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ALTERNATIVE DERIVATION AND NUMERICAL ANALYSIS OF AN OPTIMAL INTERPOLATION FORMULA IN THE HILBERT SPACE

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Background: The problem of optimal interpolation from function values prescribed at the nodes of a uniform grid is considered in the space $W_2^{(1,0)}(0,1)$. The coefficients of this formula are known as the $m = 1$ special case of a general result obtained for the space $W_2^{(m,m-1)}(0,1)$. In the previous work, they were found by the Sobolev method using an extremal function and a discrete analogue of a differential operator.

Method: Here the same result is derived by a different approach. The integrating factor $g(x) = e^x f(x)$ and the monotone coordinate transformation $s(x) = (e^{2x} - 1)/2$ reduce the original seminorm to a simple first-derivative energy.

Results: Direct minimization of the Peano-type kernel of the transformed problem proves the locality of the optimal formula. Thus, only the coefficients associated with the two endpoints of the interval containing z are nonzero. A separate theorem proves that the resulting local formula involving hyperbolic sines coincides exactly with the sign-function formula for the $m = 1$ case of the previous general result. Exact and uniform estimates of the error-functional norm are obtained.

Convergence analysis: A second-order asymptotic convergence result is established for smooth functions. Numerical experiments for z^2 , e^{2z} , and $\sin z$ confirm that the maximum error decreases as the number of nodes increases and corroborate the theoretical $O(h^2)$ order.

Keywords: optimal interpolation, Sobolev space, alternative method, error functional, hyperbolic sine, numerical experiment, order of convergence

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1 Introduction

Recovering the value of a function at an intermediate point from a finite number of nodal values is one of the central problems of interpolation theory. In the functional-analytic approach, the difference between the exact value and an interpolation formula is viewed as a linear error functional. For fixed nodes, minimizing the norm of this functional with respect to the coefficients leads to the construction of an optimal formula [1]. Methods for constructing optimal interpolation formulas in Sobolev spaces were developed in [2, 3].

The functional-analytic foundations of this direction go back to the work on interpolation of functions of several variables in [4]. The Sobolev method was subsequently used to construct optimal grid interpolation formulas for periodic functions of several variables

in [5]. Closely related interpolation splines minimizing seminorms were obtained in [6, 7], and D^m -splines in $L_2^{(m)}(0, 1)$ were constructed by the Sobolev method in [8].

In [3], a general form of the coefficients of an optimal interpolation formula in the space $W_2^{(m, m-1)}(0, 1)$ was obtained by the Sobolev method. That approach employs an extremal function, a Green function, and a discrete analogue of the differential operator

$$\frac{d^{2m}}{dx^{2m}} - \frac{d^{2m-2}}{dx^{2m-2}}.$$

The $m = 1$ case of this general result belongs precisely to the space $W_2^{(1,0)}(0, 1)$. Therefore, the coefficient formula obtained in this paper is not claimed to be new.

Subsequent studies extended optimal formulas to data involving derivatives. In particular, the variational formulation in the space $W_2^{(2,1)}(0, 1)$ and the norm of its error functional were studied in [9]; the system defining the optimal coefficients was investigated by means of the Riesz theorem and Lagrange multipliers in [10]. Further optimal interpolation formulas in Hilbert and Sobolev spaces, including formulas involving derivative data, were developed in [11–13]. Error-functional norms in higher-order spaces were studied in [11, 14].

Nuraliev [15] studied optimal interpolation formulas using function values at the nodes and first-derivative values at the endpoints of the interval.

Recent studies have further developed optimal interpolation formulas in Hilbert spaces. Shadimetov and Boltaev [16, 17] constructed exponential-trigonometric optimal interpolation formulas by minimizing the norm of the error functional. Applications of optimal interpolation to integral equations were considered by Babaev et al. [18], who used the coefficients of an optimal interpolation formula in $W_2^{(1,0)}$ to construct basis functions for the Galerkin approximation of Fredholm integral equations of the second kind.

Hayotov and Doniyorov [19] constructed basis functions from optimal interpolation coefficients and applied them to the finite element approximation of boundary value problems for second-order ordinary differential equations.

The aim of this work is to rederive the known $m = 1$ result by an alternative method based on an integrating factor and a monotone coordinate transformation. This method reduces the original optimality problem to a first-derivative energy problem and directly explains why the optimal formula is local. In addition, the resulting formula is compared algebraically with the $m = 1$ case of the general formula in [3], a simple expression for the error norm is obtained, and the convergence order is examined numerically.

2 Problem statement and previous result

Let $N \in \mathbb{N}$, $h = 1/N$, and

$$x_\beta = h\beta, \quad \beta = 0, 1, \dots, N \quad (1)$$

be the nodes of a uniform grid. At an arbitrary point $z \in [0, 1]$, consider the formula

$$f(z) \approx P_{0,f}(z) = \sum_{\beta=0}^N A_{\beta,0}(z) f(x_\beta). \quad (2)$$

Denote by $AC[0, 1]$ the class of absolutely continuous functions and consider the space

$$W_2^{(1,0)}(0, 1) = \{f \in AC[0, 1] : f' + f \in L_2(0, 1)\} \quad (3)$$

with the semi-inner product and seminorm

$$\langle f, g \rangle_{W_2^{(1,0)}} = \int_0^1 (f'(x) + f(x))(g'(x) + g(x)) dx, \quad (4)$$

$$\|f\|_{W_2^{(1,0)}}^2 = \int_0^1 (f'(x) + f(x))^2 dx, \quad (5)$$

respectively. General information on Sobolev spaces is given in [20]. The null space of the seminorm is

$$\mathcal{N} = \ker(D + I) = \text{span}\{e^{-x}\} \quad (6)$$

and hence formula (2) must be exact for e^{-x} , i.e., the following equality must hold:

$$\sum_{\beta=0}^N A_{\beta,0}(z) e^{-x_\beta} = e^{-z}. \quad (7)$$

We define the error functional by

$$\ell_z(f) = f(z) - \sum_{\beta=0}^N A_{\beta,0}(z) f(x_\beta). \quad (8)$$

The optimality problem consists in finding, among all coefficients satisfying (7), those $A_{\beta,0}(z)$ that minimize the quantity

$$\|\ell_z\|_{W_2^{(1,0)*}} = \sup_{f \in W_2^{(1,0)}(0,1), f \neq 0} \frac{|\ell_z(f)|}{\|f\|_{W_2^{(1,0)}}}. \quad (9)$$

This problem was solved, for example, in [3]. For the coefficients of the optimal interpolation formula, we have the following result for $m = 1$ (see [3, Theorem 3]):

$$\begin{aligned} \tilde{C}_\beta(z) = \frac{1}{2(1 - e^{2h})} & \left[\text{sgn}(z - x_\beta - h) (e^{x_\beta + 2h - z} - e^{z - x_\beta}) + \right. \\ & + \text{sgn}(z - x_\beta + h) (e^{x_\beta - z} - e^{z - x_\beta + 2h}) + \\ & \left. + (1 + e^{2h}) \text{sgn}(z - x_\beta) (e^{z - x_\beta} - e^{x_\beta - z}) \right], \end{aligned} \quad (10)$$

where $\beta = 0, 1, \dots, N$. In the present work, we rederive result (10) by a different method.

3 Alternative method and optimal coefficients

We introduce the following notation:

$$g(x) = e^x f(x), \quad s = s(x) = \frac{e^{2x} - 1}{2}, \quad G(s(x)) = g(x).$$

Then we have

$$\|f\|_{W_2^{(1,0)}}^2 = \int_0^S (G'(s))^2 ds, \quad \text{where} \quad S = \frac{e^2 - 1}{2}. \quad (11)$$

Theorem 1. *If*

$$s_\beta = s(x_\beta), \quad s_z = s(z), \quad B_\beta(z) = A_{\beta,0}(z)e^{z-x_\beta}, \quad (12)$$

then (8) and (7) become, respectively,

$$\ell_z(f) = e^{-z} \left(G(s_z) - \sum_{\beta=0}^N B_\beta(z) G(s_\beta) \right), \quad (13)$$

$$\sum_{\beta=0}^N B_\beta(z) = 1. \quad (14)$$

Proof. The identities $g'(x) = e^x(f'(x) + f(x))$ and $ds = e^{2x} dx$ imply

$$f'(x) + f(x) = e^{-x} g'(x) = e^x G'(s(x)).$$

Substitution into (5) yields (11). Substituting $f(x) = e^{-x} G(s(x))$ into (8) gives (13). Finally, multiplying (7) by e^z gives (14).

Theorem 2. *Let $z \in [x_k, x_{k+1}]$. The optimal coefficients of (2) in the space $W_2^{(1,0)}(0, 1)$ are*

$$\begin{aligned} A_{k,0}(z) &= \frac{\sinh(x_{k+1} - z)}{\sinh h}, \\ A_{k+1,0}(z) &= \frac{\sinh(z - x_k)}{\sinh h}, \\ A_{\beta,0}(z) &= 0, \quad \beta \neq k, k+1. \end{aligned} \quad (15)$$

Proof. From (14) and the identity $G(s) = G(0) + \int_0^s G'(t) dt$, we obtain

$$G(s_z) - \sum_{\beta=0}^N B_\beta G(s_\beta) = \int_0^s K_B(t) G'(t) dt, \quad (16)$$

where

$$K_B(t) = \mathbf{1}_{[0, s_z]}(t) - \sum_{\beta=0}^N B_\beta \mathbf{1}_{[0, s_\beta]}(t).$$

By the Cauchy–Schwarz inequality, with equality for the choice $G' = K_B$, the optimality problem is equivalent to minimizing $\|K_B\|_{L_2(0, S)}^2$.

Set $C_j = \sum_{\beta=0}^j B_\beta$. Since $\sum_{\beta=0}^N B_\beta = 1$, for $t \in (s_j, s_{j+1})$ we have

$$\sum_{\beta=0}^N B_\beta \mathbf{1}_{[0, s_\beta]}(t) = \sum_{\beta=j+1}^N B_\beta = 1 - C_j.$$

If $j < k$, then $t < s_z$ and $\mathbf{1}_{[0, s_z]}(t) = 1$; hence $K_B(t) = C_j$. On the subinterval containing s_z , the value of the first indicator changes at $t = s_z$, so that $K_B(t) = C_k$ for $t \in (s_k, s_z)$ and $K_B(t) = C_k - 1$ for $t \in (s_z, s_{k+1})$. Finally, if $j > k$, then $t > s_z$ and $\mathbf{1}_{[0, s_z]}(t) = 0$; hence $K_B(t) = C_j - 1$. Therefore, apart from the endpoints, which do not affect the integral,

$$K_B(t) = \begin{cases} C_j, & t \in (s_j, s_{j+1}), \quad j = 0, \dots, k-1, \\ C_k, & t \in (s_k, s_z), \\ C_k - 1, & t \in (s_z, s_{k+1}), \\ C_j - 1, & t \in (s_j, s_{j+1}), \quad j = k+1, \dots, N-1. \end{cases}$$

Consequently, splitting the integral over these subintervals gives

$$\begin{aligned}\|K_B\|_{L_2(0,S)}^2 &= \int_0^S K_B^2(t) dt = \\ &= \sum_{j=0}^{k-1} \int_{s_j}^{s_{j+1}} C_j^2 dt + \int_{s_k}^{s_z} C_k^2 dt + \\ &\quad + \int_{s_z}^{s_{k+1}} (C_k - 1)^2 dt + \sum_{j=k+1}^{N-1} \int_{s_j}^{s_{j+1}} (C_j - 1)^2 dt.\end{aligned}$$

Since every integrand in the last expression is constant on its corresponding subinterval, evaluating these integrals yields

$$\begin{aligned}\|K_B\|_{L_2(0,S)}^2 &= \sum_{j=0}^{k-1} C_j^2 (s_{j+1} - s_j) + C_k^2 (s_z - s_k) + \\ &\quad + (C_k - 1)^2 (s_{k+1} - s_z) + \sum_{j=k+1}^{N-1} (C_j - 1)^2 (s_{j+1} - s_j).\end{aligned}\quad (17)$$

Since every $s_{j+1} - s_j > 0$, at the minimum we have

$$C_j = 0 \quad (j < k), \quad C_j = 1 \quad (j > k),$$

and

$$C_k = \frac{s_{k+1} - s_z}{s_{k+1} - s_k}.$$

Thus, only

$$B_k = \frac{s_{k+1} - s_z}{s_{k+1} - s_k}, \quad B_{k+1} = \frac{s_z - s_k}{s_{k+1} - s_k} \quad (18)$$

are nonzero. Since (12) gives $A_{\beta,0} = e^{x\beta-z} B_\beta$, substituting $s(x) = (e^{2x} - 1)/2$ into (18) and simplifying yields (15).

4 Equivalence with the previous result

The following theorem shows that the formula obtained by the alternative method is exactly the same as the result of [3, Theorem 3].

Theorem 3. For all $z \in [0, 1]$ and $\beta = 0, 1, \dots, N$,

$$\tilde{C}_\beta(z) = A_{\beta,0}(z), \quad (19)$$

where the left-hand side is defined by (10) and the right-hand side by (15).

Proof. Let $z \in (x_k, x_{k+1})$, $\theta = z - x_k$, and $0 < \theta < h$. For $\beta < k$, all three sign functions in (10) are positive; for $\beta > k + 1$, they are all negative. Grouping the exponential terms gives $\tilde{C}_\beta(z) = 0$ in both cases.

For $\beta = k$, the signs are $(-, +, +)$, and

$$\begin{aligned}\tilde{C}_k(z) &= \frac{e^{2h-\theta} - e^\theta}{e^{2h} - 1} = \\ &= \frac{\sinh(h - \theta)}{\sinh h}.\end{aligned}$$

For $\beta = k + 1$, the signs are $(-, +, -)$, and

$$\begin{aligned}\tilde{C}_{k+1}(z) &= \frac{e^h(e^\theta - e^{-\theta})}{e^{2h} - 1} = \\ &= \frac{\sinh \theta}{\sinh h}.\end{aligned}$$

These values agree with (15). The equality at the nodes follows by continuity or by directly setting $\text{sgn}(0) = 0$.

Remark 1. In [3], (10) was derived by using a discrete operator and convolution. Theorem 3 shows that the transformation in Theorem 1 and the direct minimization of the kernel norm lead to the same optimal coefficients. Thus, the result is the same, whereas the derivations are different.

5 Error norm and order of convergence

Theorem 4. Let $z = x_k + \theta$, $0 \leq \theta \leq h$. The square of the norm of the error functional of the optimal formula is

$$\|\ell_z\|_{W_2^{(1,0)*}}^2 = \frac{\sinh \theta \sinh(h - \theta)}{\sinh h}. \quad (20)$$

Proof. Substitution of the optimal values into (17) gives

$$\min \|K_B\|_{L_2(0,S)}^2 = \frac{(s_z - s_k)(s_{k+1} - s_z)}{s_{k+1} - s_k}.$$

Taking into account the factor e^{-z} in (13), we obtain

$$\|\ell_z\|_{W_2^{(1,0)*}}^2 = e^{-2z} \frac{(s_z - s_k)(s_{k+1} - s_z)}{s_{k+1} - s_k}.$$

Substitution of $s(x) = (e^{2x} - 1)/2$, $z = x_k + \theta$, and $x_{k+1} = x_k + h$, followed by simplification, gives (20).

Corollary 1. For every $f \in W_2^{(1,0)}(0, 1)$,

$$|f(z) - P_{0,f}(z)| \leq \left(\frac{\sinh \theta \sinh(h - \theta)}{\sinh h} \right)^{1/2} \|f\|_{W_2^{(1,0)}}. \quad (21)$$

Moreover,

$$\max_{z \in [0,1]} \|\ell_z\|_{W_2^{(1,0)*}} = \left(\frac{1}{2} \tanh \frac{h}{2} \right)^{1/2} = \frac{\sqrt{h}}{2} + O(h^{5/2}). \quad (22)$$

Proof. Estimate (21) follows from the definition of the dual-space norm. The right-hand side of (20) attains its maximum at $\theta = h/2$. Substitution of this value and the Maclaurin expansion of $\tanh(h/2)$ give (22).

Theorem 5. Let $f \in C^3[0, 1]$ and $z = x_k + \theta$, where $0 \leq \theta \leq h$. Then, as $h \rightarrow 0$, uniformly with respect to $\theta/h \in [0, 1]$,

$$f(z) - P_{0,f}(z) = \frac{\theta(h - \theta)}{2} (f(z) - f''(z)) + O(h^3). \quad (23)$$

In particular,

$$\max_{z \in [0,1]} |f(z) - P_{0,f}(z)| = O(h^2). \quad (24)$$

Proof. Put $\rho = \theta/h$. The Maclaurin expansion of the hyperbolic sine gives

$$A_{k,0}(z) = (1 - \rho) - \frac{\rho(1 - \rho)(2 - \rho)}{6}h^2 + O(h^4),$$

$$A_{k+1,0}(z) = \rho - \frac{\rho(1 - \rho)(1 + \rho)}{6}h^2 + O(h^4).$$

Expanding $f(x_k)$ and $f(x_{k+1})$ about z by Taylor's formula and substituting these expressions into (2) gives (23). Since $0 \leq \theta(h - \theta) \leq h^2/4$, relation (24) follows.

Remark 2. Relation (22) gives the $O(h^{1/2})$ estimate for the worst case on the unit ball of the whole space $W_2^{(1,0)}(0, 1)$. Relation (24) gives the $O(h^2)$ order for functions with one additional degree of smoothness. These two estimates concern different classes of functions and therefore do not contradict one another.

6 Numerical results

For the numerical experiments, the functions used in [3] are adopted:

$$f_1(z) = z^2, \quad f_2(z) = e^{2z}, \quad f_3(z) = \sin z.$$

For every N , the quantity

$$E_N(f) = \max_{0 \leq z \leq 1} |f(z) - P_{0,f}(z)| \quad (25)$$

was computed at $10^6 + 1$ uniformly spaced test points in $[0, 1]$. Here, N is the number of subintervals and $N + 1$ is the number of nodes.

Table 1 Decrease of the maximum error as the number of nodes increases.

N	Number of nodes	$E_N(z^2)$	$E_N(e^{2z})$	$E_N(\sin z)$
5	6	$9.9004 \cdot 10^{-3}$	$9.0771 \cdot 10^{-2}$	$7.7956 \cdot 10^{-3}$
10	11	$2.4938 \cdot 10^{-3}$	$2.5074 \cdot 10^{-2}$	$2.0311 \cdot 10^{-3}$
20	21	$6.2461 \cdot 10^{-4}$	$6.5895 \cdot 10^{-3}$	$5.1716 \cdot 10^{-4}$
40	41	$1.5623 \cdot 10^{-4}$	$1.6891 \cdot 10^{-3}$	$1.3040 \cdot 10^{-4}$

The observed order of convergence is computed as

$$p_N(f) = \log_2 \frac{E_N(f)}{E_{2N}(f)}. \quad (26)$$

Table 2 Observed order of convergence.

$N \rightarrow 2N$	$p_N(z^2)$	$p_N(e^{2z})$	$p_N(\sin z)$
5 $\rightarrow$ 10	1.9892	1.8560	1.9404
10 $\rightarrow$ 20	1.9973	1.9279	1.9736
20 $\rightarrow$ 40	1.9993	1.9640	1.9876

Table 1 shows that the maximum error decreases monotonically as the number of nodes increases for all three functions. The values of p_N in Table 2 approach 2. These results confirm the $O(h^2)$ order in Theorem 5.

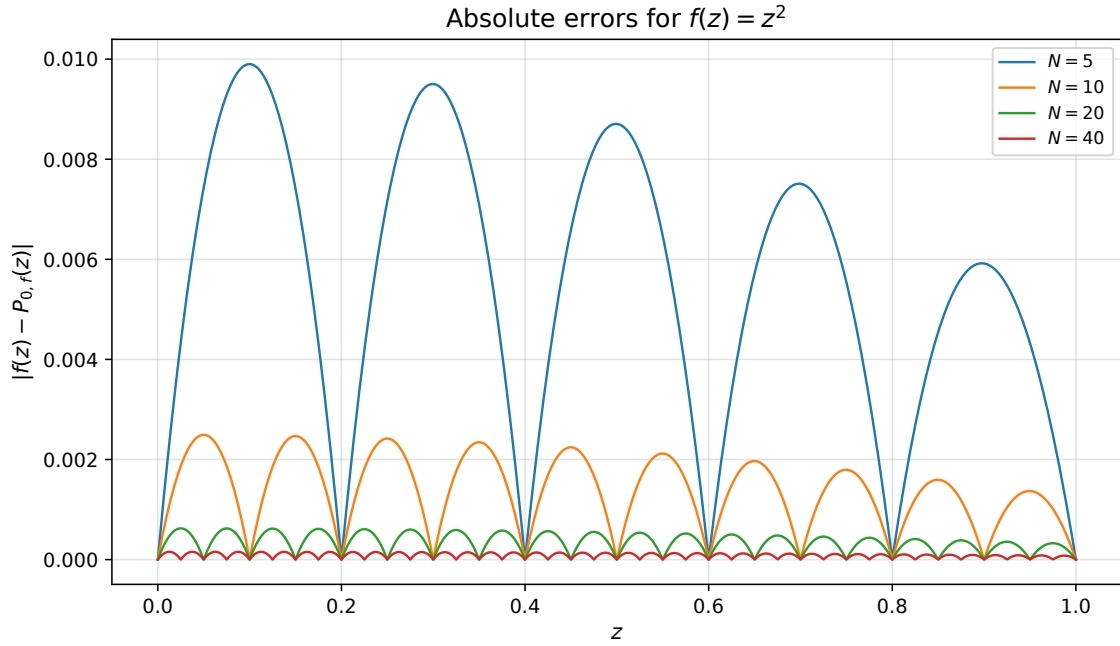


Figure 1 Absolute errors for $f(z) = z^2$ and $N = 5, 10, 20, 40$.

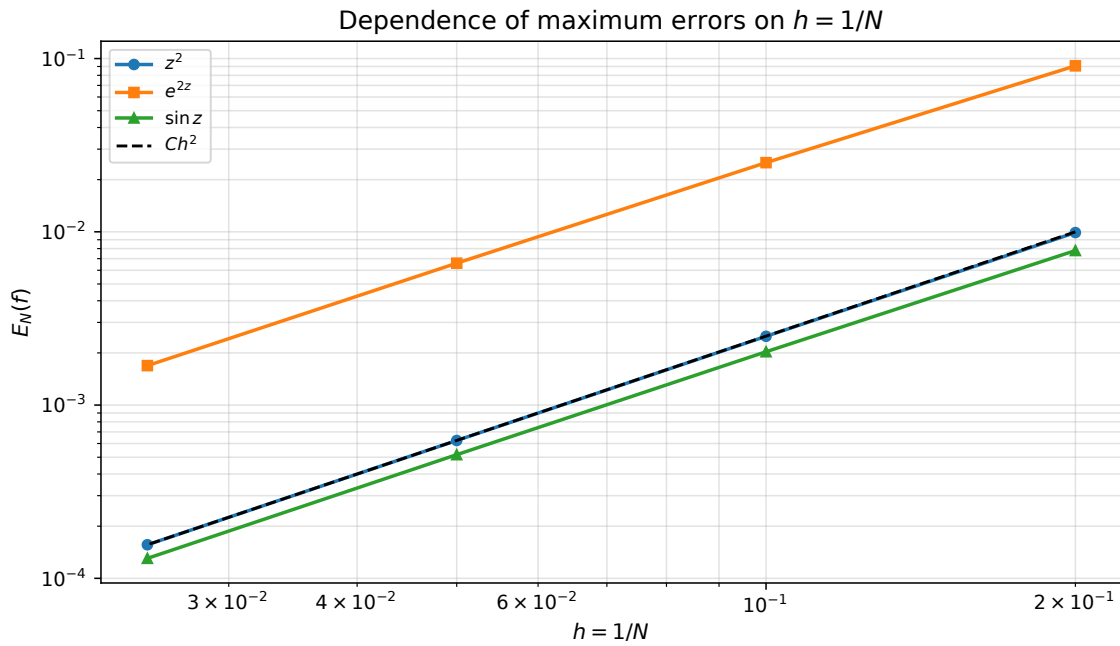


Figure 2 Dependence of the maximum errors on the step $h = 1/N$.

Figure 1 shows that the error is zero at every interval boundary and that its amplitude inside each interval decreases as N increases.

The curves in Figure 2 are parallel to the reference line Ch^2 , which indicates second-order convergence. Moreover, direct substitution of (15) gives

$$P_{0,e^{-x}}(z) = e^{-z}, \quad P_{0,e^x}(z) = e^z.$$

In the numerical computation, the maximum difference for e^{-x} did not exceed $3.4 \cdot 10^{-16}$; this is the machine-precision representation of zero.

7 Conclusion

The coefficients of the optimal interpolation formula in $W_2^{(1,0)}(0,1)$ were derived by an alternative method based on an integrating factor and a monotone coordinate transformation. This transformation reduced the optimality problem to minimizing a kernel norm in a simple first-derivative energy and directly established the locality of the formula. The resulting two-node hyperbolic formula was proved to coincide exactly with the $m = 1$ case of the general result in [3]. Thus, the main contribution of this article is not a new coefficient formula, but an alternative derivation of a known result, an explanation of its local form, and a demonstration of the equivalence of the two methods.

An exact expression for the error-functional norm and its uniform estimate were obtained. The $O(h^2)$ convergence order was derived for functions with additional smoothness. Numerical experiments for z^2 , e^{2z} , and $\sin z$ show that the maximum error decreases as the number of nodes grows and that the observed order approaches two.

Declarations

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Author contributions. The authors prepared, read, and approved the final manuscript.

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АЛЬТЕРНАТИВНЫЙ ВЫВОД И ЧИСЛЕННЫЙ АНАЛИЗ ОПТИМАЛЬНОЙ ИНТЕРПОЛЯЦИОННОЙ ФОРМУЛЫ В ГИЛЬБЕРТОВОМ ПРОСТРАНСТВЕ

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Предпосылки: В статье рассматривается задача оптимальной интерполяции по значениям функции в узлах равномерной сетки в пространстве $W_2^{(1,0)}(0,1)$. Коэффициенты этой формулы ранее были получены как частный случай $m = 1$ общего результата для пространства $W_2^{(m,m-1)}(0,1)$ методом Соболева с использованием дискретного аналога дифференциального оператора.

Метод: В настоящей работе тот же результат выводится альтернативным способом. Интегрирующий множитель $g(x) = e^x f(x)$ и монотонная замена координаты $s(x) = (e^{2x} - 1)/2$ преобразуют исходную полунорму в простой энергетический интеграл.

Результаты: Непосредственная минимизация нормы ядра погрешности доказывает локальность оптимальной формулы. Установлено, что полученная локальная формула с гиперболическими синусами точно совпадает с ранее полученной общей формулой при $m = 1$. Получены точная норма функционала погрешности и ее равномерная оценка.

Анализ сходимости: Для гладких функций доказан второй порядок сходимости. Численные результаты для функций z^2 , e^{2z} и $\sin z$ подтверждают уменьшение максимальной погрешности при увеличении числа узлов и теоретический порядок $O(h^2)$.

Ключевые слова: оптимальная интерполяция, пространство Соболева, альтернативный метод, функционал погрешности, гиперболический синус, численный эксперимент, порядок сходимости

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