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OPTIMAL NUMERICAL INTEGRATION RULES BASED ON ALGEBRAIC AND TRIGONOMETRIC FUNCTIONS

^{1,2}*Khaitov T.O.*

tojiboy.xaitov.77@mail.ru

¹V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, 9, University str., Tashkent, 100174 Uzbekistan;²Tashkent State Technical University named after Islam Karimov, 2, University str., Tashkent, Uzbekistan

In this paper, we conduct a comprehensive investigation into the computation of coefficients for derivative-based quadrature formulas by employing the so-called φ -function technique. This approach offers a structured and efficient methodology for the construction of optimal quadrature formulas within the framework of numerical integration. Notably, the φ -function method not only streamlines the derivation of the associated coefficients but also guarantees a high level of precision in the resulting formulas.

Moreover, a detailed analysis of the error functional associated with the proposed quadrature formula is presented. The error estimates are substantiated by rigorous analytical expressions, thereby confirming the accuracy and robustness of the obtained results. By examining quadrature formulas with arbitrarily prescribed nodes, the optimality criteria are systematically established, and the procedures for computing the relevant coefficients and structural components are presented in detail.

As a principal result of this study, closed-form analytical representations of the coefficients of the optimal quadrature formula are derived. A special case of particular interest is that of uniformly distributed nodes, where the resulting quadrature formula coincides with the classical Euler–Maclaurin type formula. This outcome highlights both the general applicability and the practical relevance of the φ -function methodology.

Keywords: the φ -function method, quadrature formula, definite integral, nodal points, optimality

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1 Introduction

The definite integral is one of the fundamental concepts of mathematical analysis and, in particular, it can be used to get the area or volume of a solid under the graph of a function. However, in many cases, finding an antiderivative function of an integrand is difficult or even impossible. Therefore, several approximate calculation methods for numerical integration have been developed. It is known from the course of mathematical analysis that if the antiderivative function of the function under the integral is known, the integral can be calculated using the Newton–Leibniz formula. For cases where it is difficult to find the antiderivative of the functions under the integral, the problem of approximating the value of the definite integral arises. To solve this problem, various formulas have been found in mathematics, which are generally called quadrature and cubature formulas [1–8].

In this work, we construct optimal quadrature formulas in the space $K_2^{(4,2)}$ using the φ -function method. The space $K_2^{(4,2)}$ that we are looking at is a Hilbert space. This

space is a special case of the spaces considered in the works [4, 9, 10]. If the nodal points of the quadrature formula are arbitrarily fixed, then the Spline method, the φ -function method, and the Sobolev method are available to construct quadrature formulas obtained by minimizing the norm of the error functional depending on the coefficients.

Using the φ -function method in several spaces, A. Ghizzetti and A. Ossicini [11], F. Lanzara [5], T. Catinaş and G. Coman [12] constructed optimal quadrature formulas.

In [10], the authors studied the problem of constructing optimal quadrature formulas in the sense of Sard. The coefficients of the optimal quadrature formula were calculated using the resulting φ -function. The optimal quadrature formula in that work is exact for the functions $e^{\sigma x}$ and $e^{-\sigma x}$, where σ is a nonzero real parameter. Quadrature formulas involving the first and second order derivatives were developed in the following works [13–15].

Based on them, in this work we consider the following types of quadrature formulas

$$\int_a^b f(x)dx = \sum_{k=0}^n A_{0k}f(x_k) + \sum_{k=0}^n A_{1k}f'(x_k) + \sum_{k=0}^n A_{2k}f''(x_k) + \sum_{k=0}^n A_{3k}f'''(x_k) + R_n(f), \quad (1)$$

where A_{0k} , A_{1k} , A_{2k} , A_{3k} and x_k are the coefficients and nodes of the quadrature formula (1), respectively. Let the nodal points be partitioned on the interval $[a, b]$ as follows

$$a = x_0 < x_1 < \dots < x_n = b, \quad (2)$$

where $R_n(f)$ is the error of the quadrature formula (1). Let us assume that the function f under the integral we are looking at is taken from the space $K_2^{(4,2)}$, where

$$K_2^{(4,2)} := \{f : [a, b] \rightarrow \mathbb{R} \mid f''' - \text{absolutely continuous and } f^{IV} \in L_2(a, b)\}.$$

In this space, the inner product for arbitrary functions $f(x)$ and $g(x)$ is defined as

$$\langle f, g \rangle_{K_2^{(4,2)}} = \int_a^b (f^{IV}(x) + f''(x))(g^{IV}(x) + g''(x))dx.$$

The corresponding norm in this space is determined as follows

$$\|f\|_{K_2^{(4,2)}} = \left(\int_a^b (f^{IV}(x) + f''(x))^2 dx \right)^{\frac{1}{2}}.$$

We consider the construction of the quadrature formula of the form (1) with the smallest error for functions of the space $K_2^{(4,2)}$ considered in a certain form in the space of quadrature formulas. For convenience, we introduce the following designations

$$\begin{aligned} A_0 &= (A_{00}, A_{01}, \dots, A_{0n}), & A_1 &= (A_{10}, A_{11}, \dots, A_{1n}), & A_2 &= (A_{20}, A_{21}, \dots, A_{2n}), \\ & & A_3 &= (A_{30}, A_{31}, \dots, A_{3n}) \end{aligned}$$

and

$$X = (x_0, x_1, \dots, x_n). \quad (3)$$

Below we give the definitions of optimality, of the quadrature formulas [5, 12, 13].

Definition 1. A quadrature formula of the form (1) is called optimal in the sense of Nikolskii in the space $K_2^{(4,2)}$ if the quantity

$$F_n(K_2^{(4,2)}, A_0, A_1, A_2, A_3, X) = \sup_{f \in K_2^{(4,2)}} |R_n(f)|,$$

reaches its smallest value with respect to A_0, A_1, A_2, A_3 and X , and A_0, A_1, A_2, A_3 and X are determined by equality (3).

Definition 2. A quadrature formula of the form (1) is called optimal in the sense of Sard in the space $K_2^{(4,2)}$ if the quantity

$$F_n(K_2^{(4,2)}, A_0, A_1, A_2, A_3) = \sup_{f \in K_2^{(4,2)}} |R_n(f)|,$$

reaches its smallest value relative to A_0, A_1, A_2, A_3 for fixed X , where A_0, A_1, A_2, A_3 and X are defined in (3).

In this paper we construct an optimal quadrature formula of the form (1) in the sense of Sard in the Hilbert space $K_2^{(4,2)}$ which is exact for trigonometric functions $\sin(x), \cos(x)$ and any linear function.

Next, we discuss the φ -function method for construction of quadrature formulas of the form (1) in the space $K_2^{(4,2)}$. In Section 3, we consider the optimization of the quadrature formulas in the form (1). We obtain explicit expressions for the coefficients of the optimal quadrature formula. In particular, we derive a new optimal quadrature formula in the space $K_2^{(4,2)}$.

2 A method for constructing quadrature formulas

Let a function f be taken from the space $K_2^{(4,2)}$ and the nodes for the given natural numbers n be distributed as in the form (2). Then, for each interval $[x_{k-1}, x_k]$ ($k = 1, 2, \dots, n$), we consider the functions φ_k , $k = 1, 2, \dots, n$ with the following property

$$\varphi_k^{IV}(x) + \varphi_k''(x) = 1, \quad k = 1, 2, \dots, n. \quad (4)$$

Then the φ -function is defined as follows

$$\varphi|_{[x_{k-1}, x_k]} = \varphi_k, \quad k = 1, 2, \dots, n.$$

Therefore, the reduction of the φ -function on the interval $[x_{k-1}, x_k]$ is equal to φ_k .

We introduce the following notations:

$$I(f) := \int_a^b f(x) dx,$$

$$Q_{n,3}(f) = \sum_{k=0}^n A_{0k} f(x_k) + \sum_{k=0}^n A_{1k} f'(x_k) + \sum_{k=0}^n A_{2k} f''(x_k) + \sum_{k=0}^n A_{3k} f'''(x_k).$$

Now, using the property of addition of definite integrals and taking into account equality (4), we get

$$I(f) = \int_a^b f(x) dx = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} 1 \cdot f(x) dx = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} (\varphi_k^{IV}(x) + \varphi_k''(x)) f(x) dx.$$

Then from here integrating by parts we have

$$\begin{aligned}
I(f) &= [\varphi_n'''(x_n) + \varphi_n'(x_n)]f(x_n) + \\
&+ \sum_{k=1}^{n-1} [\varphi_k'''(x_k) - \varphi_{k+1}'''(x_k) + \varphi_k'(x_k) - \varphi_{k+1}'(x_k)]f(x_k) - \\
&- [\varphi_1'''(x_0) + \varphi_1'(x_0)]f(x_0) - [\varphi_n''(x_n) + \varphi_n(x_n)]f'(x_n) - \\
&- \sum_{k=1}^{n-1} [\varphi_k''(x_k) - \varphi_{k+1}''(x_k) + \varphi_k(x_k) - \varphi_{k+1}(x_k)]f'(x_k) + [\varphi_1''(x_0) + \\
&+ \varphi_1(x_0)]f'(x_0) + \varphi_n'(x_n)f''(x_n) + \sum_{k=1}^{n-1} [\varphi_k'(x_k) - \varphi_{k+1}'(x_k)]f''(x_k) - \varphi_1'(x_0)f''(x_0) - \\
&- \varphi_n(x_n)f'''(x_n) - \sum_{k=1}^{n-1} [\varphi_k(x_k) - \varphi_{k+1}(x_k)]f'''(x_k) + \varphi_1(x_0)f'''(x_0) + \\
&+ \sum_{k=1}^n \int_{x_{k-1}}^{x_k} (f^{(IV)}(x) + f^{(II)}(x))\varphi_k(x)dx.
\end{aligned} \tag{5}$$

From equality (5) we get

$$\begin{aligned}
A_{00} &= -\varphi_1'''(x_0) - \varphi_1'(x_0), \\
A_{0k} &= \varphi_k'''(x_k) - \varphi_{k+1}'''(x_k) + \varphi_k'(x_k) - \varphi_{k+1}'(x_k), \quad k = 1, 2, \dots, n-1, \\
A_{0n} &= \varphi_n'''(x_n) + \varphi_n'(x_n), \\
A_{10} &= \varphi_1''(x_0) + \varphi_1(x_0), \\
A_{1k} &= -\varphi_k''(x_k) + \varphi_{k+1}''(x_k) - \varphi_k(x_k) + \varphi_{k+1}(x_k), \quad k = 1, 2, \dots, n-1, \\
A_{1n} &= -\varphi_n''(x_n) - \varphi_n(x_n), \\
A_{20} &= -\varphi_1'(x_0), \\
A_{2k} &= \varphi_k'(x_k) - \varphi_{k+1}'(x_k), \quad k = 1, 2, \dots, n-1, \\
A_{2n} &= \varphi_n'(x_n), \\
A_{30} &= \varphi_1(x_0), \\
A_{3k} &= -\varphi_k(x_k) + \varphi_{k+1}(x_k), \quad k = 1, 2, \dots, n-1, \\
A_{3n} &= -\varphi_n(x_n).
\end{aligned} \tag{6}$$

and the error formula is as follows

$$\begin{aligned}
R_n(f) &= \sum_{k=1}^n \int_{x_{k-1}}^{x_k} (f^{IV}(x) + f''(x))\varphi_k(x)dx \\
&= \int_a^b (f^{IV}(x) + f''(x))\varphi(x)dx.
\end{aligned} \tag{7}$$

Now we solve this nonhomogeneous fourth-order differential equation:

$$y^{IV} + y'' = 1 \tag{8}$$

and obtain a general solution in the following form

$$y = C_1 \cos(x) + C_2 \sin(x) + C_3 + C_4 x + \frac{x^2}{2}. \tag{9}$$

Remark 1. Knowing the φ -function, one can find the coefficients A_{0k} , A_{1k} , A_{2k} and A_{3k} , $k = 0, 1, \dots, n$ from equality (6). This is called the φ -function method of constructing a quadrature formula.

Remark 2. As can be seen from expression (7) above, the quadrature formula (1) is exact to the functions that are solutions of the equation

$$f^{IV}(x) + f''(x) = 0.$$

In the following sections, we deal with finding the coefficients of the optimal quadrature formula (1) in the space $K_2^{(4,2)}$.

3 An optimal quadrature formula based on the φ -function method

In this section, we examine the optimality of the quadrature formula (1) within the space $K_2^{(4,2)}$. By applying the Cauchy–Schwarz inequality, the absolute value of the error corresponding to expression (7) for the quadrature formula (1) can be estimated as

$$|R_n(f)| \leq \|f^{IV} + f''\|_{L_2(a,b)} \cdot \left(\int_a^b \varphi^2(x) dx \right)^{\frac{1}{2}} = \|f(x)\|_{K_2^{(4,2)}} \cdot \|\varphi(x)\|_{L_2(a,b)}.$$

Now we consider the function $\varphi_k(x)$ in the interval $[x_{k-1}, x_k]$ ($k = 1, 2, \dots, n$) as a solution of equation (8). Then based on (9) for φ_k , we have

$$\varphi_k(x) = C_1^{(k)} \cos x + C_2^{(k)} \sin x + C_3^{(k)} + C_4^{(k)} x + \frac{x^2}{2}, \quad (10)$$

where $C_1^{(k)}$, $C_2^{(k)}$, $C_3^{(k)}$, $C_4^{(k)}$, ($k = 1, 2, \dots, n$) are arbitrary constants.

In conclusion, to find the $\varphi_k(x)$ functions, it is necessary to find all unknowns $C_1^{(k)}$, $C_2^{(k)}$, $C_3^{(k)}$ and $C_4^{(k)}$, $k = 1, 2, \dots, n$. We find $C_1^{(k)}$, $C_2^{(k)}$, $C_3^{(k)}$ and $C_4^{(k)}$ such that when the integral of the square of the $\varphi_k(x)$ function takes the smallest value. Let's look at this function that is related to these,

$$\mathcal{F}_k(C_1^{(k)}, C_2^{(k)}, C_3^{(k)}, C_4^{(k)}) = \int_{x_{k-1}}^{x_k} (\varphi_k(x))^2 dx, \quad k = 1, 2, \dots, n.$$

Then, taking into account (10), we have the following.

$$\mathcal{F}_k(C_1^{(k)}, C_2^{(k)}, C_3^{(k)}, C_4^{(k)}) = \int_{x_{k-1}}^{x_k} \left(C_1^{(k)} \cos(x) + C_2^{(k)} \sin(x) + C_3^{(k)} + C_4^{(k)} x + \frac{x^2}{2} \right)^2 dx.$$

We calculate the first partial derivatives of this function with respect to $C_1^{(k)}$, $C_2^{(k)}$, $C_3^{(k)}$ and $C_4^{(k)}$ and equating them to zero we have the following system of linear equations, and for convenience we also introduce this notation $P_k = C_1^{(k)} \cos x + C_2^{(k)} \sin x + C_3^{(k)} + C_4^{(k)} x +$

$$+ \frac{x^2}{2}:$$

$$\begin{aligned}\mathcal{F}'_k(C_1^{(k)}, C_2^{(k)}, C_3^{(k)}, C_4^{(k)})_{C_1^{(k)}} &= 2 \int_{x_{k-1}}^{x_k} P_k \cos x dx, \\ \mathcal{F}'_k(C_1^{(k)}, C_2^{(k)}, C_3^{(k)}, C_4^{(k)})_{C_2^{(k)}} &= 2 \int_{x_{k-1}}^{x_k} P_k \sin x dx, \\ \mathcal{F}'_k(C_1^{(k)}, C_2^{(k)}, C_3^{(k)}, C_4^{(k)})_{C_3^{(k)}} &= 2 \int_{x_{k-1}}^{x_k} P_k dx, \\ \mathcal{F}'_k(C_1^{(k)}, C_2^{(k)}, C_3^{(k)}, C_4^{(k)})_{C_4^{(k)}} &= 2 \int_{x_{k-1}}^{x_k} P_k x dx.\end{aligned}$$

We can write the system in the following form by introducing notations, and we present these notations below:

$$A_k = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \quad C_k = \begin{pmatrix} C_1^{(k)} \\ C_2^{(k)} \\ C_3^{(k)} \\ C_4^{(k)} \end{pmatrix}, \quad B_k = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}, \quad k = 1, 2, 3, 4.$$

$$A_k C_k = B_k, \quad (k = 1, 2, 3, 4)$$

where

$$\begin{aligned}a_{11} &= \int_{x_{k-1}}^{x_k} 2 \cos^2(x) dx, \\ a_{12} = a_{21} &= \int_{x_{k-1}}^{x_k} 2 \sin(x) \cos(x) dx, \\ a_{13} = a_{31} &= \int_{x_{k-1}}^{x_k} 2 \cos(x) dx, \\ a_{14} = a_{41} &= \int_{x_{k-1}}^{x_k} 2x \cos(x) dx, \\ a_{22} &= \int_{x_{k-1}}^{x_k} 2 \sin^2(x) dx, \\ a_{23} = a_{32} &= \int_{x_{k-1}}^{x_k} 2 \sin(x) dx, \\ a_{24} = a_{42} &= \int_{x_{k-1}}^{x_k} 2x \sin(x) dx, \\ a_{33} &= \int_{x_{k-1}}^{x_k} 2 dx, \\ a_{34} = a_{43} &= \int_{x_{k-1}}^{x_k} 2x dx, \\ a_{44} &= \int_{x_{k-1}}^{x_k} 2x^2 dx,\end{aligned}$$

$$\begin{aligned}
b_1 &= - \int_{x_{k-1}}^{x_k} x^2 \cos(x) dx, \\
b_2 &= - \int_{x_{k-1}}^{x_k} x^2 \sin(x) dx, \\
b_3 &= - \int_{x_{k-1}}^{x_k} x^2 dx, \\
b_4 &= - \int_{x_{k-1}}^{x_k} x^3 dx.
\end{aligned}$$

We provide the computations of these terms below

$$\begin{aligned}
a_{11} &= x_k - x_{k-1} + \sin(x_k - x_{k-1}) \cos(x_k + x_{k-1}), \\
a_{12} &= a_{21} = \sin(x_k - x_{k-1}) \sin(x_k + x_{k-1}), \\
a_{13} &= a_{31} = 2(\sin x_k - \sin x_{k-1}), \\
a_{14} &= a_{41} = 2(x_k \sin x_k - x_{k-1} \sin x_{k-1} + \cos x_k - \cos x_{k-1}), \\
a_{22} &= x_k - x_{k-1} - \sin(x_k - x_{k-1}) \cos(x_k + x_{k-1}), \\
a_{23} &= a_{32} = -2(\cos x_k - \cos x_{k-1}), \\
a_{24} &= a_{42} = 2(-x_k \cos x_k + x_{k-1} \cos x_{k-1} + \sin x_k - \sin x_{k-1}), \\
a_{33} &= 2(x_k - x_{k-1}), \\
a_{34} &= a_{43} = x_k^2 - x_{k-1}^2, \\
a_{44} &= \frac{2}{3}(x_k^3 - x_{k-1}^3), \\
b_1 &= -x_k^2 \sin x_k + x_{k-1}^2 \sin x_{k-1} - 2x_k \cos x_k + 2x_{k-1} \cos x_{k-1} + 2(\sin x_k - \sin x_{k-1}), \\
b_2 &= x_k^2 \cos x_k - x_{k-1}^2 \cos x_{k-1} - 2x_k \sin x_k + 2x_{k-1} \sin x_{k-1} - 2(\cos x_k - \cos x_{k-1}), \\
b_3 &= -\frac{1}{3}(x_k^3 - x_{k-1}^3), \\
b_4 &= -\frac{1}{4}(x_k^4 - x_{k-1}^4).
\end{aligned}$$

Based on the solution of the obtained system, we derive the following expression. For convenience, we also introduce the notation $h_k = x_k - x_{k-1}$, $m_k = x_k + x_{k-1}$:

$$\begin{aligned}
C_1^{(k)} &= \frac{h_k \cdot \cos(\frac{m_k}{2}) \cdot [12 \cdot \sin(\frac{h_k}{2}) - 6 \cdot h_k \cdot \cos(\frac{h_k}{2}) - h_k^2 \cdot \sin(\frac{h_k}{2})]}{3[h_k^2 + h_k \cdot \sin(h_k) - 4 \cdot (1 - \cos(h_k))]}, \\
C_2^{(k)} &= \frac{h_k \cdot \sin(\frac{m_k}{2}) \cdot [12 \cdot \sin(\frac{h_k}{2}) - 6 \cdot h_k \cdot \cos(\frac{h_k}{2}) - h_k^2 \cdot \sin(\frac{h_k}{2})]}{3[h_k^2 + h_k \cdot \sin(h_k) - 4 \cdot (1 - \cos(h_k))]}, \\
C_3^{(k)} &= \frac{(m_k)^2}{8} - \frac{(h_k)^2}{24} - \frac{2 \sin(\frac{h_k}{2}) \cdot [12 \cdot \sin(\frac{h_k}{2}) - 6 \cdot h_k \cdot \cos(\frac{h_k}{2}) - h_k^2 \cdot \sin(\frac{h_k}{2})]}{3[h_k^2 + h_k \cdot \sin(h_k) - 4 \cdot (1 - \cos(h_k))]}, \\
C_4^{(k)} &= -\frac{1}{2}m_k.
\end{aligned} \tag{11}$$

Then using (11) from (10) we come

$$\begin{aligned}
\varphi_k(x) &= h_k \cdot \cos(x - \frac{m_k}{2}) \\
&\cdot \frac{[12 \sin(\frac{h_k}{2}) - 6 \cdot h_k \cdot \cos(\frac{h_k}{2}) - h_k^2 \sin(\frac{h_k}{2})]}{3[h_k^2 + h_k \cdot \sin(h_k) - 4(1 - \cos(h_k))]} + \frac{m_k^2}{8} - \\
&- \frac{h_k^2}{24} - \frac{2 \sin(\frac{h_k}{2})(12 \sin(\frac{h_k}{2}) - 6 \cdot h_k \cdot \cos(\frac{h_k}{2}) - h_k^2 \sin(\frac{h_k}{2}))}{3[h_k^2 + h_k \cdot \sin(h_k) - 4(1 - \cos(h_k))]} - \\
&- \frac{1}{2}(m_k)x + \frac{x^2}{2}.
\end{aligned} \tag{12}$$

For the first, second and third derivatives of φ_k we have

$$\varphi'_k(x) = -h_k \cdot \sin\left(x - \frac{m_k}{2}\right) \cdot \frac{[12 \sin(\frac{h_k}{2}) - 6 \cdot h_k \cdot \cos(\frac{h_k}{2}) - h_k^2 \sin(\frac{h_k}{2})]}{3[h_k^2 + h_k \cdot \sin(h_k) - 4(1 - \cos(h_k))]} - \frac{1}{2}(m_k) + x, \quad (13)$$

$$\varphi''_k(x) = -h_k \cdot \cos\left(x - \frac{m_k}{2}\right) \cdot \frac{[12 \sin(\frac{h_k}{2}) - 6 \cdot h_k \cdot \cos(\frac{h_k}{2}) - h_k^2 \sin(\frac{h_k}{2})]}{3[h_k^2 + h_k \cdot \sin(h_k) - 4(1 - \cos(h_k))]} + 1, \quad (14)$$

$$\varphi'''_k(x) = h_k \cdot \sin\left(x - \frac{m_k}{2}\right) \cdot \frac{[12 \sin(\frac{h_k}{2}) - 6 \cdot h_k \cdot \cos(\frac{h_k}{2}) - h_k^2 \sin(\frac{h_k}{2})]}{3[h_k^2 + h_k \cdot \sin(h_k) - 4(1 - \cos(h_k))]} . \quad (15)$$

We are now in a position to determine the coefficients $A_{0k}, A_{1k}, A_{2k}, A_{3k}$, $k = 1, 2, \dots, n$, by means of equation (6) for the optimal quadrature formula of the form (1) in the space $K_2^{(4,2)}$. Consequently, we arrive at the following principal result of the present study.

Theorem 1. *For fixed nodes $a = x_0 < x_1 < \dots < x_n = b$ there exists a unique optimal quadrature formula of the form*

$$\int_a^b f(x) dx \cong \sum_{k=0}^n A_{0k} f(x_k) + \sum_{k=0}^n A_{1k} f'(x_k) + \sum_{k=0}^n A_{2k} f''(x_k) + \sum_{k=0}^n A_{3k} f'''(x_k), \quad (16)$$

in the sense of Sard in the space $K_2^{(4,2)}$ with coefficients $A_{0k}, A_{1k}, A_{2k}, A_{3k}$ that have the form

$$\begin{aligned} A_{00} &= \frac{1}{2}h_1, \\ A_{0k} &= \frac{1}{2}(h_{k+1} + h_k), \quad k = 1, 2, \dots, n-1, \\ A_{0n} &= \frac{1}{2}h_n, \\ A_{10} &= \frac{h_1^4 + h_1^3 \cdot \sin(h_1) + 12h_1^2 + 12 \cdot h_1 \sin(h_1)}{12[h_1^2 + h_1 \cdot \sin(h_1) - 4(1 - \cos(h_1))]}, \\ A_{1k} &= \frac{h_{k+1}^2 - h_k^2}{12} + \frac{12(1 - \cos(h_k)) - 6 \cdot h_k \cdot \cos(h_k) - h_k^2(1 - \cos(h_k))}{3[h_k^2 + h_k \cdot \sin(h_k) - 4(1 - \cos(h_k))]} - \\ &\quad - \frac{12(1 - \cos(h_{k+1})) - 6 \cdot h_{k+1} \cdot \cos(h_{k+1}) - h_{k+1}^2(1 - \cos(h_{k+1}))}{3[h_{k+1}^2 + h_{k+1} \cdot \sin(h_{k+1}) - 4(1 - \cos(h_{k+1}))]}, \\ &\quad k = 1, 2, \dots, n-1, \\ A_{1n} &= -\frac{h_n^4 + h_n^3 \cdot \sin(h_n) + 12h_n^2 + 12 \cdot h_n \sin(h_n)}{12[h_n^2 + h_n \cdot \sin(h_n) - 4(1 - \cos(h_n))]}, \\ A_{20} &= \frac{h_1^3(4 - \cos(h_1)) + 9h_1^2 \cdot \sin(h_1) - 24h_1(1 - \cos(h_1))}{3[h_1^2 + h_1 \cdot \sin(h_1) - 4(1 - \cos(h_1))]}, \\ A_{2k} &= \frac{h_k^3(4 - \cos(h_k)) + 9h_k^2 \cdot \sin(h_k) - 24h_k(1 - \cos(h_k))}{3[h_k^2 + h_k \cdot \sin(h_k) - 4(1 - \cos(h_k))]} + \\ &\quad + \frac{h_{k+1}^3(4 - \cos(h_{k+1})) + 9h_{k+1}^2 \cdot \sin(h_{k+1}) - 24h_{k+1}(1 - \cos(h_{k+1}))}{3[h_{k+1}^2 + h_{k+1} \cdot \sin(h_{k+1}) - 4(1 - \cos(h_{k+1}))]}, \\ &\quad k = 1, 2, \dots, n-1, \\ A_{2n} &= -\frac{h_n^3(4 - \cos(h_n)) + 9h_n^2 \cdot \sin(h_n) - 24h_n(1 - \cos(h_n))}{3[h_n^2 + h_n \cdot \sin(h_n) - 4(1 - \cos(h_n))]}, \end{aligned}$$

$$\begin{aligned}
A_{30} &= \frac{h_1^2}{12} + (h_1 \cos(\frac{h_1}{2}) - 2 \sin(\frac{h_1}{2})) \frac{12 \sin(\frac{h_1}{2}) - 6 \cdot \cos(\frac{h_1}{2}) - h_1^2 \sin(\frac{h_1}{2})}{3[h_1^2 + h_1 \cdot \sin(h_1) - 4(1 - \cos(h_1))]}, \\
A_{3k} &= \frac{h_{k+1}^2 - h_k^2}{12} - (h_k \cos(\frac{h_k}{2}) - 2 \sin(\frac{h_k}{2})) \frac{12 \sin(\frac{h_k}{2}) - 6 \cdot \cos(\frac{h_k}{2}) - h_k^2 \sin(\frac{h_k}{2})}{3[h_k^2 + h_k \cdot \sin(h_k) - 4(1 - \cos(h_k))]} + \\
&+ (h_{k+1} \cos(\frac{h_{k+1}}{2}) - 2 \sin(\frac{h_{k+1}}{2})) \frac{12 \sin(\frac{h_{k+1}}{2}) - 6 \cdot \cos(\frac{h_{k+1}}{2}) - h_{k+1}^2 \sin(\frac{h_{k+1}}{2})}{3[h_{k+1}^2 + h_{k+1} \cdot \sin(h_{k+1}) - 4(1 - \cos(h_{k+1}))]}, \\
&k = 1, 2, \dots, n-1, \\
A_{3n} &= -\frac{h_n^2}{12} - (h_n \cos(\frac{h_n}{2}) - 2 \sin(\frac{h_n}{2})) \frac{12 \sin(\frac{h_n}{2}) - 6 \cdot \cos(\frac{h_n}{2}) - h_n^2 \sin(\frac{h_n}{2})}{3[h_n^2 + h_n \cdot \sin(h_n) - 4(1 - \cos(h_n))]} .
\end{aligned}$$

The formula is exact for trigonometric functions $\sin(x)$, $\cos(x)$ and any linear function.

The proof of Theorem 1 is obtained by putting (11), (12), (13), (14) to (6) and doing some calculations.

From Theorem 1 when the nodes are equally spaced we get a more simple optimal quadrature formula in the space $K_2^{(4,2)}$. That is the following holds.

Corollary 1. For equally spaced nodes $x_k = a + kh$, $k = 0, 1, \dots, n$, $h = \frac{b-a}{n}$, there exists a unique optimal quadrature formula

$$\begin{aligned}
\int_a^b f(x) dx &\cong \sum_{k=0}^n A_{0k} f(a + kh) + \sum_{k=0}^n A_{1k} f'(a + kh) \\
&+ \sum_{k=0}^n A_{2k} f''(a + kh) + \sum_{k=0}^n A_{3k} f'''(a + kh)
\end{aligned}$$

with coefficients

$$\begin{aligned}
A_{00} &= \frac{h}{2}, \\
A_{0k} &= h, \quad k = 1, 2, \dots, n-1, \\
A_{0n} &= \frac{h}{2}, \\
A_{10} &= \frac{h^4 + h^3 \cdot \sin(h) + 12h^2 + 12 \cdot h \sin(h)}{12[h^2 + h \cdot \sin(h) - 4(1 - \cos(h))]}, \\
A_{1k} &= 0, \quad k = 1, 2, \dots, n-1, \\
A_{1n} &= -\frac{h^4 + h^3 \cdot \sin(h) + 12h^2 + 12 \cdot h \sin(h)}{12[h^2 + h \cdot \sin(h) - 4(1 - \cos(h))]}, \\
A_{20} &= \frac{h^3(4 - \cos(h)) + 9h^2 \cdot \sin(h) - 24h(1 - \cos(h))}{3[h^2 + h \cdot \sin(h) - 4(1 - \cos(h))]}, \\
A_{2k} &= \frac{2h^3(4 - \cos(h)) + 18h^2 \cdot \sin(h) - 48h(1 - \cos(h))}{3[h^2 + h \cdot \sin(h) - 4(1 - \cos(h))]}, \quad k = 1, 2, \dots, n-1, \\
A_{2n} &= -\frac{h^3(4 - \cos(h)) + 9h^2 \cdot \sin(h) - 24h(1 - \cos(h))}{3[h^2 + h \cdot \sin(h) - 4(1 - \cos(h))]}, \\
A_{30} &= \frac{h^2}{12} + (h \cos(\frac{h}{2}) - 2 \sin(\frac{h}{2})) \frac{12 \sin(\frac{h}{2}) - 6 \cdot \cos(\frac{h}{2}) - h^2 \sin(\frac{h}{2})}{3[h^2 + h \cdot \sin(h) - 4(1 - \cos(h))]}, \\
A_{3k} &= 0, \quad k = 1, 2, \dots, n-1, \\
A_{3n} &= -\frac{h^2}{12} - (h \cos(\frac{h}{2}) - 2 \sin(\frac{h}{2})) \frac{12 \sin(\frac{h}{2}) - 6 \cdot \cos(\frac{h}{2}) - h^2 \sin(\frac{h}{2})}{3[h^2 + h \cdot \sin(h) - 4(1 - \cos(h))]} .
\end{aligned}$$

4 A precise upper bound for the error of the quadrature formula

By substituting the obtained expressions for $C_1^{(k)}$, $C_2^{(k)}$, $C_3^{(k)}$ and $C_4^{(k)}$, we obtain the general representation of the $\varphi_k(x)$ function, where $k = 1, 2, \dots, n$,

$$\begin{aligned} \varphi_k(x) = & h_k \cdot \cos\left(x - \frac{m_k}{2}\right) \\ & \cdot \frac{[12 \sin(\frac{h_k}{2}) - 6 \cdot h_k \cdot \cos(\frac{h_k}{2}) - h_k^2 \sin(\frac{h_k}{2})]}{3[h_k^2 + h_k \cdot \sin(h_k) - 4(1 - \cos(h_k))]} + \frac{m_k^2}{8} - \\ & - \frac{h_k^2}{24} - \frac{2 \sin(\frac{h_k}{2})(12 \sin(\frac{h_k}{2}) - 6 \cdot h_k \cdot \cos(\frac{h_k}{2}) - h_k^2 \sin(\frac{h_k}{2}))}{3[h_k^2 + h_k \cdot \sin(h_k) - 4(1 - \cos(h_k))]} - \\ & - \frac{1}{2}(m_k)x + \frac{x^2}{2}. \end{aligned} \quad (17)$$

Subsequently, we evaluate $\varphi_k^2(x)$ and compute its integral over the interval $[x_{k-1}, x_k]$. For convenience, we first introduce the following notations

$$M_k = \frac{12 \sin(\frac{h_k}{2}) - 6 \cdot h_k \cdot \cos(\frac{h_k}{2}) - h_k^2 \sin(\frac{h_k}{2})}{3[h_k^2 + h_k \cdot \sin(h_k) - 4(1 - \cos(h_k))]},$$

$$N_k = \frac{m_k^2}{8} - \frac{h_k^2}{24}.$$

Accordingly, the $\varphi_k(x)$ function takes the following form:

$$\begin{aligned} \varphi_k(x) = & h_k \cos\left(x - \frac{m_k}{2}\right) \cdot M_k + N_k - \\ & - 2 \sin\left(\frac{h_k}{2}\right) \cdot M_k - \frac{1}{2}(m_k)x + \frac{x^2}{2}. \end{aligned}$$

$$\begin{aligned} \int_{x_{k-1}}^{x_k} \varphi_k^2(x) dx = & h_k^2 M_k^2 \int_{x_{k-1}}^{x_k} \cos^2\left(x - \frac{m_k}{2}\right) dx + N_k^2 \int_{x_{k-1}}^{x_k} dx + \\ & + 4M_k^2 \sin^2\left(\frac{h_k}{2}\right) \int_{x_{k-1}}^{x_k} x dx + \frac{1}{4}m_k^2 \int_{x_{k-1}}^{x_k} x^2 dx + \\ & + \frac{1}{4} \int_{x_{k-1}}^{x_k} x^4 dx + 2h_k M_k N_k \int_{x_{k-1}}^{x_k} \cos\left(x - \frac{m_k}{2}\right) dx - \\ & - 4h_k M_k^2 \sin\left(\frac{h_k}{2}\right) \int_{x_{k-1}}^{x_k} \cos\left(x - \frac{m_k}{2}\right) dx - \\ & - h_k m_k M_k \int_{x_{k-1}}^{x_k} x \cos\left(x - \frac{m_k}{2}\right) dx + h_k M_k \int_{x_{k-1}}^{x_k} x^2 \cos\left(x - \frac{m_k}{2}\right) dx - \\ & - 4M_k N_k \sin\left(\frac{h_k}{2}\right) \int_{x_{k-1}}^{x_k} dx - N_k m_k \int_{x_{k-1}}^{x_k} x dx + N_k \int_{x_{k-1}}^{x_k} x^2 dx + \\ & + 2M_k m_k \sin\left(\frac{h_k}{2}\right) \int_{x_{k-1}}^{x_k} dx - 2M_k \sin\left(\frac{h_k}{2}\right) \int_{x_{k-1}}^{x_k} x^2 dx - \frac{1}{2}m_k \int_{x_{k-1}}^{x_k} x^3 dx. \end{aligned}$$

After evaluating these integrals separately and substituting the resulting expressions into the corresponding integrals above, we obtain the following equality.

$$\int_{x_{k-1}}^{x_k} \varphi_k^2(x) dx = \frac{h_k^5}{720} - \frac{h_k}{18} \cdot \frac{[12 \sin(\frac{h_k}{2}) - 6 \cdot h_k \cdot \cos(\frac{h_k}{2}) - h_k^2 \sin(\frac{h_k}{2})]}{h_k^2 + h_k \sin(h_k) - 4(1 - \cos(h_k))}.$$

Subsequently, we estimate the norm of the $\varphi(x)$ function:

$$\|\varphi\|_{L_2(a,b)}^2 = \sum_{k=1}^n \left[\frac{h_k^5}{720} - \frac{h_k}{18} \cdot \frac{[12 \sin(\frac{h_k}{2}) - 6 \cdot h_k \cdot \cos(\frac{h_k}{2}) - h_k^2 \sin(\frac{h_k}{2})]}{h_k^2 + h_k \sin(h_k) - 4(1 - \cos(h_k))} \right].$$

We have

$$|R_n(f)| \leq \|f^{IV} + f''\|_{L_2(a,b)} \cdot \left(\int_a^b \varphi^2(x) dx \right)^{1/2} = \|f\|_{K_2^{(4,2)}} \cdot \|\varphi\|_{L_2(a,b)} =$$

$$= \|f\|_{K_2^{(4,2)}} \cdot \left(\sum_{k=1}^n \left[\frac{h_k^5}{720} - \frac{h_k}{18} \cdot \frac{[12 \sin(\frac{h_k}{2}) - 6 \cdot h_k \cdot \cos(\frac{h_k}{2}) - h_k^2 \sin(\frac{h_k}{2})]}{h_k^2 + h_k \sin(h_k) - 4(1 - \cos(h_k))} \right] \right)^{1/2}.$$

For uniformly spaced nodes, the following formula can be derived.

$$\|\varphi\|_{L_2(a,b)}^2 = \sum_{k=1}^n \left[\frac{h^5}{720} - \frac{h}{18} \cdot \frac{[12 \sin(\frac{h}{2}) - 6 \cdot h \cdot \cos(\frac{h}{2}) - h^2 \sin(\frac{h}{2})]}{h^2 + h \sin(h) - 4(1 - \cos(h))} \right]. \quad (18)$$

To compare the optimal quadrature formula with the Euler–Maclaurin quadrature formula, we consider several test functions. The obtained numerical results and corresponding errors are presented in Table 1.

Table 1 Comparison of the optimal quadrature formula and the Euler–Maclaurin quadrature formula.

Function	Definite integral $I(f)$	OQF $Q_n(f)$	Error E_n	Euler–Maclaurin $Q_N^{EM}(f)$	Error E_N^{EM}
$F_1 = \cos x + \sin x + 1 + x$	2.8011686789	2.8011686789	1.0×10^{-49}	2.301168679	1.0×10^{-10}
$F_2 = e^{-x^2} \cos(30x)$	$-1.2215039270 \times 10^{-1}$	$-1.2211406431 \times 10^{-1}$	3.63×10^{-6}	$-0.1187154094 \times 10^{-1}$	3.43×10^{-4}
$F_3 = \cos(x^3) + \sin(x^3)$	1.1655496861	1.1655496861	5.03×10^{-11}	1.165549763	7.7×10^{-8}
$F_4 = \sin(40x) \ln(1+x)$	$1.180810107 \times 10^{-2}$	$1.176004160 \times 10^{-2}$	4.80×10^{-5}	$9.446848469 \times 10^{-3}$	2.36×10^{-3}
$F_5 = \frac{1}{25x^2 + 1}$	$2.7468015338 \times 10^{-1}$	$2.7468028487 \times 10^{-1}$	1.31×10^{-7}	$2.746823439 \times 10^{-1}$	2.19×10^{-6}

5 Conclusion

In this work, an optimal quadrature formula is constructed in the Hilbert space $K_2^{(4,2)}$, consisting of absolutely continuous functions whose first derivatives belong to the space of square-integrable functions on the interval $[a, b]$. This functional framework provides a rigorous mathematical foundation for the development of highly accurate numerical integration methods.

A central aspect of the proposed approach is the application of the φ -function method, which plays a significant role in the optimization of the quadrature formula. The use of this technique improves both the accuracy and the analytical properties of the resulting formula, thereby enhancing its efficiency and practical applicability.

The obtained quadrature formula is suitable for both theoretical investigations and practical applications in numerical analysis and computational mathematics. Moreover, the proposed methodology can be extended to the construction of more general classes of optimal quadrature formulas in future research.

In conclusion, the results presented in this study contribute to the development of efficient numerical integration techniques and demonstrate considerable potential for applications in a wide range of scientific and engineering computations.

Declarations

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References

- [1] Ahlberg J. H., Nilson E. N., Walsh J. L. 1967. *The theory of splines and their applications*. New York: Academic Press.
- [2] Boltaev N. D., Hayotov A. R., Shadimetov Kh. M. 2017. Construction of optimal quadrature formulas for Fourier coefficients in Sobolev space $L_2^{(m)}(0, 1)$. *Numerical Algorithms*. 74(5): 307–336. doi: <https://doi.org/10.12691/ajna-4-1-1>.
- [3] Coman Gh. 1972. Formule de cuadratura de tip Sard. *Studia Universitatis Babeş-Bolyai Mathematica–Mechanica*. 17(2): 73–77.
- [4] Hayotov A. R., Rasulov R. G. 2020. The order of convergence of an optimal quadrature formula with derivative in the space $W_2^{(1,0)}$. *Filomat*. 34(11): 3835–3844.
- [5] Lanzara F. 2000. On optimal quadrature formulae. *Journal of Inequalities and Applications*. 5: 201–225.
- [6] Nikolsky S. M. 1950. On the issue of estimates of approximations by quadrature formulas. *Advances in Mathematical Sciences*. 5(3): 165–177. (In Russian)
- [7] Nikolsky S. M. 1988. *Quadrature formulas*. 4th ed. Moscow: Nauka. 258 p. (In Russian)
- [8] Shadimetov Kh. M., Hayotov A. R. 2022. *Optimal approximation of the error functionals of quadrature and interpolation formulas in spaces of differentiable functions*. Tashkent: Muhr Press. 246 p.
- [9] Hayotov A. R., Babaev S. S. 2023. Optimal quadrature formula for numerical integration of fractional integrals in a Hilbert space. *Journal of Mathematical Sciences*. 277(3): 403–419.
- [10] Hayotov A. R., Babaev S. S., Abduakhadov A. A., Davronov J. R. 2024. An optimal quadrature formula exact to the exponential function by the φ -function method. *Studia Universitatis Babeş-Bolyai Mathematica*. 69(3): 651–663.
- [11] Ghizzetti A., Ossicini A. 1970. *Quadrature formulae*. Berlin: Akademie Verlag. 191 p.
- [12] Catinaş T., Coman Gh. 2006. Optimal quadrature formulas based on the φ -function method. *Studia Universitatis Babeş-Bolyai Mathematica*. 51(1): 49–64.
- [13] Haitov T. O. 2023. An optimal formula for the approximate calculation of the fractional Riemann–Liouville integrals. *Problems of Computational and Applied Mathematics*. 3/1(50): 137–148.
- [14] Hayotov A. R., Khaitov T. O., Buvashero D. S. 2024. An optimal formula for numerical integration of fractional Riemann–Liouville integral. *Problems of Computational and Applied Mathematics*. 4/1(59): 142–151.
- [15] Hayotov A. R., Berdimuradova U. A. 2025. An optimal quadrature formula with derivatives for arbitrarily fixed nodes in the Sobolev space. *Problems of Computational and Applied Mathematics*. 2(64): 64–73. doi: https://doi.org/10.71310/pcam.2_64.2025.06.
- [16] Hayotov A. R. 2018. Construction of interpolation splines minimizing the semi-norm in the space $K_2(P_m)$. *Journal of Siberian Federal University. Mathematics & Physics*. 11(3): 383–396.
- [17] Hayotov A. R., Babaev S. S. 2021. Optimal quadrature formulas for computing of Fourier integrals in $W_2^{(m,m-1)}$ space. *AIP Conference Proceedings*. 2365: 1–22. doi: <https://doi.org/10.1063/5.0057127>.
- [18] Hayotov A. R., Babaev S. S. 2025. An optimal quadrature formula with derivative for weakly singular integrals. *Mathematical Methods in the Applied Sciences*. 48(17): 15591–15601.
- [19] Hayotov A. R., Jeon S., Lee C.-O. 2020. On an optimal quadrature formula for approximation of Fourier integrals in the space $L_2^{(1)}$. *Journal of Computational and Applied Mathematics*. 1–16. doi: <https://doi.org/10.1016/j.cam.2020.112713>.

- [20] Hayotov A. R., Jeon S., Shadimetov Kh. M. 2021. Application of optimal quadrature formulas for reconstruction of CT images. *Journal of Computational and Applied Mathematics*. 388: Art. 113313.

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ОПТИМАЛЬНЫЕ МЕТОДЫ ЧИСЛЕННОГО ИНТЕГРИРОВАНИЯ, ОСНОВАННЫЕ НА АЛГЕБРАИЧЕСКИХ И ТРИГОНОМЕТРИЧЕСКИХ ФУНКЦИЯХ*

^{1,2}Хаитов Т.О.

tojiboy.xaitov.77@mail.ru

¹Институт математики имени В.И. Романовского Академии наук Республики Узбекистан, 100174, Узбекистан, г. Ташкент, ул. Университетская, 9;

²Ташкентский государственный технический университет имени Ислама Каримова, Узбекистан, г. Ташкент, ул. Университетская, 2

В настоящей работе проводится всестороннее исследование вычисления коэффициентов квадратурных формул, содержащих значения производных, с использованием так называемого метода φ -функций. Данный подход представляет собой систематическую и эффективную методологию построения оптимальных квадратурных формул в рамках численного интегрирования. Следует отметить, что метод φ -функций не только существенно упрощает вывод соответствующих коэффициентов, но и обеспечивает высокую точность получаемых квадратурных формул.

Кроме того, проводится детальный анализ функциональной погрешности предлагаемой квадратурной формулы. Полученные оценки погрешности обоснованы строгими аналитическими выражениями, что подтверждает точность и надежность полученных результатов. На основе исследования квадратурных формул с произвольно заданными узлами систематически устанавливаются критерии оптимальности, а также подробно излагаются методы вычисления соответствующих коэффициентов и структурных компонентов формулы.

В качестве основного результата настоящего исследования получены аналитические представления в замкнутой форме для коэффициентов оптимальной квадратурной формулы. Особый интерес представляет случай равномерно распределённых узлов, при котором полученная квадратурная формула естественным образом совпадает с классической формулой типа Эйлера–Маклорена. Этот результат демонстрирует как общность применения, так и практическую значимость метода φ -функций.

Ключевые слова: метод φ -функций, оптимальная квадратурная формула, определённый интеграл, узлы квадратурной формулы, оптимальность

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HISOBLASH VA AMALIY MATEMATIKA MUAMMOLARI

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И ПРИКЛАДНОЙ МАТЕМАТИКИ
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Содержание

Хаётов А.Р., Мухторова М.М.

Функции гистосплайна при восстановлении функций по интегральным значениям 5

Олимов Н.Н.

Численное решение линейных интегральных уравнений Фредгольма второго рода на основе оптимальных интерполяционных формул с производными . . 16

Бабаев С.С., Элмуродова С.С.

Численное приближение дробных интегралов: оптимальный квадратурный подход, основанный на фи-функциях 29

Кулдошев Х.М.

Выражения для коэффициентов оптимальных квадратурных формул с параметром 41

Хаётов А.Р., Хусаинов А.

Альтернативный вывод и численный анализ оптимальной интерполяционной формулы в гильбертовом пространстве 61

Хаитов Т.О.

Оптимальные методы численного интегрирования, основанные на алгебраических и тригонометрических функциях 72

Шадиметов Х.М., Нафасов А.Ю.

Квадрат нормы функционала погрешности интерполяционной формулы с третьей производной в гильбертовом пространстве 85

Болтаев А.К., Пардаева О.Ф.

Интерполяционные сплайны, минимизирующие полунорму в гильбертовом пространстве Соболева 98

Бердимуродова У.А.

Явные коэффициенты и оценка погрешности оптимальной квадратурной формулы 116

Полвонов С.З.

Оптимальная квадратурная формула с кратными узлами в гильбертовом пространстве 127

Барраева С.Ш.

Оптимальные квадратурные формулы с тригонометрическими весами в пространстве Соболева 141

Нуралиев Ф.А., Абдуллаева Г.Ш.

Сравнительный анализ алгебраико-гиперболических натуральных сплайнов шестого порядка и интегральных квазиинтерполяционных сплайнов 153

Ахмедов Д.М., Шомаликова М.Ш.

Оптимальная квадратурная формула типа Эрмита со слабосингулярным весом методом РН-функций 167

Хаётов А.Р., Пирнапасов С.О.

Приближённое вычисление дробной производной Капуто 180

Буваширов Д.С., Алиев Ш.Е.

Оптимальные квадратурные формулы для интегралов типа Коши с тригонометрическими весами 190

Насруллаев П.А., Джурабоева О.С., Саманова Г.Б.

Консервативная модель переноса примесей в атмосфере над сложным рельефом 201

Contents

<i>Hayotov A.R., Mukhtorova M.M.</i>	
Histospline functions in reconstructing functions using integral values	5
<i>Olimov N.N.</i>	
Numerical solution of linear Fredholm integral equations of the second kind based on optimal interpolation formulas with derivatives	16
<i>Babaev S.S., Elmurodova S.S.</i>	
Numerical approximation of fractional integrals: an optimal quadrature approach based on phi-functions	29
<i>Kuldoshev Kh.M.</i>	
Expressions for the coefficients of optimal quadrature formulas with a parameter	41
<i>Hayotov A.R., Husainov A.</i>	
Alternative derivation and numerical analysis of an optimal interpolation formula in the Hilbert space	61
<i>Khaitov T.O.</i>	
Optimal numerical integration rules based on algebraic and trigonometric functions	72
<i>Shadimetov Kh.M., Nafasov A.Y.</i>	
The squared norm of the error functional for a third-derivative interpolation formula in a Hilbert space	85
<i>Boltaev A.K., Pardaeva O.F.</i>	
Interpolation splines minimizing a semi-norm in a Sobolev Hilbert space	98
<i>Berdimuradova U.A.</i>	
Explicit coefficients and error estimate for an optimal quadrature formula	116
<i>Polvonov S.Z.</i>	
Optimal quadrature formula with multiple nodes in a Hilbert space	127
<i>Barraeva S.Sh.</i>	
Optimal quadrature formulas with trigonometric weights in a Sobolev space	141
<i>Nuraliev F.A., Abdullaeva G.Sh.</i>	
A comparative analysis of sixth-order algebraic-hyperbolic natural splines and integral quasi-interpolating splines	153
<i>Akhmedov D.M., Shomalikova M.Sh.</i>	
Optimal Hermite-type quadrature with weakly singular weight via PHI-functions	167
<i>Hayotov A.R., Pirnapasov S.O.</i>	
Approximate calculation of the Caputo fractional-order derivative	180
<i>Buvasharov D.S., Aliev Sh.E.</i>	
Optimal quadrature formulas for highly oscillatory Cauchy-type integrals	190
<i>Nasrullayev P.A., Juraboyeva O.S., Samanova G.B.</i>	
A mass-conserving model of pollutant transport in the atmosphere over complex terrain	201