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NUMERICAL APPROXIMATION OF FRACTIONAL INTEGRALS: AN OPTIMAL QUADRATURE APPROACH BASED ON PHI-FUNCTIONS*

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Background: Riemann–Liouville fractional integrals are non-local operators that model memory effects in applied sciences, and they can rarely be evaluated analytically. Classical quadrature rules of Gregory, Newton–Cotes, Euler, Gauss and Markov type were designed for integer-order integration and cannot, in general, be applied directly to fractional-order operators with singular kernels.

Purpose: The aim is to construct a quadrature formula for the left Riemann–Liouville fractional integral that is optimal in the sense of Sard in a Hilbert space of absolutely continuous functions with square-integrable derivative, for an arbitrary fixed set of nodes, and to find its coefficients explicitly.

Method: The formula is built by the phi-function method: the error is expressed through a piecewise phi-function, and the integration error is bounded from above through the integral of its square. On each subinterval between neighboring nodes this function solves a first-order linear differential equation with a weakly singular right-hand side. The equation is solved by variation of parameters with a series expansion of the exponential, and the free constants are chosen to minimize the integral of the squared function. The coefficients then follow directly from the values of the function at the nodes.

Results: For any fixed set of nodes the optimal formula exists and is unique, and its coefficients are given explicitly by series that depend on the nodes, the upper limit of integration and the order of the fractional integral. The formula is exact for the increasing and the decreasing exponential functions of unit rate, which distinguishes it from classical rules that are exact for polynomials. For equally spaced nodes it reduces to a previously known result.

Implication: The paper extends the phi-function method to fractional integrals with arbitrarily placed nodes. The results are analytical and provide a basis for theoretical error bounds and for numerical comparison with existing approximation methods.

Keywords: Hilbert space, φ -functions method, optimal quadrature formula, error of the formula

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1 Introduction

Fractional calculus occupies a central place among the modern branches of mathematics, since it provides an effective tool for describing a wide range of applied problems

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arising in physics, mechanics, biology, and engineering. Because Riemann–Liouville fractional integrals possess a non-local character, computing them analytically is frequently impossible or requires excessive computational effort [1–3]. Traditional quadrature rules — including those attributed to Gregory, Newton–Cotes, Euler, Gauss, and Markov — work well for classical integer-order integration, but they cannot, in general, be applied directly to fractional-order operators, since fractional derivatives and integrals exhibit singular behavior and memory effects that these classical rules do not account for. For this reason, building numerical methods designed specifically for high-precision fractional calculus remains an important research direction.

In recent years, a number of numerical schemes have been developed specifically for Riemann–Liouville fractional integrals. Dimitrov [4] derived an asymptotic expansion of the trapezoidal approximation of the fractional integral; when the derivatives of the integrand at the end points are available, this expansion yields formulas of fourth order of accuracy, which were further applied to the numerical solution of fractional integral equations. Budak et al. [5] obtained bounds for the error of approximating a fractional integral by Simpson’s rule, thereby extending classical error estimates of Newton–Cotes type to the fractional setting. Ciesielski and Grodzki [6] constructed algorithms for the left- and right-sided Riemann–Liouville integrals and the Riesz integral based on linear, quadratic and cubic spline interpolation of the integrand, derived analytical error estimates, and confirmed experimentally convergence orders of up to four. Elgindy [7] proposed a Gegenbauer-based fractional approximation method with precomputable fractional-order integration matrices, which attains super-exponential convergence for smooth integrands. Within the theory of optimal quadrature formulas, Shadimetov and Toshboev [8] studied the right-sided Riemann–Liouville integral in the Sobolev space $L_2^{(m)}$ by means of the Peano kernel of the error functional and obtained the Sard-optimal coefficients in closed form for every m on a uniform grid, together with sharp asymptotics of the error norm. Most of the above approaches, however, either rely on polynomial exactness and sufficient smoothness of the integrand, or are formulated on equally spaced nodes. This motivates the construction of optimal formulas whose exactness is governed by a differential operator other than d^m/dx^m and which remain valid for an arbitrary distribution of nodes.

In the present paper, the problem of constructing quadrature formulas that are optimal in the sense of Sard is considered for the approximate computation of Riemann–Liouville fractional integrals within the Hilbert space $W_2^{(1,0)}(0, t)$. The approach adopted here is grounded in the functional framework developed by A. Sard [9, 10] and S. M. Nikolsky [11, 12], while the broader theory of optimal cubature formulas is rooted in the foundational contributions of S. L. Sobolev [13, 14]. Substantial progress in the practical realization of such algorithms within Sobolev-type spaces has further been achieved through the work of Kh. M. Shadimetov, A. R. Hayotov, S. S. Babaev and other researchers [15–20]. On the basis of these results, the φ -functions method is employed to extend the existing framework to the setting of Riemann–Liouville fractional integrals.

The remainder of the paper is organized as follows. Section 2 states the problem, presents the φ -functions method for constructing formulas of the type (2) in $W_2^{(1,0)}(0, t)$, establishes the analytical link between the coefficients and the φ -functions, and reduces the optimality problem to the minimization of the integral (9). Section 3 derives the explicit φ -functions that minimize the error of the quadrature formula (2) and utilizes them to compute the exact coefficients of the resulting optimal formula. Section 4 concludes the paper.

2 Problem statement and method

2.1 Statement of the problem

In this work, we consider the numerical evaluation of the left Riemann–Liouville fractional integral, defined as follows (see [1–3]):

$$D_{0,t}^{-\alpha} f(t) = {}_{RL}I_{0,t}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x) dx. \quad (1)$$

To approximate the integral term in Eq. (1), we construct the following quadrature formula

$$\int_0^t (t-x)^{\alpha-1} f(x) dx = \sum_{k=0}^n A_k f(x_k) + R_n(f). \quad (2)$$

Once the quadrature formula (2) is established, it can be directly applied to evaluate the fractional integral (1). Throughout this paper, $\Gamma(\cdot)$ denotes Euler’s gamma function, $t > 0$ and $0 < \alpha < 1$. Here, A_k and x_k represent the weights (coefficients) and the nodes of the quadrature formula, respectively, while $R_n(f)$ denotes the error term. We assume that the function $f(x)$ belongs to the space $W_2^{(1,0)}(0, t)$, which is defined as the Hilbert space of absolutely continuous functions whose first derivatives are square-integrable. The nodes of the quadrature formula are distributed over the closed interval $[0, t]$ in the following order:

$$0 = x_0 < x_1 < \cdots < x_n = t.$$

For any two functions $f(x)$ and $g(x)$ belonging to this space, the inner product is given by

$$\langle f(x), g(x) \rangle_{W_2^{(1,0)}} = \int_0^t (f'(x) + f(x))(g'(x) + g(x)) dx,$$

and the associated norm is defined as

$$\|f(x)\|_{W_2^{(1,0)}} = \left(\int_0^t (f'(x) + f(x))^2 dx \right)^{\frac{1}{2}}.$$

A central question in the theory of quadrature formulas concerns their optimality with respect to the error term [21, 22]. In this study, this question is addressed in the sense introduced by Sard, relying on the one-to-one correspondence that exists between quadrature formulas and φ -functions.

To simplify the notation, we introduce the following multi-indices:

$$A = (A_0, A_1, \dots, A_n) \text{ and } X = (x_0, x_1, \dots, x_n). \quad (3)$$

Definition 1. The quadrature formula (2) is said to be optimal in the sense of Nikolsky in the space $W_2^{(1,0)}(0, t)$ if the quantity

$$F_n(W_2^{(1,0)}, A, X) = \sup_{f \in W_2^{(1,0)}} |R_n(f)|$$

attains its minimum with respect to both A and X as given in (3).

Definition 2. *The quadrature formula (2) is said to be optimal in the sense of Sard in the space $W_2^{(1,0)}(0, t)$ if the quantity*

$$F_n(W_2^{(1,0)}, A) = \sup_{f \in W_2^{(1,0)}} |R_n(f)| \quad (4)$$

attains its minimum with respect to A for a fixed set of X .

Our goal in this work is to solve this optimality problem in the sense of Sard for the formula (2) within the space $W_2^{(1,0)}(0, t)$ — that is, to determine coefficients in (2) that minimize the quantity (4) for a fixed node set X . The φ -functions method is the primary mathematical tool utilized to accomplish this [23–25].

2.2 Method of φ -functions for constructing quadrature formulas in the space $W_2^{(1,0)}(0, t)$

This subsection describes the φ -functions technique employed to construct quadrature formulas of the type (2) that are optimal in the sense of Sard within $W_2^{(1,0)}(0, t)$. A more comprehensive treatment of the general φ -function method can be found in [17, 23–26].

Consider a function $f(x) \in W_2^{(1,0)}(0, t)$, and fix a positive integer n together with a set of nodes X as defined in (3). On every subinterval $[x_{k-1}, x_k]$ for $k = 1, 2, \dots, n$, we define the functions $\varphi_k(x)$ satisfying the following linear differential equation:

$$\varphi'_k(x) - \varphi_k(x) = (t - x)^{\alpha-1}, \quad k = 1, 2, \dots, n. \quad (5)$$

The global function $\varphi(x)$ is then defined piecewise as follows:

$$\varphi(x)|_{[x_{k-1}, x_k]} = \varphi_k(x), \quad k = 1, 2, \dots, n,$$

in other words, $\varphi(x)$ coincides with $\varphi_k(x)$ on each subinterval $[x_{k-1}, x_k]$.

Let us introduce the following notation:

$$I(f) := \int_0^t (t - x)^{\alpha-1} f(x) dx, \quad (6)$$

and

$$Q_n(f) := \sum_{k=0}^n A_k f(x_k).$$

By splitting the integral additively over the subintervals and applying Eq. (5) to Eq. (6), we obtain [23–25]

$$\begin{aligned}
I(f) &= \int_0^t (t-x)^{\alpha-1} f(x) dx = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} (\varphi'_k(x) - \varphi_k(x)) f(x) dx = \\
&= \sum_{k=1}^n \left(\int_{x_{k-1}}^{x_k} \varphi'_k(x) f(x) dx - \int_{x_{k-1}}^{x_k} \varphi_k(x) f(x) dx \right) = \\
&= \sum_{k=1}^n \left(\varphi_k(x) \cdot f(x) \Big|_{x_{k-1}}^{x_k} - \int_{x_{k-1}}^{x_k} \varphi_k(x) f'(x) dx - \int_{x_{k-1}}^{x_k} \varphi_k(x) \cdot f(x) dx \right) = \\
&= \sum_{k=1}^n \left(\varphi_k(x) f(x) \Big|_{x_{k-1}}^{x_k} - \int_{x_{k-1}}^{x_k} \varphi_k(x) \cdot (f'(x) + f(x)) dx \right) = \\
&= \sum_{k=1}^n (\varphi_k(x_k) f(x_k) - \varphi_k(x_{k-1}) f(x_{k-1})) - \sum_{k=1}^n \int_{x_{k-1}}^{x_k} \varphi_k(x) \cdot (f'(x) + f(x)) dx = \\
&= \sum_{k=1}^n \varphi_k(x_k) f(x_k) - \sum_{k=1}^n \varphi_k(x_{k-1}) \cdot f(x_{k-1}) - \sum_{k=1}^n \int_{x_{k-1}}^{x_k} \varphi_k(x) \cdot (f'(x) + f(x)) dx = \\
&= \sum_{k=1}^n \varphi_k(x_k) \cdot f(x_k) - \sum_{k=0}^{n-1} \varphi_{k+1}(x_k) \cdot f(x_k) - \sum_{k=1}^n \int_{x_{k-1}}^{x_k} \varphi_k(x) \cdot (f'(x) + f(x)) dx = \\
&= \varphi_n(x_n) \cdot f(x_n) + \sum_{k=1}^{n-1} \varphi_k(x_k) \cdot f(x_k) - \sum_{k=1}^{n-1} \varphi_{k+1}(x_k) \cdot f(x_k) - \varphi_1(x_0) f(x_0) - \\
&\quad - \sum_{k=1}^n \int_{x_{k-1}}^{x_k} \varphi_k(x) \cdot (f'(x) + f(x)) dx,
\end{aligned}$$

which leads to

$$I(f) = A_0 f(x_0) + \sum_{k=1}^{n-1} A_k f(x_k) + A_n f(x_n) + R_n[f],$$

which yields the following explicit expressions for the coefficients:

$$\begin{aligned}
A_0 &= -\varphi_1(x_0), \\
A_k &= \varphi_k(x_k) - \varphi_{k+1}(x_k), \quad k = 1, 2, \dots, n-1, \\
A_n &= \varphi_n(x_n),
\end{aligned} \tag{7}$$

while the corresponding error term takes the form:

$$\begin{aligned}
R_n[f] &= - \sum_{k=1}^n \int_{x_{k-1}}^{x_k} \varphi_k(x) (f'(x) + f(x)) dx = \\
&= - \int_0^t \varphi(x) (f'(x) + f(x)) dx.
\end{aligned} \tag{8}$$

Remark 1. Once the function $\varphi(x)$ in (7) is determined, the coefficients A_k ($k = 0, 1, \dots, n$) follow immediately. This systematic procedure for constructing a quadrature formula is known as the φ -functions method.

Remark 2. Formula (8) demonstrates that the quadrature rule (2) is exact (i.e., $R_n(f) = 0$) for any function satisfying the differential equation

$$f'(x) + f(x) = 0.$$

Consequently, the constructed formula is exact for functions of the form $f(x) = Ce^{-x}$.

2.3 The optimality problem for the quadrature formula

This subsection is devoted to computing the coefficients of the optimal quadrature formula (2) within the space $W_2^{(1,0)}(0, t)$.

Applying the Cauchy–Schwarz inequality to the integral representation of the error term (8) yields:

$$\begin{aligned} |R_n(f)| &\leq \|f'(x) + f(x)\|_{L_2(0,t)} \left(\int_0^t \varphi^2(x) dx \right)^{1/2} = \\ &= \|f\|_{W_2^{(1,0)}(0,t)} \|\varphi\|_{L_2(0,t)}. \end{aligned}$$

Note that constructing an optimal quadrature formula of the form (2) in the Sard sense within $W_2^{(1,0)}(0, t)$ amounts to determining coefficients $A = (A_0, A_1, \dots, A_n)$ — for a node set $X = (x_0, x_1, \dots, x_n)$ fixed as in (3) — that minimize

$$F_n(A) = \int_0^t \varphi^2(x) dx. \quad (9)$$

Equivalently, this problem reduces to finding the functions $\varphi_k(x)$ ($k = 1, 2, \dots, n$) that satisfy the differential equation (5) while minimizing the integral (9) on each respective subinterval $[x_{k-1}, x_k]$. We first determine these minimizing functions $\varphi_k(x)$; the exact coefficients A_k of the resulting optimal quadrature formula (2) are then directly obtained using relations (7).

3 Results and discussion

3.1 Finding the functions φ_k

In this subsection, we determine the explicit form of the functions $\varphi_k(x)$ on each subinterval $[x_{k-1}, x_k]$ for $k = 1, 2, \dots, n$, which satisfy the differential equation:

$$y' - y = (t - x)^{\alpha-1}. \quad (10)$$

We seek a solution using the method of variation of parameters, assuming the form $y(x) = u(x)y_1(x)$, where $y_1(x)$ represents the non-trivial solution to the corresponding homogeneous equation:

$$y' - y = 0. \quad (11)$$

It is straightforward to verify that

$$y_1(x) = e^x \quad (12)$$

is a fundamental solution to (11). This enables us to solve the non-homogeneous equation (10) by assuming a general solution of the form:

$$y(x) = u(x)y_1(x). \quad (13)$$

Differentiating the unknown function $y(x)$ with respect to x gives:

$$y' = u'y_1 + uy'_1. \quad (14)$$

Substituting (13) and (14) into (10) and utilizing the property of y_1 in (12) yields:

$$u'e^x + ue^x - ue^x = (t-x)^{\alpha-1},$$

which simplifies to:

$$u' = e^{-x}(t-x)^{\alpha-1}.$$

Integrating both sides with respect to x , and employing the Taylor series expansion for the exponential term $e^{t-x} = \sum_{i=0}^{\infty} \frac{(t-x)^i}{i!}$, we evaluate the integral as follows:

$$u(x) = -e^{-t} \sum_{i=0}^{\infty} \frac{1}{i!} \frac{(t-x)^{i+\alpha}}{i+\alpha} + C.$$

Combining this result with (12), the general solution to the differential equation (10) is obtained as:

$$y = -e^{x-t} \sum_{i=0}^{\infty} \frac{1}{i!} \frac{(t-x)^{i+\alpha}}{i+\alpha} + Ce^x. \quad (15)$$

Accordingly, on each subinterval $[x_{k-1}, x_k]$ for $k = 1, 2, \dots, n$, we define $\varphi_k(x)$ in the form of (15):

$$\varphi_k(x) = -e^{x-t} \sum_{i=0}^{\infty} \frac{1}{i!} \frac{(t-x)^{i+\alpha}}{i+\alpha} + C^{(k)}e^x, \quad x \in [x_{k-1}, x_k]. \quad (16)$$

Consequently, determining the exact functions $\varphi_k(x)$ reduces to selecting the appropriate constants $C^{(k)}$ ($k = 1, 2, \dots, n$) such that the error functional norm (9) is minimized on each subinterval. To achieve this, we look for the values of $C^{(k)}$ that minimize the integral of $\varphi_k^2(x)$ over $[x_{k-1}, x_k]$. Let us consider the following function to be minimized:

$$\mathcal{F}_k(C^{(k)}) = \int_{x_{k-1}}^{x_k} \varphi_k^2(x) dx, \quad k = 1, 2, \dots, n.$$

Substituting the expression for $\varphi_k(x)$ from (16) into the objective function yields:

$$\begin{aligned} \mathcal{F}_k(C^{(k)}) &= \int_{x_{k-1}}^{x_k} \varphi_k^2(x) dx = \int_{x_{k-1}}^{x_k} \left[e^x \left(-e^{-t} \sum_{i=0}^{\infty} \frac{1}{i!} \frac{(t-x)^{i+\alpha}}{i+\alpha} + C^{(k)} \right) \right]^2 dx = \\ &= \int_{x_{k-1}}^{x_k} e^{2x} \left(-e^{-t} \sum_{i=0}^{\infty} \frac{1}{i!} \frac{(t-x)^{i+\alpha}}{i+\alpha} \right)^2 dx - \\ &\quad - 2C^{(k)}e^{-t} \int_{x_{k-1}}^{x_k} e^{2x} \sum_{i=0}^{\infty} \frac{1}{i!} \frac{(t-x)^{i+\alpha}}{i+\alpha} dx + \\ &\quad + (C^{(k)})^2 \int_{x_{k-1}}^{x_k} e^{2x} dx, \quad k = 1, 2, \dots, n. \end{aligned}$$

To find the minimum, we differentiate $\mathcal{F}_k(C^{(k)})$ with respect to the parameter $C^{(k)}$ and equate the result to zero, which provides the necessary condition for optimality:

$$2C^{(k)} \int_{x_{k-1}}^{x_k} e^{2x} dx - 2e^{-t} \int_{x_{k-1}}^{x_k} e^{2x} \sum_{i=0}^{\infty} \frac{1}{i!} \frac{(t-x)^{i+\alpha}}{i+\alpha} dx = 0.$$

Solving this equation for $C^{(k)}$ requires evaluating the integral $\int_{x_{k-1}}^{x_k} e^{2x}(t-x)^{i+\alpha} dx$. By utilizing the Taylor series expansion $e^{2x} = e^{2t}e^{-2(t-x)} = e^{2t} \sum_{j=0}^{\infty} \frac{(-1)^j 2^j}{j!} (t-x)^j$, we can resolve this integral analytically to obtain:

$$\begin{aligned} C^{(k)} &= \frac{e^{-t} \int_{x_{k-1}}^{x_k} e^{2x} \sum_{i=0}^{\infty} \frac{1}{i!} \frac{(t-x)^{i+\alpha}}{i+\alpha} dx}{\int_{x_{k-1}}^{x_k} e^{2x} dx} = \\ &= -\frac{2e^t}{e^{2x_k} - e^{2x_{k-1}}} \sum_{i=0}^{\infty} \frac{1}{i!(i+\alpha)} \times \\ &\quad \times \sum_{j=0}^{\infty} \frac{(-1)^j 2^j}{j!} \cdot \frac{1}{i+j+\alpha+1} \left((t-x_k)^{i+j+\alpha+1} - (t-x_{k-1})^{i+j+\alpha+1} \right), \\ &\quad k = 1, 2, \dots, n. \end{aligned} \quad (17)$$

Since $\mathcal{F}_k$ is a convex quadratic function with respect to $C^{(k)}$, one can rigorously verify that this choice of $C^{(k)}$ strictly minimizes $\mathcal{F}_k(C^{(k)})$ on the subinterval $[x_{k-1}, x_k]$. Finally, substituting the optimal constant (17) back into (16) gives the exact explicit representation of the functions $\varphi_k(x)$:

$$\begin{aligned} \varphi_k(x) &= -e^{x-t} \sum_{i=0}^{\infty} \frac{1}{i!} \frac{(t-x)^{i+\alpha}}{i+\alpha} - \frac{2e^{t+x}}{e^{2x_k} - e^{2x_{k-1}}} \cdot \sum_{i=0}^{\infty} \frac{1}{i!(i+\alpha)} \times \\ &\quad \times \sum_{j=0}^{\infty} \frac{(-1)^j 2^j}{j!} \frac{1}{i+j+\alpha+1} \left((t-x_k)^{i+j+\alpha+1} - (t-x_{k-1})^{i+j+\alpha+1} \right), \\ &\quad x \in [x_{k-1}, x_k], \quad k = 1, 2, \dots, n. \end{aligned} \quad (18)$$

3.2 Coefficients of the optimal quadrature formula

With the explicit representation of $\varphi_k(x)$ given by (18), we now utilize relations (7) to compute the exact coefficients A_k ($k = 0, 1, \dots, n$) of the optimal quadrature formula (2).

We begin with the first coefficient A_0 . Evaluating $\varphi_1(x)$ at the initial node $x_0 = 0$, we obtain:

$$\begin{aligned} A_0 &= e^{-t} \sum_{i=0}^{\infty} \frac{1}{i!} \frac{(t)^{i+\alpha}}{i+\alpha} + \frac{2e^t}{e^{2x_1} - 1} \sum_{i=0}^{\infty} \frac{1}{i!(i+\alpha)} \times \\ &\quad \times \sum_{j=0}^{\infty} \frac{(-1)^j 2^j}{j!} \frac{1}{i+j+\alpha+1} \left[(t-x_1)^{i+j+\alpha+1} - (t)^{i+j+\alpha+1} \right]. \end{aligned} \quad (19)$$

Next, we compute the interior coefficients A_k for $k = 1, 2, \dots, n-1$. By evaluating the respective expressions for $\varphi_k(x)$ and $\varphi_{k+1}(x)$ from (18) at the node x_k , we notice that the first summation terms perfectly cancel each other out, yielding a simplified expression:

$$\begin{aligned} A_k &= \frac{2e^{t+x_k}}{e^{2x_{k+1}} - e^{2x_k}} \sum_{i=0}^{\infty} \frac{1}{i!(i+\alpha)} \sum_{j=0}^{\infty} \frac{(-1)^j 2^j}{j!} \frac{1}{i+j+\alpha+1} \times \\ &\quad \times \left[(t-x_{k+1})^{i+j+\alpha+1} - (t-x_k)^{i+j+\alpha+1} \right] - \\ &\quad - \frac{2e^{t+x_k}}{e^{2x_k} - e^{2x_{k-1}}} \sum_{i=0}^{\infty} \frac{1}{i!(i+\alpha)} \sum_{j=0}^{\infty} \frac{(-1)^j 2^j}{j!} \frac{1}{i+j+\alpha+1} \times \\ &\quad \times \left[(t-x_k)^{i+j+\alpha+1} - (t-x_{k-1})^{i+j+\alpha+1} \right], \\ &\quad k = 1, 2, \dots, n-1. \end{aligned} \quad (20)$$

Finally, the last coefficient A_n follows from (7) together with (18). Evaluating $\varphi_n(x)$ at the final node $x_n = t$ yields:

$$\begin{aligned} A_n = \varphi_n(x_n) = & -e^{x_n-t} \sum_{i=0}^{\infty} \frac{1}{i!} \frac{(t-x_n)^{i+\alpha}}{i+\alpha} - \frac{2e^{t+x_n}}{e^{2x_n} - e^{2x_{n-1}}} \sum_{i=0}^{\infty} \frac{1}{i!(i+\alpha)} \times \\ & \times \sum_{j=0}^{\infty} \frac{(-1)^j 2^j}{j!} \frac{1}{i+j+\alpha+1} [(t-x_n)^{i+j+\alpha+1} - (t-x_{n-1})^{i+j+\alpha+1}]. \end{aligned} \quad (21)$$

Combining the derived explicit expressions in (19), (20), and (21) provides the complete foundation for the main theorem of this work.

Theorem 1. *For every fixed set of nodes X , there exists a unique quadrature formula in the space $W_2^{(1,0)}(0, t)$ that is optimal in the sense of Sard and has the form*

$$\int_0^t (t-x)^{\alpha-1} f(x) dx = \sum_{k=0}^n A_k f(x_k) + R_n(f)$$

with the coefficients

$$\begin{aligned} A_0 = & e^{-t} \sum_{i=0}^{\infty} \frac{1}{i!} \frac{(t)^{i+\alpha}}{i+\alpha} + \frac{2e^t}{e^{2x_1} - 1} \sum_{i=0}^{\infty} \frac{1}{i!(i+\alpha)} \times \\ & \times \sum_{j=0}^{\infty} \frac{(-1)^j 2^j}{j!} \frac{1}{i+j+\alpha+1} [(t-x_1)^{i+j+\alpha+1} - (t-x_0)^{i+j+\alpha+1}], \\ A_k = & \frac{2e^{t+x_k}}{e^{2x_{k+1}} - e^{2x_k}} \sum_{i=0}^{\infty} \frac{1}{i!(i+\alpha)} \sum_{j=0}^{\infty} \frac{(-1)^j 2^j}{j!} \frac{1}{i+j+\alpha+1} \times \\ & \times [(t-x_{k+1})^{i+j+\alpha+1} - (t-x_k)^{i+j+\alpha+1}] - \\ & - \frac{2e^{t+x_k}}{e^{2x_k} - e^{2x_{k-1}}} \sum_{i=0}^{\infty} \frac{1}{i!(i+\alpha)} \sum_{j=0}^{\infty} \frac{(-1)^j 2^j}{j!} \frac{1}{i+j+\alpha+1} \times \\ & \times [(t-x_k)^{i+j+\alpha+1} - (t-x_{k-1})^{i+j+\alpha+1}], \\ A_n = \varphi_n(x_n) = & -e^{x_n-t} \sum_{i=0}^{\infty} \frac{1}{i!} \frac{(t-x_n)^{i+\alpha}}{i+\alpha} - \frac{2e^{t+x_n}}{e^{2x_n} - e^{2x_{n-1}}} \sum_{i=0}^{\infty} \frac{1}{i!(i+\alpha)} \times \\ & \times \sum_{j=0}^{\infty} \frac{(-1)^j 2^j}{j!} \frac{1}{i+j+\alpha+1} [(t-x_n)^{i+j+\alpha+1} - (t-x_{n-1})^{i+j+\alpha+1}] \end{aligned}$$

for fixed nodes x_k , $k = 0, 1, \dots, n$, satisfying the inequality $0 = x_0 < x_1 < \dots < x_n = t$.

Remark 3. Note that if we consider a uniform grid $x_k = kh$ for $k = 0, 1, \dots, n$ with the step size $h = t/n$, Theorem 1 strictly reduces to the analytical results previously obtained in [27].

4 Conclusion

In this paper, we addressed the problem of constructing an optimal quadrature formula for the approximation of Riemann–Liouville fractional integrals within the Hilbert space $W_2^{(1,0)}(0, t)$. The explicit form of the error function $\varphi(x)$ was rigorously determined on each subinterval by solving the associated linear differential equations, and a systematic

algorithm for computing the exact optimal coefficients was developed based on the resulting optimality conditions. Our methodology successfully minimized the squared L_2 -norm of the error function, yielding explicit analytical expressions for the quadrature weights. Future research will focus on establishing strict theoretical upper bounds for the optimal quadrature error and conducting comprehensive numerical experiments to compare the proposed approach with existing approximation methods in the literature.

Declarations

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Author contributions. Babaev S.S.: problem statement and construction of the φ -functions; Elmurodova S.S.: derivation of the optimal coefficients and verification of the results.

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ЧИСЛЕННОЕ ПРИБЛИЖЕНИЕ ДРОБНЫХ ИНТЕГРАЛОВ: ОПТИМАЛЬНЫЙ КВАДРАТУРНЫЙ ПОДХОД, ОСНОВАННЫЙ НА ФИ-ФУНКЦИЯХ*

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Предпосылки: Дробные интегралы Римана–Лиувилля являются нелокальными операторами, описывающими эффекты памяти в прикладных науках, и лишь изредка вычисляются аналитически. Классические квадратурные формулы типа Грегори, Ньютона–Котеса, Эйлера, Гаусса и Маркова создавались для интегрирования целого порядка и, как правило, не могут быть напрямую применены к операторам дробного порядка с сингулярными ядрами.

Цель: Цель работы — построить квадратурную формулу для левого дробного интеграла Римана–Лиувилля, оптимальную в смысле Сарда в гильбертовом пространстве абсолютно непрерывных функций с интегрируемой с квадратом производной, для произвольного фиксированного набора узлов и найти её коэффициенты в явном виде.

Метод: Формула строится методом фи-функций: погрешность выражается через кусочную фи-функцию, а сама погрешность оценивается сверху через интеграл от её квадрата. На каждом отрезке между соседними узлами эта функция удовлетворяет линейному дифференциальному уравнению первого порядка со слабо сингулярной правой частью. Уравнение решается методом вариации постоянных с разложением экспоненты в ряд, а свободные константы выбираются из условия минимума интеграла от квадрата функции. После этого коэффициенты формулы находятся непосредственно по значениям функции в узлах.

Результаты: Для любого фиксированного набора узлов оптимальная формула существует и единственна, а её коэффициенты выражаются в явном виде через ряды, зависящие от узлов, верхнего предела интегрирования и порядка дробного интеграла. Формула точна для возрастающей и убывающей экспоненциальных функций единичного показателя, что отличает её от классических формул, точных на многочленах. Для равноотстоящих узлов она сводится к ранее известному результату.

Значимость: Работа распространяет метод фи-функций на дробные интегралы с произвольным расположением узлов. Полученные результаты носят аналитический характер и создают основу для теоретических оценок погрешности и численного сравнения с существующими методами аппроксимации.

Ключевые слова: гильбертово пространство, метод фи-функций, оптимальная квадратурная формула, погрешность формулы

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