuracy of such methods depends largely on

the construction of quadrature formulas that provide reliable approximations while preserving computational efficiency. For smooth functions, the use of derivative information makes it possible to improve the approximation accuracy and to construct quadrature formulas with desirable optimality properties.

The theory of optimal quadrature formulas has been extensively developed within approximation theory and numerical analysis. Early contributions by Sard [1, 2] and Nikolskii [3, 4] established important theoretical foundations for the construction of optimal approximation and quadrature methods. These studies demonstrated that incorporating the smoothness properties of functions into the construction process makes it possible to obtain highly accurate numerical integration formulas. Later, the Sobolev method provided an effective framework for constructing optimal quadrature formulas in Sobolev spaces and has become one of the principal approaches in this area [5–9]. Optimal quadrature formulas in Sobolev spaces have also been investigated for non-periodic functions and their applications [10].

Fundamental results on optimal quadrature formulas were established by A. Ghizzetti and A. Ossicini [11]. These ideas were further developed by F. Lanzara [12], who studied optimal quadrature formulas exact for solutions of linear differential equations. Subsequently, Gh. Coman and T. Cătiņaș [13] introduced the φ -function method as an analytical framework for constructing optimal quadrature formulas. In several subsequent works [14–21], this method was successfully applied to a variety of optimal quadrature problems, leading to explicit analytical expressions for the corresponding quadrature coefficients.

Although various approaches have been proposed for constructing optimal quadrature formulas, obtaining explicit analytical expressions for both the quadrature coefficients and the corresponding error estimate remains a challenging problem, particularly in higher-order Sobolev spaces. Such explicit formulas are of considerable practical interest since they eliminate the need for solving additional systems of equations and facilitate the efficient implementation of optimal quadrature formulas in numerical integration.

Despite the progress achieved in this direction, the construction of an optimal quadrature formula in the Sobolev space $L_2^{(5)}(a, b)$ by the φ -function method has not been sufficiently investigated. In particular, the derivation of explicit analytical expressions for the coefficients and the error of the quadrature formula remains of considerable interest. These issues constitute the main objective of the present paper.

In this paper, an optimal quadrature formula is constructed in the Sobolev space $L_2^{(5)}(a, b)$ using the φ -function method. Explicit analytical expressions for the coefficients of the quadrature formula are derived, and an upper bound for the approximation error of the constructed quadrature formula is established. The accuracy of the proposed quadrature formula is illustrated by several numerical examples and compared with the classical Euler–Maclaurin formula.

The remainder of this paper is organized as follows. Section 2 formulates the problem. Section 3 presents the construction of the φ -function and the derivation of the coefficients of the optimal quadrature formula. Section 4 presents an explicit expression of optimal coefficients and error estimate of the quadrature formula. Section 5 contains numerical examples and comparisons with the classical Euler–Maclaurin formula. Finally, Section 6 concludes the paper.

2 Problem formulation

Let

$$a = x_0 < x_1 < \dots < x_n = b, \quad (1)$$

be an arbitrary partition of the interval $[a, b]$. The problem considered in this paper is the approximation of the integral

$$I(f) = \int_a^b f(x) dx,$$

by a quadrature formula involving the values of the function and its derivatives at the partition nodes. More precisely, we consider quadrature formulas of the form

$$\int_a^b f(x) dx = \sum_{i=0}^4 \sum_{k=0}^n A_{ik} f^{(i)}(x_k), \quad (2)$$

where A_{ik} are unknown coefficients to be determined.

The quadrature formula (2) involves not only the values of the integrand but also its derivatives up to the fourth order at the quadrature nodes. The inclusion of derivative information increases the flexibility of the approximation process and provides additional degrees of freedom for constructing optimal quadrature formulas with improved accuracy.

The corresponding remainder of the quadrature formula is defined by

$$R(f) = \int_a^b f(x) dx - \sum_{i=0}^4 \sum_{k=0}^n A_{ik} f^{(i)}(x_k). \quad (3)$$

The remainder $R(f)$ characterizes the approximation error of the quadrature formula. Therefore, the main objective is to determine the coefficients A_{ik} so that this error becomes as small as possible.

We assume that the integrand belongs to the Sobolev space $L_2^{(5)}(a, b)$, equipped with the inner product

$$(f, g) = \int_a^b f^{(5)}(x) g^{(5)}(x) dx$$

and the corresponding seminorm

$$\|f\|_{L_2^{(5)}(a,b)} = \left(\int_a^b (f^{(5)}(x))^2 dx \right)^{1/2}.$$

For convenience, we introduce the following notation:

$$\begin{aligned} \mathbf{A}_0 &= (A_{00}, \dots, A_{0n}), & \mathbf{A}_1 &= (A_{10}, \dots, A_{1n}), & \mathbf{A}_2 &= (A_{20}, \dots, A_{2n}), \\ \mathbf{A}_3 &= (A_{30}, \dots, A_{3n}), & \mathbf{A}_4 &= (A_{40}, \dots, A_{4n}), & \mathbf{X} &= (x_0, \dots, x_n). \end{aligned}$$

Definition 1. The quadrature formula (2) is said to be optimal in the sense of Sard in the space $L_2^{(5)}(a, b)$ if, for fixed $\mathbf{X}$, the following quantity attains its minimum with respect to $\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$:

$$F_n \left(L_2^{(5)}(a, b), \mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4 \right) = \sup_{f \in L_2^{(5)}(a,b)} |R(f)|. \quad (4)$$

The main purpose of this study is to construct an optimal quadrature formula of the considered type. Using the φ -function method, we determine the optimal coefficients $\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$ that minimize the above error quantity.

3 Representation of the quadrature formula by the φ -function method

To determine the coefficients of the quadrature formula, we employ the φ -function method. The basic idea of this approach is to introduce auxiliary functions on each subinterval of the partition and to represent both the coefficients and the remainder of the quadrature formula in terms of these functions.

For each subinterval $[x_{k-1}, x_k]$, $k = 1, 2, \dots, n$, let $\varphi_k(x)$ satisfy the differential equation

$$\varphi_k^{(5)}(x) = 1. \quad (5)$$

The corresponding piecewise-defined function is introduced by

$$\varphi(x)|_{[x_{k-1}, x_k]} = \varphi_k(x), \quad k = 1, 2, \dots, n. \quad (6)$$

Applying repeated integration by parts together with the above definition, the coefficients of the quadrature formula and the remainder can be expressed through the values of the φ_k functions and their derivatives at the quadrature nodes. Since this derivation follows the standard φ -function technique, only the resulting expressions are presented below:

$$\begin{aligned} A_{00} &= -\varphi_1^{(4)}(x_0), & A_{0k} &= \varphi_k^{(4)}(x_k) - \varphi_{k+1}^{(4)}(x_k), & A_{0n} &= \varphi_n^{(4)}(x_n), \\ A_{10} &= \varphi_1^{(3)}(x_0), & A_{1k} &= \varphi_{k+1}^{(3)}(x_k) - \varphi_k^{(3)}(x_k), & A_{1n} &= -\varphi_n^{(3)}(x_n), \\ A_{20} &= -\varphi_1^{(2)}(x_0), & A_{2k} &= \varphi_k^{(2)}(x_k) - \varphi_{k+1}^{(2)}(x_k), & A_{2n} &= \varphi_n^{(2)}(x_n), \\ A_{30} &= \varphi_1'(x_0), & A_{3k} &= \varphi_{k+1}'(x_k) - \varphi_k'(x_k), & A_{3n} &= -\varphi_n'(x_n), \\ A_{40} &= -\varphi_1(x_0), & A_{4k} &= \varphi_k(x_k) - \varphi_{k+1}(x_k), & A_{4n} &= \varphi_n(x_n). \end{aligned} \quad (7)$$

The remainder of the quadrature formula is represented by

$$R(f) = - \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f^{(5)}(x) \varphi_k(x) dx. \quad (8)$$

Therefore, the construction of the quadrature formula reduces to determining the explicit form of the auxiliary functions $\varphi_k(x)$. Since $\varphi_k^{(5)}(x) = 1$, the corresponding differential equation can be integrated successively five times. Consequently, its general solution is given by

$$\varphi_k(x) = \frac{x^5}{5!} + C_{1k} \frac{x^4}{4!} + C_{2k} \frac{x^3}{3!} + C_{3k} \frac{x^2}{2!} + C_{4k}x + C_{5k}, \quad k = 1, 2, \dots, n. \quad (9)$$

The general solution contains five arbitrary constants on each subinterval. These constants are determined by minimizing the corresponding quadratic functional. As a result, explicit analytical expressions for the constants are obtained, which subsequently lead to explicit formulas for the coefficients of the optimal quadrature formula. The details of this optimization procedure are presented in the following section.

4 Determination of the optimal coefficients and error estimate

The construction of an optimal quadrature formula is based on minimizing the norm of the corresponding error of the quadrature formula. According to the representation

obtained in the previous section, the remainder of the quadrature formula can be written as

$$R(f) = - \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f^{(5)}(x) \varphi_k(x) dx. \quad (10)$$

Applying the Cauchy–Schwarz inequality to each subinterval yields

$$\begin{aligned} |R(f)| &\leq \sum_{k=1}^n \left(\int_{x_{k-1}}^{x_k} (f^{(5)}(x))^2 dx \right)^{1/2} \left(\int_{x_{k-1}}^{x_k} \varphi_k^2(x) dx \right)^{1/2} \leq \\ &\leq \left(\int_a^b (f^{(5)}(x))^2 dx \right)^{1/2} \left(\int_a^b \varphi^2(x) dx \right)^{1/2}. \end{aligned} \quad (11)$$

Hence,

$$|R(f)| \leq \|f^{(5)}\|_{L_2(a,b)} \|\varphi\|_{L_2(a,b)}. \quad (12)$$

Since the function f is fixed, the quantity $\|f^{(5)}\|_{L_2(a,b)}$ does not depend on the choice of the $\varphi(x)$ function. Therefore, minimizing the upper bound of the quadrature error is equivalent to minimizing the norm of $\varphi(x)$. Consequently, the construction of an optimal quadrature formula is reduced to the following optimization problem

$$\|\varphi\|_{L_2(a,b)} = \left(\int_a^b \varphi^2(x) dx \right)^{1/2} \longrightarrow \min. \quad (13)$$

Since the $\varphi_k(x)$ function is defined piecewise on the partition (1), the minimization problem can be written as

$$\sum_{k=1}^n \int_{x_{k-1}}^{x_k} \varphi_k^2(x) dx \longrightarrow \min. \quad (14)$$

For each subinterval $[x_{k-1}, x_k]$, the $\varphi_k(x)$ function has the general form

$$\varphi_k(x) = \frac{x^5}{5!} + C_{1k} \frac{x^4}{4!} + C_{2k} \frac{x^3}{3!} + C_{3k} \frac{x^2}{2!} + C_{4k}x + C_{5k}, \quad (15)$$

where $C_{1k}, C_{2k}, C_{3k}, C_{4k}$ and C_{5k} are unknown constants. To determine these constants, we minimize the quadratic functional

$$J_k = \int_{x_{k-1}}^{x_k} \varphi_k^2(x) dx, \quad k = 1, 2, \dots, n. \quad (16)$$

Substituting (15) into (14), the functional becomes a quadratic function of the unknown coefficients. Differentiating (14) with respect to each unknown coefficient and setting the

obtained derivatives equal to zero, we arrive at the following system of integral equations:

$$\left\{ \begin{array}{l} \int_{x_{k-1}}^{x_k} \left(\frac{x^5}{5!} + C_{1k} \frac{x^4}{4!} + C_{2k} \frac{x^3}{3!} + C_{3k} \frac{x^2}{2!} + C_{4k}x + C_{5k} \right) \frac{x^4}{4!} dx = 0, \\ \int_{x_{k-1}}^{x_k} \left(\frac{x^5}{5!} + C_{1k} \frac{x^4}{4!} + C_{2k} \frac{x^3}{3!} + C_{3k} \frac{x^2}{2!} + C_{4k}x + C_{5k} \right) \frac{x^3}{3!} dx = 0, \\ \int_{x_{k-1}}^{x_k} \left(\frac{x^5}{5!} + C_{1k} \frac{x^4}{4!} + C_{2k} \frac{x^3}{3!} + C_{3k} \frac{x^2}{2!} + C_{4k}x + C_{5k} \right) \frac{x^2}{2!} dx = 0, \\ \int_{x_{k-1}}^{x_k} \left(\frac{x^5}{5!} + C_{1k} \frac{x^4}{4!} + C_{2k} \frac{x^3}{3!} + C_{3k} \frac{x^2}{2!} + C_{4k}x + C_{5k} \right) x dx = 0, \\ \int_{x_{k-1}}^{x_k} \left(\frac{x^5}{5!} + C_{1k} \frac{x^4}{4!} + C_{2k} \frac{x^3}{3!} + C_{3k} \frac{x^2}{2!} + C_{4k}x + C_{5k} \right) dx = 0. \end{array} \right. \quad (17)$$

Solving system (17), we determine the coefficients $C_{1k}, C_{2k}, C_{3k}, C_{4k}$ and C_{5k} . Substituting these coefficients into (15), we obtain the explicit expression of the $\varphi_k(x)$ function.

Theorem 1. *For the optimal quadrature formula of the form (2) in the Sobolev space $L_2^{(5)}(a, b)$, the auxiliary function $\varphi_k(x)$ is given by*

$$\begin{aligned} \varphi_k(x) = & \frac{x^5}{120} - \frac{(x_k - x_{k-1}) \cdot x^4}{48} + \frac{(x_k + 2x_{k-1})(2x_k + x_{k-1}) \cdot x^3}{108} - \\ & - \frac{(x_k - x_{k-1})(x_k^2 + 4x_kx_{k-1} + x_{k-1}^2) \cdot x^2}{144} + \\ & + \frac{1}{1008}(x_k^4 + 10x_k^3x_{k-1} + 20x_k^2x_{k-1}^2 + 10x_kx_{k-1}^3 + x_{k-1}^4) \cdot x - \\ & - \frac{(x_k - x_{k-1})(x_k^4 + 24x_k^3x_{k-1} + 76x_k^2x_{k-1}^2 + 24x_kx_{k-1}^3 + x_{k-1}^4)}{30240}, \end{aligned} \quad (18)$$

for every subinterval $[x_{k-1}, x_k]$, $k = 1, 2, \dots, n$.

Theorem 1 establishes an explicit representation of the function $\varphi_k(x)$ on each subinterval of the partition. This representation plays a fundamental role in the subsequent construction of the optimal quadrature formula, since the coefficients of the quadrature formula are obtained directly from the values of $\varphi_k(x)$ and its derivatives at the partition nodes.

Substituting the explicit representation of the $\varphi_k(x)$ function into relations (7), we obtain the coefficients of the optimal quadrature formula.

Theorem 2. *The coefficients of the optimal quadrature formula (2) in the Sobolev space $L_2^{(5)}(a, b)$ are given by the following formulas, where $k = 1, 2, \dots, n-1$:*

$$\begin{aligned} A_{00} &= \frac{x_1 - x_0}{2}, \quad A_{0k} = \frac{x_{k+1} - x_{k-1}}{2}, \quad A_{0n} = \frac{x_n - x_{n-1}}{2}, \\ A_{10} &= \frac{(x_1 - x_0)^2}{9}, \quad A_{1k} = \frac{(x_{k+1} - x_k)^2 - (x_k - x_{k-1})^2}{9}, \quad A_{1n} = -\frac{(x_n - x_{n-1})^2}{9}, \\ A_{20} &= \frac{(x_1 - x_0)^3}{72}, \quad A_{2k} = \frac{(x_{k+1} - x_k)^3 + (x_k - x_{k-1})^3}{72}, \quad A_{2n} = \frac{(x_n - x_{n-1})^3}{72}, \\ A_{30} &= \frac{(x_1 - x_0)^4}{1008}, \quad A_{3k} = \frac{(x_{k+1} - x_k)^4 - (x_k - x_{k-1})^4}{1008}, \quad A_{3n} = -\frac{(x_n - x_{n-1})^4}{1008}, \\ A_{40} &= \frac{(x_1 - x_0)^5}{30240}, \quad A_{4k} = \frac{(x_{k+1} - x_k)^5 + (x_k - x_{k-1})^5}{30240}, \quad A_{4n} = \frac{(x_n - x_{n-1})^5}{30240}. \end{aligned} \quad (19)$$

An important feature of the obtained coefficients is that they are expressed in closed form for an arbitrary partition of the integration interval. Consequently, once the partition is specified, all coefficients of the proposed quadrature formula can be evaluated directly. This result completes the construction of the optimal quadrature formula for arbitrary distributions of quadrature nodes. According to inequality (12), the error bound of the optimal quadrature formula depends on the L_2 -norm of the $\varphi_k(x)$ function. Therefore, it remains to evaluate this norm explicitly. The following theorem provides the corresponding result.

Theorem 3. *The error of the optimal quadrature formula satisfies the estimate*

$$|R(f)| \leq \|f^{(5)}\|_{L_2(a,b)} \|\varphi\|_{L_2(a,b)}, \quad (20)$$

where

$$\|\varphi\|_{L_2(a,b)} = \frac{\sqrt{\sum_{k=1}^n (x_k - x_{k-1})^{11}}}{30240\sqrt{11}}. \quad (21)$$

In the case of equally spaced nodes, where $x_k - x_{k-1} = h$ for all $k = 1, 2, \dots, n$, the above estimate takes a simpler form.

Corollary 1. *For equally spaced nodes, where*

$$h = \frac{b-a}{n}, \quad (22)$$

the estimate in Theorem 3 reduces to

$$\|\varphi\|_{L_2(a,b)} = \frac{\sqrt{b-a} h^5}{30240\sqrt{11}}, \quad (23)$$

and consequently,

$$|R(f)| \leq \frac{\sqrt{b-a} h^5}{30240\sqrt{11}} \|f^{(5)}\|_{L_2(a,b)}. \quad (24)$$

Corollary 1 shows that, for a uniform partition of the integration interval, the general error estimate is reduced to a particularly simple form depending only on the step size h . This representation is convenient for practical applications, since the error can be estimated directly once the step size is specified. Moreover, it demonstrates that the general estimate obtained for arbitrary partitions naturally includes the case of uniformly distributed quadrature nodes as a special case. Consequently, the obtained estimate provides a convenient theoretical tool for assessing the approximation error in the case of equally spaced quadrature nodes. It also establishes a direct connection between the theoretical analysis and the numerical experiments presented in the next section.

5 Numerical results

To illustrate the accuracy of the constructed quadrature formula, five sufficiently smooth test functions are considered. The absolute approximation errors obtained by the constructed quadrature formula are compared with those of the classical Euler–Maclaurin quadrature formula. The corresponding results are presented in Table 1.

Since the classical Euler–Maclaurin quadrature formula is formulated for equally spaced nodes, the comparison is carried out using the proposed quadrature formula specialized to the case of a uniform partition of the integration interval. This ensures that both quadrature formulas are evaluated under identical conditions, providing a fair and

Table 1 Comparison of the absolute approximation errors obtained by the classical Euler–Maclaurin formula and the constructed quadrature formula.

Test function	n	Euler–Maclaurin	Constructed formula
$x^8 - 2x^5 + x^2 - 1$	10	$2.21888 \cdot 10^{-7}$	0
	50	$1.42213 \cdot 10^{-11}$	$1.00000 \cdot 10^{-30}$
	100	$2.22218 \cdot 10^{-13}$	$3.00000 \cdot 10^{-30}$
e^{15x}	10	77.72090	$0.11643 \cdot 10^{-2}$
	50	$0.52419 \cdot 10^{-2}$	$1.27564 \cdot 10^{-10}$
	100	$0.82044 \cdot 10^{-4}$	$1.24844 \cdot 10^{-13}$
$\frac{1}{1+25x^2}$	10	$2.19052 \cdot 10^{-6}$	$1.41321 \cdot 10^{-8}$
	50	$4.18075 \cdot 10^{-14}$	$5.57993 \cdot 10^{-23}$
	100	$6.53456 \cdot 10^{-16}$	$5.4533 \cdot 10^{-26}$
$\ln(1+x)$	10	$7.63025 \cdot 10^{-10}$	$3.88883 \cdot 10^{-16}$
	50	$4.91912 \cdot 10^{-14}$	$4.09179 \cdot 10^{-23}$
	100	$7.68790 \cdot 10^{-16}$	$3.99950 \cdot 10^{-26}$
$\sin(x)$	10	$1.52054 \cdot 10^{-11}$	$4.57146 \cdot 10^{-21}$
	50	$9.72914 \cdot 10^{-16}$	$4.670000 \cdot 10^{-28}$
	100	$1.52016 \cdot 10^{-17}$	$2.000000 \cdot 10^{-30}$

meaningful comparison of their approximation accuracy. Therefore, the observed differences in the numerical results are attributed to the properties of the quadrature formulas themselves rather than to the distribution of the quadrature nodes.

It is observed from Table 1 that the constructed quadrature formula yields more accurate approximations than the classical Euler–Maclaurin formula for all considered test functions. The improvement is particularly noticeable for the exponential and rational test functions, where the obtained absolute errors are several orders of magnitude smaller than those produced by the Euler–Maclaurin formula. Similar results are also observed for the logarithmic and trigonometric functions, indicating that the proposed quadrature formula consistently provides high accuracy for all test functions considered.

Furthermore, increasing the number of subintervals from $n = 10$ to $n = 100$ leads to a substantial reduction in the approximation error for both quadrature formulas. However, throughout all numerical experiments, the constructed quadrature formula preserves its advantage over the classical Euler–Maclaurin formula. These observations are in good agreement with the theoretical error estimate established in Section 4 and further confirm the effectiveness of the proposed quadrature formula.

6 Conclusion

In this paper, a new optimal quadrature formula in the Sobolev space $L_2^{(5)}(a, b)$ was constructed by means of the φ -function method. Explicit formulas for all coefficients of the quadrature formula were derived in terms of the subinterval lengths, and an explicit estimate for the remainder was obtained. In the case of equally spaced nodes, the error bound was simplified to a compact form.

The numerical examples demonstrated that the constructed quadrature formula provides more accurate approximations than the classical Euler–Maclaurin quadrature formula for the considered test functions. The obtained results confirm the theoretical error

estimate and indicate that the proposed approach is effective for the numerical approximation of definite integrals. The developed analytical approach provides a basis for further investigation of optimal quadrature formulas in Sobolev spaces. The proposed methodology can be extended to the construction of optimal quadrature formulas in higher-order Sobolev spaces and to related problems of numerical integration.

Declarations

Funding. The authors received no financial support for the research, authorship, or publication of this article.

Conflicts of interest. The author declares that there are no competing interests.

Data availability. The data are available from the author on reasonable request.

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УДК 519.644

ЯВНЫЕ КОЭФФИЦИЕНТЫ И ОЦЕНКА ПОГРЕШНОСТИ ОПТИМАЛЬНОЙ КВАДРАТУРНОЙ ФОРМУЛЫ

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Точное численное интегрирование имеет важное значение во многих вычислительных приложениях, особенно в случаях, когда подынтегральные функции обладают высокой степенью гладкости. Оптимальные квадратурные формулы в пространствах Соболева обеспечивают строгую основу для контроля погрешностей аппроксимации, тогда как классические формулы, такие как формула Эйлера–Маклорена, могут быть менее гибкими для общего распределения квадратурных узлов. Это обосновывает необходимость разработки точных квадратурных формул с явными оценками погрешности, применимых за пределами равномерного разбиения.

Целью данной работы является построение оптимальной квадратурной формулы в пространстве Соболева $L_2^{(5)}(a, b)$ и получение явных выражений для её коэффициентов и характеристик погрешности с использованием φ -функционального метода.

Предлагаемый подход представляет погрешность квадратурной формулы с помощью φ -функции и определяет коэффициенты квадратурной формулы путём решения соответствующих условий оптимальности. Данная процедура позволяет получить явные аналитические выражения для квадратурных коэффициентов и явную оценку погрешности аппроксимации. Кроме того, получено замкнутое выражение для нормы погрешности квадратурной формулы, а для равноотстоящих квадратурных узлов выведена её упрощённая форма. Построение сформулировано для произ-

вольного распределения узлов и, следовательно, не требует равномерного разбиения интервала интегрирования.

Полученная квадратурная формула обеспечивает строгую оценку погрешности и демонстрирует высокую численную точность. Численные эксперименты, проведённые для нескольких тестовых функций, показывают, что предложенная формула обеспечивает меньшие абсолютные погрешности аппроксимации по сравнению с классической квадратурной формулой Эйлера–Маклорена. Численные результаты также согласуются с теоретически установленными оценками погрешности.

Полученные результаты расширяют область применения оптимальных квадратурных формул в пространствах Соболева и предлагают гибкую и точную альтернативу классическим методам численного интегрирования, особенно в случаях, когда требуется использование неравномерно распределённых квадратурных узлов.

Ключевые слова: оптимальная квадратурная формула, метод φ -функций, пространство Соболева, оценка погрешности, численное интегрирование, формула Эйлера–Маклорена

Цитирование: Бердимуратова У.А. Явные коэффициенты и оценка погрешности оптимальной квадратурной формулы // Проблемы вычислительной и прикладной математики. – 2026. – № 4/1(75). – С. 116–126.

DOI: 10.71310/psam.4-1_75.2026.09

HISOBLASH VA AMALIY MATEMATIKA MUAMMOLARI

ПРОБЛЕМЫ ВЫЧИСЛИТЕЛЬНОЙ
И ПРИКЛАДНОЙ МАТЕМАТИКИ
PROBLEMS OF COMPUTATIONAL
AND APPLIED MATHEMATICS

ПРОБЛЕМЫ ВЫЧИСЛИТЕЛЬНОЙ И ПРИКЛАДНОЙ МАТЕМАТИКИ

№ 4/1(75) 2026

Журнал основан в 2015 году.

Издается 6 раз в год.

Учредитель:

Научно-исследовательский институт развития цифровых технологий и
искусственного интеллекта.

Главный редактор:

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Арипов М.М., Шадиметов Х.М., Ахмедов Д.Д.

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Mascagni M. (США), Min A. (Германия), Singh M. (Южная Корея).

Журнал зарегистрирован в Агентстве информации и массовых коммуникаций при
Администрации Президента Республики Узбекистан.

Свидетельство №0856 от 5 августа 2015 года.

ISSN 2181-8460, eISSN 2181-046X

При перепечатке материалов ссылка на журнал обязательна.

За точность фактов и достоверность информации ответственность несут авторы.

Адрес редакции:

100125, г. Ташкент, м-в. Буз-2, 17А.

Тел.: +(998) 71 263-41-98.

Э-почта: journals@airi.uz.

Веб-сайт: <https://journals.airi.uz>.

Дизайн и вёрстка:

Шарипов Х.Д.

Отпечатано в типографии НИИ РЦТИИ.

Подписано в печать 05.10.2026 г.

Формат 60x84 1/8. Заказ № 5. Тираж 100 экз.

PROBLEMS OF COMPUTATIONAL AND APPLIED MATHEMATICS

No. 4/1(75) 2026

The journal was established in 2015.
6 issues are published per year.

Founder:

Digital Technologies and Artificial Intelligence Development Research Institute.

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The journal is registered by Agency of Information and Mass Communications under the
Administration of the President of the Republic of Uzbekistan.

Certificate of Registration No. 0856 of 5 August 2015.

ISSN 2181-8460, eISSN 2181-046X

At a reprint of materials the reference to the journal is obligatory.
Authors are responsible for the accuracy of the facts and reliability of the information.

Address:

100125, Tashkent, Buz-2, 17A.

Tel.: +(998) 71 263-41-98.

E-mail: journals@airi.uz.

Web-site: <https://journals.airi.uz>.

Layout design:

Sharipov Kh.D.

DTAIRI printing office.

Signed for print 05.10.2026

Format 60x84 1/8. Order No. 5. Print run of 100 copies.

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