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THE SQUARED NORM OF THE ERROR FUNCTIONAL FOR A THIRD-DERIVATIVE INTERPOLATION FORMULA IN A HILBERT SPACE

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This work is devoted to the construction of an optimal interpolation formula for functions defined on a finite interval in the Hilbert space, which is characterized by the requirement that the third-order derivative of the function in this space needs to be absolutely continuous and integrable by the square of its fourth-order derivative. The interpolation formula under consideration is constructed so that it is exact both for polynomials up to the second degree and for an invariant exponential function, and the subspace covered by these functions plays a central role throughout the construction. The accuracy of the interpolation formula is characterized by its corresponding error functional, and the quality of the formula is measured by the norm of this function in the conjugate of the underlying Hilbert space. Based on the variational approach developed by S. L. Sobolev and his school for constructing optimal quadrature and interpolation formulas, an extremal function related to the error functional is introduced, and its exact expression is obtained by convolution with a special kernel function determined by the structure of the space. Using this extremal function together with the discrete orthogonality conditions satisfied by the interpolation formula, a precise closed-form expression for the square of the norm of the error functional is obtained in terms of the unknown coefficients of the formula. This expression serves as the main initial result on which the subsequent minimization problem is based to find the optimal coefficients of the interpolation formula.

Keywords: optimal interpolation, Hilbert space, error minimization, variational methods, optimal coefficients

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1 Introduction

The problem of approximating a function by an interpolation formula that uses not only its values but also the values of its derivatives at a discrete set of nodes arises naturally in numerical analysis, numerical differentiation and integration, the numerical solution of differential and integral equations, and computer-aided design. Such interpolation formulas, often referred to as Hermite-type formulas [1], allow one to incorporate additional smoothness information about the function into the approximation and, when properly constructed, yield a substantially higher order of accuracy than formulas based on function values alone.

A general and rigorous approach to the construction of optimal interpolation and quadrature formulas was developed by S. L. Sobolev, who proposed seeking formulas whose error, regarded as a linear functional on an appropriate Hilbert space of functions, has the smallest possible norm in the conjugate space [2]. This variational approach reduces the construction of an optimal formula to two problems: first, deriving an explicit expression for the squared norm of the error functional in terms of the unknown coefficients of the formula, and second, minimizing this expression, typically by means of Lagrange's method, subject to the constraints under which the formula reproduces a given finite-dimensional subspace exactly. Since Sobolev's foundational work, this method has been developed and extended by many researchers to various classes of Hilbert and Sobolev spaces [3, 4], including spaces defined with respect to differential operators other than the classical derivative [5], and to interpolation formulas that use both function values and derivative values at the nodes [6].

The present paper continues this line of research to the case $m = 4$, that is, to the Hilbert space $W_2^{(4,3)}(0, 1)$ of functions whose third derivative is absolutely continuous and whose fourth derivative is square-integrable on $[0, 1]$. The corresponding interpolation formula uses values of the function together with its first, second, and third derivatives at the nodes and is required to be exact for functions $1, x, x^2$ and e^{-x} . Compared with the lower cases, the presence of the third-derivative term introduces substantial additional algebraic complexity into the construction of the extremal function and the resulting norm expression. In this paper, we carry out the first of the two problems described above: we introduce the error functional associated with the interpolation formula, construct the corresponding extremal function by means of Sobolev's method, and derive an explicit closed-form expression for the squared norm of the error functional in terms of the coefficients of the formula. The resulting expression provides the necessary foundation for the subsequent construction of the optimal coefficients, which will be addressed in forthcoming work.

The remainder of the paper is organized as follows. In Section 2, the Hilbert space $W_2^{(4,3)}(0, 1)$ and the interpolation formula under consideration are introduced. In Section 3, the error functional of the interpolation formula is defined, and its extremal function is constructed. In Section 4, an explicit closed-form expression for the squared norm of the error functional is derived.

2 The function space and the interpolation formula

Let N be a positive integer, $h = 1/N$, and $x_\beta = h\beta$, $\beta = 0, 1, \dots, N$, be the corresponding equally spaced nodes on the interval $[0, 1]$. We consider the Hilbert space [7]

$$W_2^{(4,3)}(0, 1) = \{ \varphi : [0, 1] \rightarrow \mathbb{R} \mid \varphi^{(3)} \text{ is absolutely continuous and } \varphi^{(4)} \in L_2(0, 1) \},$$

that is, the space of functions on $[0, 1]$ whose third derivative is absolutely continuous and whose fourth derivative is square-integrable. For $f, g \in W_2^{(4,3)}(0, 1)$, the pseudo inner product is defined by

$$\langle f, g \rangle_{W_2^{(4,3)}} = \int_0^1 (f^{(4)}(x) + f^{(3)}(x))(g^{(4)}(x) + g^{(3)}(x)) dx,$$

and the corresponding seminorm is

$$\|f\|_{W_2^{(4,3)}}^2 = \int_0^1 (f^{(4)}(x) + f^{(3)}(x))^2 dx. \quad (1)$$

Since (1) vanishes on every function of the form $P_2(x) + d e^{-x}$, where P_2 is an arbitrary polynomial of degree at most 2 and $d \in \mathbb{R}$, it is a seminorm rather than a norm on $W_2^{(4,3)}(0,1)$ itself. We therefore set

$$K = \text{span}\{1, x, x^2, e^{-x}\}$$

and pass to the factor space of $W_2^{(4,3)}(0,1)$ with respect to K : two functions $f_1, f_2 \in W_2^{(4,3)}(0,1)$ are identified, $f_1 \sim f_2$, whenever $f_1 - f_2 \in K$, that is, whenever $f_1(x) = f_2(x) + P_2(x) + d e^{-x}$ for some polynomial P_2 of degree at most 2 and some $d \in \mathbb{R}$. The corresponding equivalence class is denoted $\bar{f}(x) = f(x) + K$, and in particular the zero element of the factor space is $\bar{0}(x) = P_2(x) + d e^{-x}$. For notational convenience, the resulting factor space $W_2^{(4,3)}(0,1)/K$ is again denoted by $W_2^{(4,3)}(0,1)$ throughout the paper; on this factor space, (1) is a norm.

We consider an approximation formula that approximates a function $\varphi \in W_2^{(4,3)}(0,1)$ using the values of φ and of its first three derivatives at the nodes x_β :

$$P_\varphi(x) = \sum_{\beta=0}^N C_{\beta,0}(x)\varphi(x_\beta) + \sum_{\beta=0}^N C_{\beta,1}(x)\varphi'(x_\beta) + \sum_{\beta=0}^N C_{\beta,2}(x)\varphi''(x_\beta) + \sum_{\beta=0}^N C_{\beta,3}(x)\varphi'''(x_\beta), \quad (2)$$

or, more compactly,

$$P_\varphi(x) = \sum_{\beta=0}^N \sum_{\alpha=0}^3 C_{\beta,\alpha}(x)\varphi^{(\alpha)}(x_\beta).$$

Here $C_{\beta,\alpha}(x)$, $\alpha = 0, 1, 2, 3$, are coefficients to be determined, and the formula is required to be exact on K , that is, to reproduce 1, x , x^2 , and e^{-x} without error. The formula is called *optimal* if the coefficients are chosen so as to minimize the norm of the corresponding error functional in the conjugate space $W_2^{(4,3)*}(0,1)$.

The coefficients $C_{\beta,\alpha}$ for $\alpha = 0, 1, 2$ coincide with the coefficients of the optimal interpolation formulas obtained for the cases $m = 1, 2, 3$ in [8–10], respectively, and are used here without modification. They admit the following closed-form representations, valid for all $\beta = 0, 1, \dots, N$:

$$C_{\beta,0}(x) = \frac{1}{2h} \left(|x - h(\beta - 1)| + |x - h(\beta + 1)| - 2|x - h\beta| \right). \quad (3)$$

$$\begin{aligned} C_{\beta,1}(x) = & \frac{1}{4h} \left(h|x - h(\beta + 1)| - h|x - h(\beta - 1)| + \text{sgn}(x - h(\beta - 1))(x - h(\beta - 1))^2 \right. \\ & \left. - 2\text{sgn}(x - h\beta)(x - h\beta)^2 + \text{sgn}(x - h(\beta + 1))(x - h(\beta + 1))^2 \right), \end{aligned} \quad (4)$$

$\beta = 0, 1, \dots, N.$

$$\begin{aligned} C_{\beta,2}(x) = & \frac{1}{24h} \left[2|x - h(\beta - 1)|^3 + h^2|x - h(\beta - 1)| - 4|x - h\beta|^3 - 2h^2|x - h\beta| \right. \\ & \left. + 2|x - h(\beta + 1)|^3 + 3h\text{sgn}(x - h(\beta + 1))(x - h(\beta + 1))^2 + h^2|x - h(\beta + 1)| \right. \\ & \left. - 3h\text{sgn}(x - h(\beta - 1))(x - h(\beta - 1))^2 \right]. \end{aligned} \quad (5)$$

Before proceeding to the unknown coefficients $C_{\beta,3}(x)$, we record the following properties of the known coefficients (3)–(5), which follow from the exactness of the interpolation formula (2) on the subspace K and will be used repeatedly in the sequel.

Theorem 1. For every $x \in [0, 1]$, the coefficients (3)–(5) satisfy the following equalities:

$$\sum_{\beta=0}^N C_{\beta,0}(x) = 1, \quad (6)$$

$$\sum_{\beta=0}^N C_{\beta,0}(x) (x_\beta) = x, \quad (7)$$

$$\sum_{\beta=0}^N C_{\beta,1}(x) = 0, \quad (8)$$

$$\sum_{\beta=0}^N C_{\beta,2}(x) = 0. \quad (9)$$

Proof. We verify (6) directly, without auxiliary notation. Writing out $2h C_{\beta,0}(x)$ for a few consecutive β ,

$$\begin{aligned} \beta = 0 : & \quad |x + h| + |x - h| - 2|x|, \\ \beta = 1 : & \quad |x| + |x - 2h| - 2|x - h|, \\ \beta = 2 : & \quad |x - h| + |x - 3h| - 2|x - 2h|, \dots, \end{aligned}$$

we see that each interior term $|x - kh|$, $1 \leq k \leq N - 1$, receives coefficient $+1$ from its two neighboring β 's and -2 from its own β , i.e. a total of 0 ; the same cancellation occurs at the other end ($\beta = N - 2, N - 1, N$). Summing over $\beta = 0, \dots, N$, all interior terms cancel and only the outermost ones survive:

$$2h \sum_{\beta=0}^N C_{\beta,0}(x) = |x + h| - |x| - |x - h(N - 1)| + |x - h(N + 1)|.$$

Since $hN = 1$ and $x \in [0, 1]$, we have $|x + h| - |x| = h$ and $|x - h(N + 1)| - |x - h(N - 1)| = h$, hence

$$2h \sum_{\beta=0}^N C_{\beta,0}(x) = 2h, \quad \text{i.e.} \quad \sum_{\beta=0}^N C_{\beta,0}(x) = 1.$$

Let us also verify (7) directly. Multiplying $C_{\beta,0}(x)$ by β and $2h$ and summing, the interior terms $|x - kh|$, $1 \leq k \leq N - 1$, cancel (coefficient $(k + 1) + (k - 1) - 2k = 0$), and only the outermost terms survive:

$$2h \sum_{\beta=0}^N \beta C_{\beta,0}(x) = x - (N + 1)|x - 1| + N|x - h(N + 1)|.$$

For $x \in [0, 1]$, using $|x - 1| = 1 - x$, $|x - h(N + 1)| = 1 + h - x$ and $Nh = 1$, this equals $Nh + 2x - 1 = 2x$, hence

$$\sum_{\beta=0}^N C_{\beta,0}(x) x_\beta = h \sum_{\beta=0}^N \beta C_{\beta,0}(x) = x, \quad x \in [0, 1].$$

Similarly, writing $4h C_{\beta,1}(x) = hA_\beta + B_\beta$ with $A_\beta = |x - h(\beta + 1)| - |x - h(\beta - 1)|$ and $B_\beta = \text{sgn}(x - h(\beta - 1))(x - h(\beta - 1))^2 - 2\text{sgn}(x - h\beta)(x - h\beta)^2 + \text{sgn}(x - h(\beta + 1))(x - h(\beta + 1))^2$,

the same cancellation pattern gives

$$\sum_{\beta=0}^N A_\beta = |x - hN| + |x - h(N+1)| - |x + h| - |x| = 2 - 4x,$$

$$\sum_{\beta=0}^N B_\beta = [(x+h)^2 - x^2] + [-(x-1-h)^2 + (x-1)^2] = 4hx - 2h,$$

for $x \in (0, 1)$, using $Nh = 1$. Hence

$$4h \sum_{\beta=0}^N C_{\beta,1}(x) = h(2 - 4x) + (4hx - 2h) = 0, \quad \text{i.e.} \quad \sum_{\beta=0}^N C_{\beta,1}(x) = 0, \quad x \in [0, 1].$$

Finally, we verify (9). Writing $24h C_{\beta,2}(x) = P_\beta + h^2 Q_\beta + 3h R_\beta$ with

$$P_\beta = 2|x - h(\beta - 1)|^3 - 4|x - h\beta|^3 + 2|x - h(\beta + 1)|^3,$$

$$Q_\beta = |x - h(\beta - 1)| - 2|x - h\beta| + |x - h(\beta + 1)|,$$

$$R_\beta = \operatorname{sgn}(x - h(\beta + 1))(x - h(\beta + 1))^2 - \operatorname{sgn}(x - h(\beta - 1))(x - h(\beta - 1))^2,$$

we note that $Q_\beta = 2h C_{\beta,0}(x)$, so that $\sum_{\beta=0}^N Q_\beta = 2h$ by (6). For P_β and R_β , the same telescoping cancellation of interior terms as above leaves only the outermost contributions:

$$\sum_{\beta=0}^N P_\beta = 2[(x+h)^3 - x^3] + 2[(1+h-x)^3 - (1-x)^3],$$

$$\sum_{\beta=0}^N R_\beta = [-(1-x)^2 - x^2] + [-(1+h-x)^2 - (x+h)^2],$$

for $x \in (0, 1)$, using $Nh = 1$. Expanding both sums and combining,

$$\sum_{\beta=0}^N P_\beta = 12hx^2 - 12hx + 6h + 6h^2 + 4h^3, \quad 3 \sum_{\beta=0}^N R_\beta = -12hx^2 + 12hx - 6h - 6h^2 - 4h^3,$$

so that

$$24h \sum_{\beta=0}^N C_{\beta,2}(x) = \sum_{\beta=0}^N P_\beta + h^2 \cdot 2h + 3h \sum_{\beta=0}^N R_\beta = 0,$$

that is, $\sum_{\beta=0}^N C_{\beta,2}(x) = 0$, $x \in [0, 1]$.

The coefficients $C_{\beta,3}(x)$, $\beta = 0, 1, \dots, N$, corresponding to the third-derivative data $\varphi'''(x_\beta)$, are unknown and are to be determined by minimizing the norm of the error functional; this is the main object of the remainder of the paper.

The error of the interpolation formula (2) is

$$R_\varphi(x) = \varphi(x) - P_\varphi(x). \quad (10)$$

In the next section, this error is treated as a continuous linear functional on $W_2^{(4,3)}(0, 1)$, and its associated extremal function is constructed by means of Sobolev's method.

3 The error functional and an extremal function

Fix $z \in [0, 1]$. The error of the approximation formula (2) at the point z , defined in (10), can be represented as a continuous linear functional acting on φ by means of the Dirac delta function δ and its derivatives:

$$\begin{aligned} R(x, z) = & \delta(x - z) - \sum_{\beta=0}^N C_{\beta,0}(z) \delta(x - x_\beta) + \sum_{\beta=0}^N C_{\beta,1}(z) \delta'(x - x_\beta) \\ & - \sum_{\beta=0}^N C_{\beta,2}(z) \delta''(x - x_\beta) + \sum_{\beta=0}^N C_{\beta,3}(z) \delta'''(x - x_\beta). \end{aligned} \quad (11)$$

We verify that pairing $R(x, z)$ with φ indeed reproduces the error $R_\varphi(z)$. Using the identities $\int_{-\infty}^{\infty} \delta^{(k)}(x - a) \varphi(x) dx = (-1)^k \varphi^{(k)}(a)$, we compute

$$\begin{aligned} (R, \varphi) &= \int_{-\infty}^{\infty} R(x, z) \varphi(x) dx \\ &= \int_{-\infty}^{\infty} \delta(x - z) \varphi(x) dx - \sum_{\beta=0}^N C_{\beta,0}(z) \int_{-\infty}^{\infty} \delta(x - x_\beta) \varphi(x) dx \\ &\quad + \sum_{\beta=0}^N C_{\beta,1}(z) \int_{-\infty}^{\infty} \delta'(x - x_\beta) \varphi(x) dx - \sum_{\beta=0}^N C_{\beta,2}(z) \int_{-\infty}^{\infty} \delta''(x - x_\beta) \varphi(x) dx \\ &\quad + \sum_{\beta=0}^N C_{\beta,3}(z) \int_{-\infty}^{\infty} \delta'''(x - x_\beta) \varphi(x) dx \\ &= \varphi(z) - \sum_{\beta=0}^N C_{\beta,0}(z) \varphi(x_\beta) - \sum_{\beta=0}^N C_{\beta,1}(z) \varphi'(x_\beta) \\ &\quad - \sum_{\beta=0}^N C_{\beta,2}(z) \varphi''(x_\beta) - \sum_{\beta=0}^N C_{\beta,3}(z) \varphi'''(x_\beta) = \varphi(z) - P_\varphi(z) = R_\varphi(z). \end{aligned}$$

Since the coefficients $C_{\beta,0}, \dots, C_{\beta,3}$ are chosen so that the approximation formula reproduces $1, x, x^2, e^{-x}$ exactly, the functional $R(x, z)$ annihilates every element of $K = \text{span}\{1, x, x^2, e^{-x}\}$. For brevity, in what follows we write $R(x, z)$ simply as $R(x)$.

To construct the optimal approximation formula, we must compute the norm $\|R\|_{W_2^{(4,3)*}}$ of the error functional and then choose the unknown coefficients $C_{\beta,3}$, $\beta = 0, 1, \dots, N$ to minimize this norm; the coefficients (3)–(5) given in Section 2 remain fixed throughout this process. By the Cauchy–Schwarz inequality, the error of the formula at any $\varphi \in W_2^{(4,3)}(0, 1)$ is bounded by

$$|(R, \varphi)| \leq \|R\|_{W_2^{(4,3)*}} \cdot \|\varphi\|_{W_2^{(4,3)}}. \quad (12)$$

Inequality (12) shows that $\|R\|_{W_2^{(4,3)*}}$ depends on the coefficients $C_{\beta,3}$, $\beta = 0, 1, \dots, N$, and that the accuracy of the approximation formula is governed by this norm. It is therefore natural to seek the coefficients $C_{\beta,3}$ that minimize $\|R\|_{W_2^{(4,3)*}}$, that is, to compute

$$\inf_{C_{\beta,3}} \|R\|_{W_2^{(4,3)}}. \quad (13)$$

The coefficients $\overset{\circ}{C}_{\beta,3}$ attaining the infimum in (13) are called the *optimal coefficients*, and the corresponding formula is then called the *optimal interpolation formula*. Thus, the construction of the optimal formula reduces to solving the following two problems.

Problem 1. Compute $\|R\|_{W_2^{(4,3)}}$ explicitly in terms of the unknown coefficients $C_{\beta,3}$.

Problem 2. Find the coefficients $\overset{\circ}{C}_{\beta,3}$, $\beta = 0, 1, \dots, N$, satisfying (13).

In the remainder of this paper we solve Problem 1, deriving an explicit closed-form expression for $\|R\|_{W_2^{(4,3)}}^2$. Problem 2 will be addressed in a forthcoming work.

To solve Problem 1, we make use of the extremal function associated with the error functional R , following S. L. Sobolev's approach [2] to the construction of optimal quadrature and interpolation formulas in Hilbert spaces of this type. The extremal function $\psi_R(x)$ corresponding to R is characterized by the equality

$$(R, \psi_R) = \|R\|_{W_2^{(4,3)*}} \cdot \|\psi_R\|_{W_2^{(4,3)}}, \quad (14)$$

that is, equality is attained in the Cauchy–Schwarz inequality (12) for $\varphi = \psi_R$. For a continuous linear functional on a Hilbert space of the type considered here, the corresponding extremal function was found by S. L. Sobolev; in the present setting it takes the form

$$\psi_R(x) = -R(x) * G_4(x) + P_2(x) + d e^{-x}, \quad (15)$$

where $*$ denotes convolution, P_2 is an arbitrary polynomial of degree at most 2, $d \in \mathbb{R}$ is arbitrary, and G_m is the function [11]

$$G_m(x) = \frac{\operatorname{sgn}(x)}{2} \left(\frac{e^x - e^{-x}}{2} - \sum_{k=1}^{m-1} \frac{x^{2k-1}}{(2k-1)!} \right). \quad (16)$$

For $m = 4$, (16) gives

$$G_4(x) = \frac{\operatorname{sgn}(x)}{2} \left(\frac{e^x - e^{-x}}{2} - x - \frac{x^3}{6} - \frac{x^5}{120} \right). \quad (17)$$

In what follows, since z is fixed, we write $C_{\beta,\alpha}$ for the coefficients $C_{\beta,\alpha}(z)$, $\alpha = 0, 1, 2, 3$, in order to simplify the notation.

Theorem 2. The extremal function $\psi_R(x)$ corresponding to the error functional $R(x, z)$ in (11) is given by

$$\begin{aligned} \psi_R(x) = & G_4(x - z) - \sum_{\beta=0}^N C_{\beta,0} \cdot G_4(x - h\beta) + \sum_{\beta=0}^N C_{\beta,1} \cdot G_4'(x - h\beta) \\ & - \sum_{\beta=0}^N C_{\beta,2} \cdot G_4''(x - h\beta) + \sum_{\beta=0}^N C_{\beta,3} \cdot G_4'''(x - h\beta) + P_2(x) + d e^{-x}. \end{aligned} \quad (18)$$

Proof. For two functions f, g , the convolution $*$ is defined by

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(x - y)g(y) dy = \int_{-\infty}^{\infty} f(y)g(x - y) dy.$$

To apply (15), we first compute $R(x) * G_4(x)$. Using the definition (11) of $R(x, z)$ together with the sifting property of the delta function and its derivatives,

$$\begin{aligned}
R(x) * G_4(x) &= \int_{-\infty}^{\infty} R(y) \cdot G_4(x-y) dy = \int_{-\infty}^{\infty} R(y, z) \cdot G_4(x-y) dy \\
&= \int_{-\infty}^{\infty} \delta(y-z) \cdot \frac{\operatorname{sgn}(x-y)}{2} \left(\frac{e^{x-y} - e^{y-x}}{2} - (x-y) - \frac{(x-y)^3}{6} - \frac{(x-y)^5}{120} \right) dy \\
&\quad - \sum_{\beta=0}^N C_{\beta,0} \int_{-\infty}^{\infty} \delta(y-h\beta) \cdot \frac{\operatorname{sgn}(x-y)}{2} \left(\frac{e^{x-y} - e^{y-x}}{2} - (x-y) - \frac{(x-y)^3}{6} - \frac{(x-y)^5}{120} \right) dy \\
&\quad + \sum_{\beta=0}^N C_{\beta,1} \int_{-\infty}^{\infty} \delta'(y-h\beta) \cdot \frac{\operatorname{sgn}(x-y)}{2} \left(\frac{e^{x-y} - e^{y-x}}{2} - (x-y) - \frac{(x-y)^3}{6} - \frac{(x-y)^5}{120} \right) dy \\
&\quad - \sum_{\beta=0}^N C_{\beta,2} \int_{-\infty}^{\infty} \delta''(y-h\beta) \cdot \frac{\operatorname{sgn}(x-y)}{2} \left(\frac{e^{x-y} - e^{y-x}}{2} - (x-y) - \frac{(x-y)^3}{6} - \frac{(x-y)^5}{120} \right) dy \\
&\quad + \sum_{\beta=0}^N C_{\beta,3} \int_{-\infty}^{\infty} \delta'''(y-h\beta) \cdot \frac{\operatorname{sgn}(x-y)}{2} \left(\frac{e^{x-y} - e^{y-x}}{2} - (x-y) - \frac{(x-y)^3}{6} - \frac{(x-y)^5}{120} \right) dy \\
&= \frac{\operatorname{sgn}(x-z)}{2} \left(\frac{e^{x-z} - e^{-(x-z)}}{2} - (x-z) - \frac{(x-z)^3}{6} - \frac{(x-z)^5}{120} \right) \\
&\quad - \sum_{\beta=0}^N C_{\beta,0} \cdot \frac{\operatorname{sgn}(x-h\beta)}{2} \left(\frac{e^{x-h\beta} - e^{-(x-h\beta)}}{2} - (x-h\beta) - \frac{(x-h\beta)^3}{6} - \frac{(x-h\beta)^5}{120} \right) \\
&\quad + \sum_{\beta=0}^N C_{\beta,1} \cdot \frac{\operatorname{sgn}(x-h\beta)}{2} \left(\frac{e^{x-h\beta} + e^{-(x-h\beta)}}{2} - 1 - \frac{(x-h\beta)^2}{2} - \frac{(x-h\beta)^4}{24} \right) \\
&\quad - \sum_{\beta=0}^N C_{\beta,2} \cdot \frac{\operatorname{sgn}(x-h\beta)}{2} \left(\frac{e^{x-h\beta} - e^{-(x-h\beta)}}{2} - (x-h\beta) - \frac{(x-h\beta)^3}{6} \right) \\
&\quad + \sum_{\beta=0}^N C_{\beta,3} \cdot \frac{\operatorname{sgn}(x-h\beta)}{2} \left(\frac{e^{x-h\beta} + e^{-(x-h\beta)}}{2} - 1 - \frac{(x-h\beta)^2}{2} \right).
\end{aligned}$$

Comparing each term with (16)–(17) and its derivatives, the right-hand side is recognized as $G_4(x-z)$, $G_4(x-h\beta)$, $G_4'(x-h\beta)$, $G_4''(x-h\beta)$, and $G_4'''(x-h\beta)$, respectively, so that the last equality can be written compactly as

$$\begin{aligned}
R(x) * G_4(x) &= G_4(x-z) - \sum_{\beta=0}^N C_{\beta,0} \cdot G_4(x-h\beta) + \sum_{\beta=0}^N C_{\beta,1} \cdot G_4'(x-h\beta) \\
&\quad - \sum_{\beta=0}^N C_{\beta,2} \cdot G_4''(x-h\beta) + \sum_{\beta=0}^N C_{\beta,3} \cdot G_4'''(x-h\beta).
\end{aligned}$$

Substituting this into (15) yields (18), as claimed.

4 The squared norm of the error functional

By the Riesz representation theorem, since ψ_R is the extremal function corresponding to R , equality holds in the Cauchy–Schwarz type relation (14), and moreover

$$\|\psi_R\|_{W_2^{(4,3)}} = \|R\|_{W_2^{(4,3)*}},$$

so that

$$(R, \psi_R) = \|R\|_{W_2^{(4,3)*}} \cdot \|\psi_R\|_{W_2^{(4,3)}} = \|R\|_{W_2^{(4,3)}}^2. \quad (19)$$

Substituting the error functional (11) and the defining relation (15) for the extremal function into (19), we obtain

$$\begin{aligned} \|R\|^2 &= \int_{-\infty}^{\infty} R(x) \psi_R(x) dx = \int_{-\infty}^{\infty} R(x, z) (-R(x) * G_4(x) + P_2(x) + d e^{-x}) dx \\ &= \int_{-\infty}^{\infty} R(x) (-R(x) * G_4(x)) dx + (R, P_2(x)) + d (R, e^{-x}). \end{aligned} \quad (20)$$

We first examine the terms $(R, 1)$, (R, x) , (R, x^2) arising in (20), since $P_2(x)$ is an arbitrary polynomial of degree at most 2. By (11), $(R, \varphi) = \varphi(z) - \sum_{\beta=0}^N C_{\beta,0} \varphi(x_\beta) - \sum_{\beta=0}^N C_{\beta,1} \varphi'(x_\beta) - \sum_{\beta=0}^N C_{\beta,2} \varphi''(x_\beta) - \sum_{\beta=0}^N C_{\beta,3} \varphi'''(x_\beta)$, and since R annihilates every element of $K \supset \{1, x, x^2\}$, taking $\varphi = 1$ gives

$$(R, 1) = 1 - \sum_{\beta=0}^N C_{\beta,0} = 0. \quad (21)$$

Taking $\varphi = x$ gives

$$(R, x) = z - \sum_{\beta=0}^N C_{\beta,0} x_\beta - \sum_{\beta=0}^N C_{\beta,1} = 0. \quad (22)$$

Taking $\varphi = x^2$ gives the following equality

$$(R, x^2) = z^2 - \sum_{\beta=0}^N C_{\beta,0} x_\beta^2 - 2 \sum_{\beta=0}^N C_{\beta,1} x_\beta - 2 \sum_{\beta=0}^N C_{\beta,2} = 0. \quad (23)$$

We next consider (R, e^{-x}) . Taking $\varphi(x) = e^{-x}$, the same identity gives

$$(R, e^{-x}) = e^{-z} - \sum_{\beta=0}^N C_{\beta,0} e^{-x_\beta} + \sum_{\beta=0}^N C_{\beta,1} e^{-x_\beta} - \sum_{\beta=0}^N C_{\beta,2} e^{-x_\beta} + \sum_{\beta=0}^N C_{\beta,3} e^{-x_\beta} = 0, \quad (24)$$

since $e^{-x} \in K$.

Relations (21)–(24) express the exactness of the approximation formula on K in terms of the coefficients and will be used as constraints in the construction of the optimal coefficients $\hat{C}_{\beta,3}$. By properties (6)–(9) established in Theorem 1, conditions (21)–(23) are satisfied automatically, since they involve only the already known coefficients $C_{\beta,0}$, $C_{\beta,1}$, $C_{\beta,2}$; these identities will nevertheless be used later in deriving the explicit form of the unknown coefficients. Thus, the unknown coefficients $C_{\beta,3}$ appear only in condition (24), and it is this condition alone that will serve as a constraint in the minimization of the norm of the error functional.

We now return to (20) and compute the remaining term $\int_{-\infty}^{\infty} R(x)(-R(x) * G_4(x)) dx$ explicitly. Substituting the definition (11) of $R(x, z)$ together with the explicit form of $R(x) * G_4(x)$ obtained in the proof of Theorem 2, and simplifying, the squared norm of the error functional of the approximation formula (2) takes the form

$$\begin{aligned}
\|R\|^2 = & -2 \sum_{\beta=0}^N C_{\beta,0} \cdot \frac{\operatorname{sgn}(z-h\beta)}{2} \left(\frac{e^{z-h\beta} - e^{-(z-h\beta)}}{2} - (z-h\beta) - \frac{(z-h\beta)^3}{6} - \frac{(z-h\beta)^5}{120} \right) \\
& + 2 \sum_{\beta=0}^N C_{\beta,1} \cdot \frac{\operatorname{sgn}(z-h\beta)}{2} \left(\frac{e^{z-h\beta} + e^{-(z-h\beta)}}{2} - 1 - \frac{(z-h\beta)^2}{2} - \frac{(z-h\beta)^4}{24} \right) \\
& - 2 \sum_{\beta=0}^N C_{\beta,2} \cdot \frac{\operatorname{sgn}(z-h\beta)}{2} \left(\frac{e^{z-h\beta} - e^{-(z-h\beta)}}{2} - (z-h\beta) - \frac{(z-h\beta)^3}{6} \right) \\
& + 2 \sum_{\beta=0}^N C_{\beta,3} \cdot \frac{\operatorname{sgn}(z-h\beta)}{2} \left(\frac{e^{z-h\beta} + e^{-(z-h\beta)}}{2} - 1 - \frac{(z-h\beta)^2}{2} \right) \\
& + \sum_{\beta=0}^N \sum_{\gamma=0}^N C_{\beta,0} C_{\gamma,0} \frac{\operatorname{sgn}(h\beta-h\gamma)}{2} \left(\frac{e^{h\beta-h\gamma} - e^{-(h\beta-h\gamma)}}{2} - (h\beta-h\gamma) - \frac{(h\beta-h\gamma)^3}{6} - \frac{(h\beta-h\gamma)^5}{120} \right) \\
& + 2 \sum_{\beta=0}^N \sum_{\gamma=0}^N C_{\beta,1} C_{\gamma,0} \frac{\operatorname{sgn}(h\beta-h\gamma)}{2} \left(\frac{e^{h\beta-h\gamma} + e^{-(h\beta-h\gamma)}}{2} - 1 - \frac{(h\beta-h\gamma)^2}{2} - \frac{(h\beta-h\gamma)^4}{24} \right) \\
& + 2 \sum_{\beta=0}^N \sum_{\gamma=0}^N C_{\beta,2} C_{\gamma,0} \frac{\operatorname{sgn}(h\beta-h\gamma)}{2} \left(\frac{e^{h\beta-h\gamma} - e^{-(h\beta-h\gamma)}}{2} - (h\beta-h\gamma) - \frac{(h\beta-h\gamma)^3}{6} \right) \\
& + 2 \sum_{\beta=0}^N \sum_{\gamma=0}^N C_{\beta,3} C_{\gamma,0} \frac{\operatorname{sgn}(h\beta-h\gamma)}{2} \left(\frac{e^{h\beta-h\gamma} + e^{-(h\beta-h\gamma)}}{2} - 1 - \frac{(h\beta-h\gamma)^2}{2} \right) \\
& - \sum_{\beta=0}^N \sum_{\gamma=0}^N C_{\beta,1} C_{\gamma,1} \frac{\operatorname{sgn}(h\beta-h\gamma)}{2} \left(\frac{e^{h\beta-h\gamma} - e^{-(h\beta-h\gamma)}}{2} - (h\beta-h\gamma) - \frac{(h\beta-h\gamma)^3}{6} \right) \\
& - 2 \sum_{\beta=0}^N \sum_{\gamma=0}^N C_{\beta,2} C_{\gamma,1} \frac{\operatorname{sgn}(h\beta-h\gamma)}{2} \left(\frac{e^{h\beta-h\gamma} + e^{-(h\beta-h\gamma)}}{2} - 1 - \frac{(h\beta-h\gamma)^2}{2} \right) \\
& - 2 \sum_{\beta=0}^N \sum_{\gamma=0}^N C_{\beta,3} C_{\gamma,1} \frac{\operatorname{sgn}(h\beta-h\gamma)}{2} \left(\frac{e^{h\beta-h\gamma} - e^{-(h\beta-h\gamma)}}{2} - (h\beta-h\gamma) \right) \\
& + \sum_{\beta=0}^N \sum_{\gamma=0}^N C_{\beta,2} C_{\gamma,2} \frac{\operatorname{sgn}(h\beta-h\gamma)}{2} \left(\frac{e^{h\beta-h\gamma} - e^{-(h\beta-h\gamma)}}{2} - (h\beta-h\gamma) \right) \\
& + 2 \sum_{\beta=0}^N \sum_{\gamma=0}^N C_{\beta,3} C_{\gamma,2} \frac{\operatorname{sgn}(h\beta-h\gamma)}{2} \left(\frac{e^{h\beta-h\gamma} + e^{-(h\beta-h\gamma)}}{2} - 1 \right) \\
& - \sum_{\beta=0}^N \sum_{\gamma=0}^N C_{\beta,3} C_{\gamma,3} \frac{\operatorname{sgn}(h\beta-h\gamma)}{2} \cdot \frac{e^{h\beta-h\gamma} - e^{-(h\beta-h\gamma)}}{2}.
\end{aligned} \tag{25}$$

Recalling the definition (16)–(17) of G_4 , expression (25) can be written compactly as follows.

Theorem 3. *The squared norm of the error functional $R(x, z)$ of the approximation formula (2) admits the closed-form expression*

$$\begin{aligned}
\|R\|^2 = & -2 \sum_{\beta=0}^N C_{\beta,0} \cdot G_4(z-h\beta) + 2 \sum_{\beta=0}^N C_{\beta,1} \cdot G_4'(z-h\beta) - 2 \sum_{\beta=0}^N C_{\beta,2} \cdot G_4''(z-h\beta) \\
& + 2 \sum_{\beta=0}^N C_{\beta,3} \cdot G_4'''(z-h\beta) + \sum_{\beta=0}^N \sum_{\gamma=0}^N C_{\beta,0} C_{\gamma,0} G_4(h\beta-h\gamma) \\
& + 2 \sum_{\beta=0}^N \sum_{\gamma=0}^N C_{\beta,1} C_{\gamma,0} G_4'(h\beta-h\gamma) \\
& + 2 \sum_{\beta=0}^N \sum_{\gamma=0}^N C_{\beta,2} C_{\gamma,0} G_4''(h\beta-h\gamma) + 2 \sum_{\beta=0}^N \sum_{\gamma=0}^N C_{\beta,3} C_{\gamma,0} G_4'''(h\beta-h\gamma) \\
& - \sum_{\beta=0}^N \sum_{\gamma=0}^N C_{\beta,1} C_{\gamma,1} G_4''(h\beta-h\gamma) - 2 \sum_{\beta=0}^N \sum_{\gamma=0}^N C_{\beta,2} C_{\gamma,1} G_4'''(h\beta-h\gamma) \\
& - 2 \sum_{\beta=0}^N \sum_{\gamma=0}^N C_{\beta,3} C_{\gamma,1} G_4^{(4)}(h\beta-h\gamma) + \sum_{\beta=0}^N \sum_{\gamma=0}^N C_{\beta,2} C_{\gamma,2} G_4^{(4)}(h\beta-h\gamma)
\end{aligned}$$

$$+2 \sum_{\beta=0}^N \sum_{\gamma=0}^N C_{\beta,3} C_{\gamma,2} G_4^{(5)}(h\beta - h\gamma) - \sum_{\beta=0}^N \sum_{\gamma=0}^N C_{\beta,3} C_{\gamma,3} G_4^{(6)}(h\beta - h\gamma). \quad (26)$$

A similar approach to deriving the squared norm of the error functional was used in the lower-dimensional cases [12, 13].

Thus, (25) and (26) together give the solution to Problem 1: an explicit closed-form expression for the squared norm of the error functional in terms of the unknown coefficients $C_{\beta,3}$, $\beta = 0, 1, \dots, N$. Minimizing this expression subject to the constraint (24) to obtain the optimal coefficients $\overset{\circ}{C}_{\beta,3}$, that is, the solution of Problem 2, will be addressed in a forthcoming work, following the same approach as in the case $m = 2$ [14].

5 Concluding remarks

In this paper we constructed an approximation formula in the Hilbert space $W_2^{(4,3)}(0, 1)$, which uses the values of a function together with its first, second, and third derivatives at equally spaced nodes and is required to be exact on the subspace $K = \text{span}\{1, x, x^2, e^{-x}\}$. Using the known coefficients from the space $L_2^{(m)}$ when $m = 1, 2, 3$ and their summation properties (Theorem 1), we showed that only one exactness condition (24) genuinely constrains the unknown coefficients $C_{\beta,3}$. Following S. L. Sobolev's variational method, we constructed the extremal function of the error functional by means of the kernel G_4 (Theorem 2) and derived an explicit closed-form expression for the squared norm of the error functional, both directly (25) and compactly through G_4 (26) (Theorem 3), thus solving Problem 1.

Minimizing (26) subject to the constraint (24) by the method of Lagrange multipliers, that is, solving Problem 2, and establishing the resulting order of accuracy of the optimal formula, will be addressed in a forthcoming work.

Declarations

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КВАДРАТ НОРМЫ ФУНКЦИОНАЛА ПОГРЕШНОСТИ ИНТЕРПОЛЯЦИОННОЙ ФОРМУЛЫ С ТРЕТЬЕЙ ПРОИЗВОДНОЙ В ГИЛЬБЕРТОВОМ ПРОСТРАНСТВЕ

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Данная работа посвящена построению оптимальной интерполяционной формулы для функций, определенных на конечном интервале в гильбертовом пространстве, которая характеризуется требованием абсолютной непрерывности и интегрируемости производной третьего порядка функции в этом пространстве на квадрат ее производной четвертого порядка. Рассматриваемая интерполяционная формула постро-

ена таким образом, чтобы она была точной как для полиномов до второй степени, так и для инвариантной экспоненциальной функции, при этом подпространство, покрываемое этими функциями, играет центральную роль на протяжении всего построения. Точность интерполяционной формулы характеризуется соответствующим функционалом ошибки, а качество формулы измеряется нормой этой функции в сопряженном с базовым гильбертовым пространством. На основе вариационного подхода, разработанного С. Л. Соболевым и его школой для построения оптимальных квадратурных и интерполяционных формул, вводится экстремальная функция, связанная с функционалом ошибки, и ее точное выражение получается путем свертки со специальной ядерной функцией, определяемой структурой пространства. Используя эту экстремальную функцию вместе с условиями дискретной ортогональности, которым удовлетворяет интерполяционная формула, получают точное аналитическое выражение для квадрата нормы функционала ошибки через неизвестные коэффициенты формулы. Это выражение служит основным начальным результатом, на котором основывается последующая задача минимизации для нахождения оптимальных коэффициентов интерполяционной формулы.

Ключевые слова: оптимальная интерполяция, гильбертово пространство, минимизация ошибок, вариационные методы, оптимальные коэффициенты

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