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NUMERICAL SOLUTION OF LINEAR FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND BASED ON OPTIMAL INTERPOLATION FORMULAS WITH DERIVATIVES

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Background: Linear Fredholm integral equations of the second kind are fundamental in modeling various scientific and engineering phenomena. Traditional numerical methods for solving these equations often rely solely on nodal function values for approximation. However, these conventional approaches may struggle to accurately capture the local behavior and smoothness of the unknown solution, potentially limiting their approximation precision and numerical stability in complex scenarios.

Purpose: The primary objective of this study is to develop a novel and highly accurate numerical scheme that significantly enhances approximation precision. We aim to achieve this by effectively incorporating both function values and their higher-order derivatives at discrete nodes, thereby fully utilizing the inherent smoothness of the solution.

Method: To accomplish this, we propose an innovative approach applying an optimal interpolation formula with derivatives specifically constructed within the Sobolev space. By utilizing the exact coefficients of this optimal interpolation formula, we systematically transform the continuous integral equation into a finite, closed system of linear algebraic equations. This transformation is achieved through a rigorous collocation approach. The novelty lies in our derivative-based scheme, which, unlike standard techniques, actively integrates higher-order derivative information into the discrete approximation process.

Results: Our theoretical analysis successfully establishes strict error bounds, confirming the mathematical soundness of the proposed scheme. The integration of derivative information leads to superior convergence rates and enhanced numerical stability compared to traditional methods. Furthermore, computational experiments on a numerical example with a known exact solution demonstrate that our method reconstructs the solution with exceptionally high fidelity and minimal computational error, validating our theoretical claims.

Implication: Consequently, this work provides a highly robust and reliable computational tool for advanced mathematical and engineering applications. It establishes an improved standard among numerical solvers for integral equations where preserving local behavior and derivative information is critical for maintaining overall numerical accuracy.

Keywords: approximation, numerical solution, interpolation formula, Sobolev space, linear Fredholm integral equation

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1 Introduction

Linear integral equations serve as a critical mathematical apparatus for characterizing complex phenomena across physics, engineering, and population dynamics [1]. Specifically, Fredholm integral equations of the second kind are indispensable in modeling problems such as quantum mechanics scattering, diffraction theory, and image deblurring. Since obtaining an exact closed-form solution is often mathematically intractable for complicated kernels, the development of robust, high-order numerical approximation techniques has become a central focus of modern computational mathematics [2].

To date, a diverse spectrum of numerical schemes has been developed to address these equations. Classical approaches frequently utilize quadrature-based discretization, such as modified Simpson's rules or repeated trapezoidal and midpoint formulas [1]. For equations defined over large intervals, specialized iterative methods like the generalized Gauss–Seidel or product integration schemes are employed to manage the resulting large-scale algebraic systems [3]. Furthermore, semi-analytical techniques, including the Homotopy Analysis Method (HAM) and the Adomian Decomposition Method (ADM), provide rapid convergence and flexibility in choosing initial approximations (see [4–9]). Another effective approach is the Weighted Mean-Value Method (WMVM), which reduces the integral problem to a system of algebraic equations by leveraging the mean-value theorem for integrals [2, 10].

In recent years, the integration of orthogonal basis functions and meshless frameworks has significantly enhanced approximation accuracy. Wavelet-based methods, particularly those employing Legendre and Chebyshev wavelets, utilize operational matrices to solve systems of linear integral equations efficiently [11]. Similarly, Block Pulse Functions (BPF) and their modern extensions, such as ε -Modified Block Pulse Functions (ε MBPF), allow for the transformation of integral operators into sparse matrices, thereby optimizing computational speed [12]. In scenarios where domain geometries are irregular or non-rectangular, meshless methods based on Radial Basis Functions (RBF), such as Gaussian and Multiquadric bases, provide high-order convergence without requiring traditional grid structures [4]. Additionally, Nystrom and collocation methods based on the zeros of Jacobi orthogonal polynomials have been successfully applied to square domains [13].

A significant contemporary advancement involves the construction of optimal numerical schemes within specific Hilbert or Sobolev functional spaces [14]. Recent studies have demonstrated that weighted optimal quadrature formulas, constructed to minimize the error functional, yield superior precision compared to classical rules by treating the kernel as a weight function [15]. Parallel to this, the Galerkin method has been refined using basis functions derived from the coefficients of optimal interpolation formulas [14]. For instance, exponential bases constructed in $W_2^{(1,0)}$ space have shown significantly lower approximation errors than traditional piecewise linear bases by more effectively capturing the analytical nature of the solution.

Despite these advancements, many existing optimal interpolation and quadrature schemes rely solely on nodal function values and may not fully utilize the inherent smoothness or higher-order derivative properties of the solution. To address this limitation and bridge the gap between interpolation accuracy and computational efficiency, this paper proposes a novel numerical scheme for solving Fredholm integral equations of the second kind by applying an interpolation formula with derivatives constructed in $L_2^{(3)}(0, 1)$ Sobolev space [16, 17]. By incorporating information from both the function values and its derivatives at discrete nodes, the proposed method aims to achieve a higher order of convergence and enhanced numerical stability. This approach leverages the specific

geometric and functional properties of $L_2^{(3)}(0, 1)$ space to provide a more refined representation of the solution's local behavior, offering a viable and accurate alternative to standard nodal-based techniques.

2 Problem statement and method

Let the following linear Fredholm integral equation of the second kind [18, 19] be given on $[0, 1]$:

$$\varphi(x) = f(x) + \lambda \int_0^1 K(x, t) \varphi(t) dt, \quad x \in [0, 1], \quad (1)$$

where $K(x, t)$ is a given kernel (assumed to be at least m times continuously differentiable with respect to x , i.e. $K \in C^m([0, 1] \times [0, 1])$), $f(x) \in L_2^{(m)}(0, 1)$ is a given function, λ is a parameter, and $\varphi(x)$ is the unknown function to be found.

The goal is to approximate $\varphi(x)$ on the node set $x_\beta = h\beta$, $\beta = 0, 1, \dots, N$, $h = 1/N$, using the optimal interpolation formula in the space $L_2^{(m)}$:

$$\varphi(x) \cong P_\varphi(x) = \sum_{\beta=0}^N \sum_{\alpha=0}^{m-1} C_{\beta,\alpha}(x) \varphi^{(\alpha)}(x_\beta), \quad (2)$$

and then to use this formula to reduce equation (1) to a system of linear algebraic equations. In our case $m = 3$ (i.e. $\alpha = 0, 1, 2$: $\varphi, \varphi', \varphi''$ are involved), and the coefficients $C_{\beta,0}(x)$, $C_{\beta,1}(x)$, $C_{\beta,2}(x)$ are given explicitly.

3 Brief background on the optimal interpolation formula with derivatives

Let $L_2^{(3)}(0, 1)$ be the Sobolev space equipped with the norm

$$\|\varphi\|_{L_2^{(3)}}^2 = \int_0^1 (\varphi'''(x))^2 dx. \quad (3)$$

The error functional of the interpolation formula (2) is defined as [20]

$$\begin{aligned} R(x, t) = & \delta(x - t) - \sum_{\beta=0}^N C_{\beta,0}(t) \cdot \delta(x - x_\beta) + \sum_{\beta=0}^N C_{\beta,1}(t) \cdot \delta'(x - x_\beta) - \\ & - \sum_{\beta=0}^N C_{\beta,2}(t) \cdot \delta''(x - x_\beta), \end{aligned} \quad (4)$$

and the coefficients $C_{\beta,\alpha}(x)$ are obtained by minimizing the norm of the interpolation error

$$\inf_{C_{\beta,\alpha}} \|R\|. \quad (5)$$

Each of the coefficients $C_{\beta,\alpha}(x)$ is a piecewise-polynomial function. We have already found these coefficients for the particular cases $m = 1, 2, 3, 4$ (see [20–22]) and below we use the explicit form given for $m = 3$ case ($\alpha = 0, 1, 2$):

$$C_{\beta,0}(x) = \frac{1}{2h} (|x - h(\beta - 1)| - 2|x - h\beta| + |x - h(\beta + 1)|), \quad \beta = 0, 1, \dots, N,$$

$$\begin{aligned}
C_{0,1}(x) &= \frac{1}{2h} \begin{cases} x(h-x), & 0 \leq x \leq h, \\ 0, & h < x \leq 1, \end{cases} \\
C_{\beta,1}(x) &= \frac{1}{2h} \begin{cases} (x-h\beta)^2 + h(x-h\beta), & h(\beta-1) \leq x < h\beta, \\ -(x-h\beta)^2 + h(x-h\beta), & h\beta \leq x \leq h(\beta+1), \\ 0, & \text{otherwise,} \end{cases} \quad \beta = 1, 2, \dots, N-1, \\
C_{N,1}(x) &= \frac{1}{2h} \begin{cases} 0, & 0 \leq x \leq h(N-1), \\ (x-1)(x-1+h), & h(N-1) \leq x \leq 1, \end{cases}
\end{aligned}$$

$$\begin{aligned}
C_{0,2}(x) &= \frac{1}{12h} \begin{cases} x(h-x)(2x-h), & 0 \leq x \leq h, \\ 0, & h < x \leq 1, \end{cases} \\
C_{\beta,2}(x) &= \frac{1}{12h} \begin{cases} (x-h(\beta-1))(x-h\beta)(2x-2h\beta+h), & h(\beta-1) \leq x \leq h\beta, \\ (h\beta-x)(2h\beta+h-2x)(h\beta+h-x), & h\beta < x \leq h(\beta+1), \\ 0, & \text{otherwise,} \end{cases} \\
&\quad \beta = 1, 2, \dots, N-1, \\
C_{N,2}(x) &= \frac{1}{12h} \begin{cases} 0, & 0 \leq x < (N-1)h, \\ (x-1)(x-1+h)(2x+h-2), & h(N-1) \leq x \leq 1. \end{cases}
\end{aligned}$$

4 Applying the interpolation formula with derivatives to the Fredholm integral equation

Basic idea

In equation (1) we replace the unknown $\varphi(t)$ by the interpolation formula (2):

$$\varphi(x) \cong f(x) + \lambda \int_0^1 K(x, t) \left[\sum_{\beta=0}^N \sum_{\alpha=0}^2 C_{\beta,\alpha}(t) \varphi^{(\alpha)}(x_\beta) \right] dt. \quad (6)$$

Since summation and integration are both linear operations, we interchange their order:

$$\varphi(x) \cong f(x) + \lambda \sum_{\beta=0}^N \sum_{\alpha=0}^2 \left[\int_0^1 K(x, t) C_{\beta,\alpha}(t) dt \right] \varphi^{(\alpha)}(x_\beta). \quad (7)$$

5 Kernel moments

We introduce the following notation: the kernel moments:

$$A_{\beta,\alpha}(x) = \int_0^1 K(x, t) C_{\beta,\alpha}(t) dt, \quad \beta = 0, \dots, N, \quad \alpha = 0, 1, 2. \quad (8)$$

Because of the compact-support property [10] of $C_{\beta,\alpha}(t)$, this integral is in fact computed only over $[h(\beta-1), h(\beta+1)]$ (for interior β) or over $[0, h]$, $[h(N-1), 1]$ (for the

boundary values $\beta = 0, N$):

$$A_{0,\alpha}(x) = \int_0^h K(x, t) C_{0,\alpha}(t) dt, \quad (9)$$

$$A_{\beta,\alpha}(x) = \int_{h(\beta-1)}^{h(\beta+1)} K(x, t) C_{\beta,\alpha}(t) dt, \quad \beta = 1, \dots, N-1, \quad (10)$$

$$A_{N,\alpha}(x) = \int_{h(N-1)}^1 K(x, t) C_{N,\alpha}(t) dt. \quad (11)$$

Then (7) takes the compact form

$$\varphi(x) \cong f(x) + \lambda \sum_{\beta=0}^N \sum_{\alpha=0}^2 A_{\beta,\alpha}(x) \varphi^{(\alpha)}(x_\beta). \quad (12)$$

This is a functional equation in x containing $3(N+1)$ unknown numbers:

$$c_\beta^{(0)} = \varphi(x_\beta), \quad c_\beta^{(1)} = \varphi'(x_\beta), \quad c_\beta^{(2)} = \varphi''(x_\beta), \quad \beta = 0, 1, \dots, N.$$

5.1 Additional equations: the differentiation approach

To determine $3(N+1)$ unknowns we need $3(N+1)$ linear equations. For this purpose we collocate equation (12) at $x = x_\gamma$, $\gamma = 0, \dots, N$, obtaining $(N+1)$ equations; the remaining $2(N+1)$ equations are obtained by successively differentiating the original equation (1) with respect to x , and then collocating each differentiated equation as well. This is the key idea of the whole procedure.

First (original equation)

$$\varphi(x) = f(x) + \lambda \int_0^1 K(x, t) \varphi(t) dt. \quad (13)$$

Second (first-order derivative with respect to x). Assuming $K(x, t)$ is differentiable in x :

$$\varphi'(x) = f'(x) + \lambda \int_0^1 K_x(x, t) \varphi(t) dt, \quad K_x(x, t) = \frac{\partial K(x, t)}{\partial x}. \quad (14)$$

Third (second-order derivative with respect to x).

$$\varphi''(x) = f''(x) + \lambda \int_0^1 K_{xx}(x, t) \varphi(t) dt, \quad K_{xx}(x, t) = \frac{\partial^2 K(x, t)}{\partial x^2}. \quad (15)$$

In each of the three equations we substitute $\varphi(t)$ by the same interpolation formula (2), introducing the corresponding kernel moments:

$$A_{\beta,\alpha}^{(0)}(x) = \int_0^1 K(x, t) C_{\beta,\alpha}(t) dt, \quad (16)$$

$$A_{\beta,\alpha}^{(1)}(x) = \int_0^1 K_x(x, t) C_{\beta,\alpha}(t) dt, \quad (17)$$

$$A_{\beta,\alpha}^{(2)}(x) = \int_0^1 K_{xx}(x, t) C_{\beta,\alpha}(t) dt. \quad (18)$$

This produces three functional equations:

$$\varphi(x) \cong f(x) + \lambda \sum_{\beta, \alpha} A_{\beta, \alpha}^{(0)}(x) c_{\beta}^{(\alpha)}, \quad (19)$$

$$\varphi'(x) \cong f'(x) + \lambda \sum_{\beta, \alpha} A_{\beta, \alpha}^{(1)}(x) c_{\beta}^{(\alpha)}, \quad (20)$$

$$\varphi''(x) \cong f''(x) + \lambda \sum_{\beta, \alpha} A_{\beta, \alpha}^{(2)}(x) c_{\beta}^{(\alpha)}. \quad (21)$$

5.2 Collocation and the system of linear algebraic equations

Now, in each of equations (19)–(21) we set $x = x_{\gamma} = h\gamma$, $\gamma = 0, 1, \dots, N$. Since $\varphi(x_{\gamma}) = c_{\gamma}^{(0)}$, $\varphi'(x_{\gamma}) = c_{\gamma}^{(1)}$, $\varphi''(x_{\gamma}) = c_{\gamma}^{(2)}$, we obtain the following closed system of linear algebraic equations:

$$c_{\gamma}^{(j)} - \lambda \sum_{\beta=0}^N \sum_{\alpha=0}^2 A_{\beta, \alpha}^{(j)}(x_{\gamma}) c_{\beta}^{(\alpha)} = f^{(j)}(x_{\gamma}), \quad \gamma = 0, \dots, N, \quad j = 0, 1, 2. \quad (22)$$

This is a system of $3(N+1)$ linear equations in $3(N+1)$ unknowns. We write it in the matrix form. The vector of unknowns:

$$\mathbf{c} = (c_0^{(0)}, c_1^{(0)}, \dots, c_N^{(0)}, c_0^{(1)}, \dots, c_N^{(1)}, c_0^{(2)}, \dots, c_N^{(2)})^T \in \mathbb{R}^{3(N+1)}. \quad (23)$$

The right-hand-side vector:

$$\mathbf{f} = (f(x_0), \dots, f(x_N), f'(x_0), \dots, f'(x_N), f''(x_0), \dots, f''(x_N))^T. \quad (24)$$

Using the block matrix $\mathbb{A}$ constructed from the kernel moments, the system can be written in the matrix form as

$$(\mathbb{I} - \lambda \mathbb{A}) \mathbf{c} = \mathbf{f}, \quad (25)$$

where $\mathbb{I}$ is the identity matrix of appropriate dimension. If $\det(\mathbb{I} - \lambda \mathbb{A}) \neq 0$ (which holds, in particular, for sufficiently small $|\lambda|$, or, by the Fredholm alternative, whenever the corresponding homogeneous equation admits only the trivial solution), the system has a unique solution:

$$\mathbf{c} = (\mathbb{I} - \lambda \mathbb{A})^{-1} \mathbf{f}. \quad (26)$$

Substituting the resulting values $c_{\beta}^{(0)}, c_{\beta}^{(1)}, c_{\beta}^{(2)}$ into (2), we obtain the continuous approximate solution of $\varphi(x)$ on the entire interval $[0, 1]$:

$$\varphi(x) \cong P_{\varphi}(x) = \sum_{\beta=0}^N \left(C_{\beta,0}(x) c_{\beta}^{(0)} + C_{\beta,1}(x) c_{\beta}^{(1)} + C_{\beta,2}(x) c_{\beta}^{(2)} \right). \quad (27)$$

6 On convergence and error estimation

Since the interpolation formula (2) is optimal in the space $L_2^{(3)}(0, 1)$, i.e. it satisfies a (sharp, minimal-constant) bound of the form [23]

$$\|\varphi - P_{\varphi}\|_{C[0,1]} \leq K_m(h) \|\varphi'''\|_{L_2[0,1]}, \quad (28)$$

Require: $K(x, t)$, $f(x)$, λ , N (number of subintervals), $h = 1/N$, nodes $x_\beta = h\beta$.

- 1: Compute $K_x(x, t)$ and $K_{xx}(x, t)$ analytically (or, if this is not possible, by finite differences).
- 2: Using the explicit piecewise-polynomial expressions for $C_{\beta,0}(t), C_{\beta,1}(t), C_{\beta,2}(t)$, compute, for every $j = 0, 1, 2$ and $\alpha = 0, 1, 2$, the moments:

$$A_{\beta,\alpha}^{(j)}(x_\gamma) = \int_{\text{finite support}} \frac{\partial^j K}{\partial x^j}(x_\gamma, t) C_{\beta,\alpha}(t) dt, \quad \gamma, \beta = 0, \dots, N.$$

// Since $C_{\beta,\alpha}(t)$ has finite support, these integrals can typically be evaluated analytically without Gauss quadrature.

- 3: Assemble the block matrix $\mathbb{A}$ and the vector $\mathbf{f}$.
- 4: Solve the linear system $(\mathbb{I} - \lambda\mathbb{A})\mathbf{c} = \mathbf{f}$ (by Gaussian elimination or LU -decomposition).
- 5: Substitute the obtained coefficients $c_\beta^{(0)}, c_\beta^{(1)}$, and $c_\beta^{(2)}$ into equation (27) to construct the continuous approximate solution $P_\varphi(x)$ for any $x \in [0, 1]$.
- 6: **if** the exact solution $\varphi(x)$ is known **then**
- 7: Estimate the error:

$$\delta_N = \max_{x \in [0,1]} |\varphi(x) - P_\varphi(x)|.$$

8: **end if**

9: **return** The approximate solution $P_\varphi(x)$ and the error estimate δ_N .

the approximate solution constructed using (12) inherits a corresponding error estimate. If $\varphi \in L_2^{(3)}(0, 1)$ and the kernel $K(x, t)$, together with its derivatives in x , is sufficiently smooth, the following bound holds:

$$\max_{0 \leq \gamma \leq N} |\varphi^{(j)}(x_\gamma) - c_\gamma^{(j)}| \leq C^* \cdot h^{3-j} \cdot \|\varphi'''\|_{L_2(0,1)}, \quad j = 0, 1, 2, \quad (29)$$

where C^* is a constant depending on K , λ , and m , but not on h (for $m = 3$, the convergence orders $O(h^3)$, $O(h^2)$, $O(h)$ are expected for $\alpha = 0, 1, 2$, respectively — this is the classical property of Sobolev-type optimal interpolation formulas). If, as $N \rightarrow \infty$ ($h \rightarrow 0$), the condition number of $(\mathbb{I} - \lambda\mathbb{A})$ remains bounded, then $P_\varphi(x) \rightarrow \varphi(x)$ uniformly.

7 Examples, numerical results and discussions

Consider the following linear Fredholm integral equation of the second kind on $[0, 1]$:

$$\varphi(x) - \frac{1}{2} \int_0^1 xt \varphi(t) dt = 1 + \frac{7}{12}x, \quad (30)$$

that is,

$$\lambda = \frac{1}{2}, \quad K(x, t) = xt, \quad f(x) = 1 + \frac{7}{12}x. \quad (31)$$

Exact solution

The kernel $K(x, t) = xt$ is degenerate (separable, rank one), so the exact solution can be found directly. Since $K(x, t) = xt$, equation (30) has the form

$$\varphi(x) = 1 + \frac{7}{12}x + \frac{x}{2}C, \quad C := \int_0^1 t \varphi(t) dt, \quad (32)$$

so $\varphi(x) = 1 + \left(\frac{7}{12} + \frac{C}{2}\right)x$ is necessarily linear. Substituting this back into the definition of C :

$$C = \int_0^1 t \left[1 + \left(\frac{7}{12} + \frac{C}{2}\right)t\right] dt = \frac{1}{2} + \left(\frac{7}{12} + \frac{C}{2}\right) \cdot \frac{1}{3}, \quad (33)$$

which gives $C = \frac{5}{6}$, hence

$$\varphi_{\text{exact}}(x) = 1 + x. \quad (34)$$

Applying the interpolation formula with derivatives

Here $K_x(x, t) = t$, $K_{xx}(x, t) = 0$. Because $K(x, t) = xt$ is separable, the kernel moments simplify:

$$A_{\beta, \alpha}^{(0)}(x) = \int_0^1 xt C_{\beta, \alpha}(t) dt = x M_{\beta, \alpha}, \quad (35)$$

$$A_{\beta, \alpha}^{(1)}(x) = \int_0^1 t C_{\beta, \alpha}(t) dt = M_{\beta, \alpha} \quad (\text{independent of } x), \quad (36)$$

$$A_{\beta, \alpha}^{(2)}(x) = 0, \quad (37)$$

where $M_{\beta, \alpha} := \int_0^1 t C_{\beta, \alpha}(t) dt$. Since $f''(x) \equiv 0$, we immediately get $c_\gamma^{(2)} = 0$ for all γ . Setting $S := \sum_\beta [M_{\beta, 0} c_\beta^{(0)} + M_{\beta, 1} c_\beta^{(1)}]$, the remaining unknowns take the form $c_\gamma^{(0)} = 1 + a x_\gamma$, $c_\gamma^{(1)} = a$, where

$$a = \frac{\frac{7}{12} + \lambda S_0}{1 - \lambda(S_1 + S_2)}. \quad (38)$$

The sums S_0 , S_1 , and S_2 are defined as follows:

$$S_0 = \sum_{\beta=0}^N M_{\beta, 0}, \quad (39)$$

$$S_1 = \sum_{\beta=0}^N M_{\beta, 0} x_\beta, \quad (40)$$

$$S_2 = \sum_{\beta=0}^N M_{\beta, 1}. \quad (41)$$

And, we compute for $N = 2$ ($h = 0.5$, nodes $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$). Using the explicit piecewise-polynomial expressions for $C_{\beta, 0}(t)$, $C_{\beta, 1}(t)$, direct integration gives: and

Table 1 Values of $M_{\beta, 0}$ and $M_{\beta, 1}$ for $N = 2$.

β	$M_{\beta, 0}$	$M_{\beta, 1}$
0	1/24	1/192
1	1/4	1/96
2	5/24	-1/64
$\sum$	$S_0 = 1/2$	$S_2 = 0$

$S_1 = \sum_{\beta} M_{\beta,0} x_{\beta} = 0 \cdot \frac{1}{24} + \frac{1}{2} \cdot \frac{1}{4} + 1 \cdot \frac{5}{24} = \frac{1}{3}$. Consistency check: $S_0 = \int_0^1 t \cdot 1 dt = \frac{1}{2}$, since $\sum_{\beta} C_{\beta,0}(t) \equiv 1$ confirmed exactly by the table. Substituting into (38) with $\lambda = \frac{1}{2}$:

$$a = \frac{\frac{7}{12} + \frac{1}{2} \cdot \frac{1}{2}}{1 - \frac{1}{2} \cdot \frac{1}{3}} = \frac{\frac{10}{12}}{\frac{5}{6}} = \frac{5/6}{5/6} = 1. \quad (42)$$

Hence the nodal values are

$$\begin{aligned} c_0^{(0)} = 1, \quad c_1^{(0)} = \frac{3}{2}, \quad c_2^{(0)} = 2, \quad c_0^{(1)} = c_1^{(1)} = c_2^{(1)} = 1, \\ c_0^{(2)} = c_1^{(2)} = c_2^{(2)} = 0. \end{aligned} \quad (43)$$

Reconstructed solution and comparison with the exact solution

Substituting these nodal values into

$$P_{\varphi}(x) = \sum_{\beta=0}^2 \left(C_{\beta,0}(x) c_{\beta}^{(0)} + C_{\beta,1}(x) c_{\beta}^{(1)} + C_{\beta,2}(x) c_{\beta}^{(2)} \right) \quad (44)$$

and simplifying piecewise on $[0, 0.5]$ and $[0.5, 1]$ gives, identically,

$$P_{\varphi}(x) \equiv 1 + x, \quad x \in [0, 1]. \quad (45)$$

Table 2 Comparison table (absolute error $E(x) = |\varphi_{\text{exact}}(x) - P_{\varphi}(x)|$).

x	$\varphi_{\text{exact}}(x) = 1 + x$	$P_{\varphi}(x)$	$E(x) = \varphi_{\text{exact}}(x) - P_{\varphi}(x) $
0.00	1.00	1.00	0
0.25	1.25	1.25	0
0.50	1.50	1.50	0
0.75	1.75	1.75	0
1.00	2.00	2.00	0

$$E_N = \max_{x \in [0,1]} |\varphi_{\text{exact}}(x) - P_{\varphi}(x)| = 0, \quad \text{already at } N = 2. \quad (46)$$

This exact match is not a numerical coincidence: the interpolation formula reproduces every polynomial of degree ≤ 1 exactly, and $\varphi_{\text{exact}}(x) = 1 + x$ is itself linear, so the method introduces zero error for any N . This example thus confirms the correctness of the derived formulas and the implementation.

Example 2. Non-degenerate kernel

To further validate the theoretical convergence order $O(h^3)$ established in Section 6, we consider the following linear Fredholm integral equation of the second kind on $[0, 1]$:

$$\varphi(x) - \int_0^1 e^{xt} \varphi(t) dt = f(x), \quad (47)$$

where $f(x)$ is chosen such that the exact solution is $\varphi_{\text{exact}}(x) = \sin(\pi x)$. Thus, the right-hand side is:

$$f(x) = \sin(\pi x) - \int_0^1 e^{xt} \sin(\pi t) dt. \quad (48)$$

We implemented the proposed collocation scheme for $N = 2, 4, 8, 16, 32$ subintervals. The maximum absolute errors E_N defined by

$$E_N = \max_{0 \leq \gamma \leq N} |\varphi_{\text{exact}}(x_\gamma) - c_\gamma^{(0)}| \quad (49)$$

are summarized in Table 3, along with the experimental order of convergence (EOC), calculated as $\text{EOC} = \log_2(E_N/E_{2N})$.

Table 3 Maximum absolute errors and convergence rates for Example 2.

N	E_N	EOC
2	1.42×10^{-2}	—
4	1.81×10^{-3}	2.97
8	2.30×10^{-4}	2.97
16	2.89×10^{-5}	2.99
32	3.62×10^{-6}	3.00

The numerical results in Table 3 confirm that the proposed algorithm attains the expected convergence rate of $O(h^3)$, validating the robustness of the derivative-based collocation approach for non-degenerate kernels.

8 Conclusion

In this research, we applied an optimal interpolation formula with derivatives constructed in $L_2^{(3)}$ Sobolev space to numerically solve linear Fredholm integral equations of the second kind. By leveraging the compact support property of the interpolation coefficients, we transformed the integral operator into a closed system of linear algebraic equations through a collocation approach that incorporates both nodal function values and their first and second derivatives.

The numerical results confirm the high efficiency and accuracy of the proposed algorithm. We demonstrated that the method achieves a theoretical and practical convergence rate of $O(h^3)$. Specifically, validation using a test problem with a linear exact solution yielded zero absolute error, even for a small number of nodes ($N = 2$), because the interpolation formula exactly reproduces polynomials of degree one. These findings establish the method as a robust computational tool that provides superior stability and precision over traditional quadrature-based approaches by effectively utilizing the solution's smoothness properties.

Declarations

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Author contributions. The author prepared, read, and approved the final manuscript.

Data availability. The datasets and code generated during the current study are available from the author on reasonable request.

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ЧИСЛЕННОЕ РЕШЕНИЕ ЛИНЕЙНЫХ ИНТЕГРАЛЬНЫХ УРАВНЕНИЙ ФРЕДГОЛЬМА ВТОРОГО РОДА НА ОСНОВЕ ОПТИМАЛЬНЫХ ИНТЕРПОЛЯЦИОННЫХ ФОРМУЛ С ПРОИЗВОДНЫМИ

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Предпосылки: Линейные интегральные уравнения Фредгольма второго рода имеют фундаментальное значение при моделировании различных научных и инженерных процессов. Традиционные численные методы решения этих уравнений часто опираются исключительно на узловые значения функции для аппроксимации. Однако эти стандартные подходы не всегда способны точно передать локальное поведение и гладкость неизвестного решения, что может ограничивать их точность и численную устойчивость в сложных сценариях.

Цель: Основная цель данного исследования заключается в разработке новой и высокоточной численной схемы, которая значительно повышает точность аппроксимации. Мы стремимся достичь этого путем эффективного использования как значений функции, так и ее производных высших порядков в дискретных узлах, тем самым в полной мере реализуя присущую решению гладкость.

Метод: Для достижения поставленной цели мы предлагаем инновационный подход, основанный на применении оптимальной интерполяционной формулы с производными, специально построенной в пространстве Соболева. Используя точные коэффициенты этой интерполяционной формулы, мы преобразуем непрерывное интегральное уравнение в конечную замкнутую систему линейных алгебраических уравнений. Данное преобразование осуществляется с помощью строгого метода коллокации. Новизна нашей схемы заключается в том, что, в отличие от стандартных

методов, она активно интегрирует информацию о производных высшего порядка в процесс дискретной аппроксимации.

Результаты: Наш теоретический анализ успешно устанавливает строгие оценки погрешности, подтверждая математическую обоснованность предложенной схемы. Использование информации о производных обеспечивает более высокую скорость сходимости и повышенную численную устойчивость по сравнению с традиционными методами. Кроме того, вычислительные эксперименты на примере с известным точным решением демонстрируют, что наш метод восстанавливает решение с исключительно высокой точностью и минимальной вычислительной ошибкой.

Значимость: Таким образом, данная работа предоставляет надежный и эффективный вычислительный инструмент для сложных математических и инженерных задач. Предложенный подход устанавливает новый стандарт для численного решения интегральных уравнений в тех случаях, когда сохранение информации о локальном поведении и производных критически важно для обеспечения общей точности вычислений.

Ключевые слова: аппроксимация, численное решение, интерполяционная формула, пространство Соболева, интегральное уравнение Фредгольма

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