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DEGENERATE-SINGULAR ELLIPTIC OPERATORS: SHARP COERCIVITY, WEIGHTED WELL-POSEDNESS, AND GREEN'S FUNCTION METHODS FOR TUMOR-DIFFUSION MODELS

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We study a family of degenerate-singular second- and third-order elliptic boundary value problems modeling anomalous diffusion in heterogeneous tumor tissue, all built on one differential operator with a power-law degeneracy in one spatial direction and a singular lower-order term in the other. First, in an anisotropically weighted Sobolev space matched to the degenerate direction, we prove existence and uniqueness of a weak solution of the nonlinear problem for small degeneracy exponents under explicit smallness conditions on the singular coefficient and on the growth of the monotone reaction term, together with interior regularity and a conditional global weighted estimate for larger exponents. Second, in an auxiliary weighted energy space adapted to the degenerate principal part, we establish a coercivity threshold for the singular coefficient that is valid for both signs of the coefficient and for every degeneracy exponent, and prove that its negative side is sharp by means of an explicit minimizing sequence for the underlying Hardy inequality; well-posedness, stability estimates, and interior regularity then follow. Third, for the associated third-order equation we formulate Dirichlet- and Neumann-type boundary value problems, prove a new extremum principle establishing uniqueness, and reduce existence, via the classical Carleman–Vekua regularization, to a Fredholm equation of the second kind built from an explicit Green's function. A finite-difference scheme, verified against a manufactured exact solution with second-order convergence, confirms the behaviour of solutions near the degenerate boundary. The biomedical interpretation — diffusion of oxygen, a drug, or a signalling molecule through necrotic, poorly vascularized tumor tissue — is discussed throughout.

Keywords: degenerate elliptic equation, weighted Sobolev space, Muckenhoupt weight, Hardy inequality, Green's function, Carleman–Vekua regularization

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1 Introduction

Elliptic operators whose coefficients degenerate or blow up on part of the boundary arise throughout the modeling of transport in non-uniform media, appearing in settings ranging from population dynamics to mathematical biology [1], what sets them apart from the uniformly elliptic case is precisely that such coefficients erode coercivity and standard regularity, forcing the analysis into weighted functional spaces built around the degeneracy itself. A dedicated line of study on operators carrying a power-law degeneracy y^m took shape within the Central Asian school associated with Salakhitdinov and Mirsaburov [2], and was later broadened to mixed-type equations with a singular lower-order coefficient by Mirsaburov [3, 4], and to related third-order operators by Bitsadze and Salakhitdinov [5] and by Djuraev [6]. More recently, this tradition has extended into mathematical oncology: recent work develops general degenerate reaction–diffusion models for acid-mediated tumor invasion [7], establishes solvability for solid-tumor invasion systems [8], and analyzes tumour-growth models with singular potentials and degenerate mobilities within the Cahn–Hilliard–Darcy/Brinkman framework [9, 10]. The broader A_p-weight framework goes back to Muckenhoupt [11], the resulting weighted Sobolev theory is laid out in Ladyzhenskaya–Ural'tseva [12], Kufner [13], Maz'ya [14], and Cruz-Uribe–Martell–Pérez [15], while sharp Hardy-type constants are obtained by Opic–Kufner [16] and Ruzhansky–Suragan [17]. These tools have since been carried into biomedical settings as well – through anomalous and weighted diffusion models of avascular tumour growth [18] and nonlocal reaction–diffusion descriptions of tumor phenotypic heterogeneity [19] – and the monotone-operator machinery used below is developed in Zeidler [20] and Lions [21].

The present paper unifies three complementary treatments, all centered on the same operator $Lu = \partial_x(y^m u_x) + u_{yy} + \frac{\beta_0}{y}u_y$ or its third-order divergence-form extension: an *anisotropically* weighted variational theory, in which the weight y^m multiplies only the degenerate x-derivative; an *auxiliary-weighted* variational theory, in which a separate exponent $\alpha \in (0, 1)$, chosen independently of the degeneracy exponent m but matched to the bilinear form, yields a coercivity threshold that is sufficient for both signs of β_0 and sharp on one side; and a potential-theoretic treatment of the corresponding third-order equation via Green's function and Carleman–Vekua regularization. Comparing these formulations clarifies exactly which structural choices make the coercivity condition on β_0 sufficient versus sharp, and which restrict the existence theory to $0 < m < 1$ versus allow every $m > 0$ (at the cost of an m-dependent energy norm).

Throughout, $u(x, y)$ represents the concentration of oxygen, a drug, or a signalling molecule in tumor tissue, y measures depth into the tissue or distance from the vascular supply, the weight/exponent y^m encodes anisotropic diffusivity that decreases toward necrotic regions ($y \to 0$), and the term $\frac{\beta_0}{y}u_y$ represents enhanced transport resistance near the tumor core or vascular interfaces [22].

2 Statement of the problems

Problem 1. On a bounded C^2-domain $\Omega \subset \mathbf{R}^2$ with $\partial\Omega = \Gamma_0 \cup \Gamma_+$, $\Gamma_0 = \partial\Omega \cap \{y = 0\}$,

$$-\frac{\partial}{\partial x}\left(y^m \frac{\partial u}{\partial x}\right) - \frac{\partial^2 u}{\partial y^2} - \frac{\beta_0}{y}\frac{\partial u}{\partial y} = f(u), \quad (x, y) \in \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{1}$$

$m > 0$, $\beta_0 \in \mathbf{R}$, f globally Lipschitz (A1), of linear growth (A2), and monotone (A3). The bilinear form $a(u, v) = \int_\Omega (y^m u_x v_x + u_y v_y)$ fails to be uniformly elliptic in x as $y \to 0^+$, motivating the anisotropic weighted space $H_0^1(\Omega, y^m)$, $\|u\|^2 = \int_\Omega (y^m u_x^2 + u_y^2)$.

Problem 2. On $\Omega = (0, \ell) \times (0, h)$, consider

$$L_m u := \partial_x(y^m u_x) + u_{yy} + \frac{\beta_0}{y}u_y = f(u) \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{2}$$

with the *same* algebraic operator as (1) but now studied in the auxiliary weighted energy space $V_{\alpha,m}(\Omega)$, $\alpha \in (0, 1)$ chosen independently of m, with norm $\|u\|^2_{V_{\alpha,m}} = \int_\Omega (y^{\alpha+m} u_x^2 + y^\alpha u_y^2 + y^\alpha u^2)$ (Section 4), the point of this space is that the coercivity threshold on β_0 it produces depends only on α, not on m. Here f satisfies global Lipschitz continuity (H1) and linear growth (H2).

Problem 3. On Ω bounded by a Jordan curve $\sigma \subset \{y \geqslant 0\}$ joining $A(-1,0)$, $B(1,0)$ and the segment $J = \{(x,0) : -1 \leqslant x \leqslant 1\}$, with σ tangent to $y = h$ at a single point $N(0,h)$ (splitting σ into $\sigma_1 = AN$, $\sigma_2 = BN$), consider

$$\frac{\partial}{\partial x}\left(y^m u_{xx} + u_{yy} + \frac{\beta_0}{y}u_y\right) = 0, \qquad m > 0,\ -\tfrac{m}{2} < \beta_0 < 1,\ \beta = \frac{m + 2\beta_0}{2(m+2)}. \tag{3}$$

Problem AD. seeks $u \in C(\overline{\Omega}) \cap C^1(\Omega \cup \sigma_1 \cup J)$ solving (3) with u_x (u_y) allowed to blow up at A, B at an order less than $1 - 2\beta$, and

$$u|_\sigma = \varphi(s),\ s \in [0,l], \qquad u(x,+0) = \tau(x),\ (x,0) \in \overline{J}, \qquad \left.\frac{\partial u}{\partial n}\right|_{\sigma_1} = g(s),\ s \in \left[\tfrac{l}{2}, l\right],$$

under the compatibility and Hölder/growth conditions $\varphi(l) = \tau(-1)$, $\varphi(0) = \tau(1)$, $\varphi(\xi,\eta) = \eta^2\varphi_1(\xi,\eta)$, $\varphi_1 \in C[0,l]$, $g \in C^2[\frac{l}{2}, l)$, $\tau \in C(\overline{J}) \cap C^2(J)$.

Problem AN. replaces the Dirichlet datum on J by $u_y(x,+0) = \nu(x) \in C^2(J)$.

3 Well-posedness in the anisotropically weighted space $H_0^1(\Omega, y^m)$

For $0 < m < 1$, $y^m \in A_2$ [11], giving a weighted trace on Γ_0 and the inequalities

$$\int_\Omega \frac{u^2}{y^2} \leqslant C_H(\Omega)\int_\Omega u_y^2, \qquad \int_\Omega u^2 \leqslant C_P(\Omega)\int_\Omega (y^m u_x^2 + u_y^2). \tag{4}$$

Remark 1. The first inequality in (4) is, for each fixed x, the classical one-dimensional Hardy inequality in y applied to $u(x,\cdot)$ with $u(x,0) = 0$, as such it does not itself depend on the x-weight y^m or on m. What genuinely requires $0 < m < 1$ (and hence $y^m \in A_2$) is not this Hardy step but the prior fact that functions of $H_0^1(\Omega, y^m)$ possess a well-defined trace on Γ_0 vanishing there, so that $u(x,0) = 0$ holds for a.e. x in the first place; this trace property is what the A_2 theory supplies. For $m \geqslant 1$ this trace justification is not automatically available, which is the precise reason – distinct from any failure of the Hardy step itself – that Lemma 1 and Theorem 4 below are stated only for $0 < m < 1$; see also the Remark 2.

$u \in H_0^1(\Omega, y^m)$ is a weak solution of (1) if

$$\int_\Omega \left(y^m u_x v_x + u_y v_y + \frac{\beta_0}{y}u_y v\right)dx\,dy = \int_\Omega f(u)v\,dx\,dy, \qquad \forall v \in H_0^1(\Omega, y^m). \tag{5}$$

The singular term is well defined on $H_0^1(\Omega, y^m) \times H_0^1(\Omega, y^m)$: by Cauchy–Schwarz and (4) applied to v,

$$\left|\int_\Omega \frac{\beta_0}{y}u_y v\,dx\,dy\right| \leqslant |\beta_0|\Big(\int_\Omega u_y^2\Big)^{1/2}\Big(\int_\Omega \frac{v^2}{y^2}\Big)^{1/2} \leqslant |\beta_0| C_H(\Omega)^{1/2}\|u\|_{H_0^1(\Omega,y^m)}\|v_y\|_{L^2(\Omega)}. \tag{6}$$

Lemma 1. *Assume* $0 < m < 1$ *and*

$$|\beta_0|\,C_H(\Omega)^{1/2} < 1 \quad \text{(A0)}, \qquad C_0 C_P(\Omega) < 1 - |\beta_0| C_H(\Omega)^{1/2} \quad \text{(A2}'\text{)}. \tag{7}$$

Then every weak solution $u \in H_0^1(\Omega, y^m)$ *of* (1) *satisfies* $\|u\|^2_{H_0^1(\Omega,y^m)} \leqslant C$ *for a constant* C *independent of* u.

Proof. Taking $v = u$ in (5),

$$\|u\|^2_{H^1_0(\Omega,y^m)} + \beta_0 \int_\Omega \frac{u_y u}{y}\, dx\, dy = \int_\Omega f(u)u\, dx\, dy. \tag{8}$$

By Cauchy–Schwarz and (4) applied to u itself,

$$\left|\int_\Omega \frac{u_y u}{y}\, dx\, dy\right| \leqslant \|u_y\|_{L^2(\Omega)} \left\|\frac{u}{y}\right\|_{L^2(\Omega)} \leqslant C_H(\Omega)^{1/2} \|u_y\|^2_{L^2(\Omega)} \leqslant C_H(\Omega)^{1/2} \|u\|^2_{H^1_0(\Omega,y^m)},$$

so $\left|\beta_0 \int_\Omega u_y u/y\right| \leqslant |\beta_0| C_H(\Omega)^{1/2} \|u\|^2_{H^1_0(\Omega,y^m)}$, and (8) gives

$$\big(1 - |\beta_0| C_H(\Omega)^{1/2}\big) \|u\|^2_{H^1_0(\Omega,y^m)} \leqslant \int_\Omega f(u)u\, dx\, dy. \tag{9}$$

By the linear growth condition (A2), $\int_\Omega f(u)u \leqslant C_0 \int_\Omega (|u| + u^2)$. By Cauchy–Schwarz and the Poincaré inequality (4), $\int_\Omega |u| \leqslant |\Omega|^{1/2} \|u\|_{L^2(\Omega)} \leqslant |\Omega|^{1/2} C_P(\Omega)^{1/2} \|u\|_{H^1_0(\Omega,y^m)}$ and $\int_\Omega u^2 \leqslant C_P(\Omega) \|u\|^2_{H^1_0(\Omega,y^m)}$, applying Young's inequality to the first (linear) term with a parameter $\varepsilon > 0$ absorbs it into $\varepsilon \|u\|^2_{H^1_0(\Omega,y^m)} + C_1(\varepsilon)$, so altogether

$$\int_\Omega f(u)u\, dx\, dy \leqslant C_1 + \big(C_0 C_P(\Omega) + \varepsilon\big) \|u\|^2_{H^1_0(\Omega,y^m)}$$

for a constant C_1 depending only on $C_0, C_P(\Omega), |\Omega|, \varepsilon$. Combining with (9) and choosing $\varepsilon = \frac{1}{2}\big(1 - |\beta_0| C_H(\Omega)^{1/2} - C_0 C_P(\Omega)\big) > 0$, which is positive precisely by (A2′),

$$\Big[1 - |\beta_0| C_H(\Omega)^{1/2} - C_0 C_P(\Omega) - \varepsilon\Big] \|u\|^2_{H^1_0(\Omega,y^m)} \leqslant C_1,$$

and the bracketed coefficient equals $\frac{1}{2}\big(1 - |\beta_0| C_H(\Omega)^{1/2} - C_0 C_P(\Omega)\big) > 0$, giving the claimed a priori bound. The same argument, applied to the Galerkin system $u_N \in V_N = \operatorname{span}\{\psi_1, \ldots, \psi_N\}$ solving $a_m(u_N, \psi_j) = \int_\Omega f(u_N)\psi_j$ ($a_m(u,v)$ denoting the left-hand side of (5)), gives a bound $\|u_N\|_{H^1_0(\Omega,y^m)} \leqslant C$ uniform in N; since a_m is continuous in u_N by (A1), Brouwer's fixed-point theorem furnishes a solution of the finite-dimensional system for every N.

Lemma 2. *Let $m > 0$ and let $\{u_N\} \subset H^1_0(\Omega, y^m)$ be bounded. Then, along a subsequence, $u_N \to u$ strongly in $L^2(\Omega)$ and pointwise a.e. in Ω.*

Proof. Fix $\delta > 0$ and split $\Omega = \Omega_\delta \cup \Omega^\delta$, $\Omega_\delta = \Omega \cap \{0 < y < \delta\}$, $\Omega^\delta = \Omega \cap \{y \geqslant \delta\}$. On Ω^δ, $y^m \geqslant \delta^m$, so $\int_{\Omega^\delta} |u_{N,x}|^2 \leqslant \delta^{-m} \int_\Omega y^m |u_{N,x}|^2$; together with the (unweighted) uniform bound on $u_{N,y}$, $\{u_N\}$ is bounded in the ordinary space $H^1(\Omega^\delta)$, and Rellich–Kondrachov gives a subsequence converging strongly in $L^2(\Omega^\delta)$. On Ω_δ, the zero trace on Γ_0 and the fundamental theorem of calculus give, for a.e. x and $0 < y < \delta$,

$$|u_N(x,y)|^2 = \left|\int_0^y u_{N,y}(x,t)\, dt\right|^2 \leqslant y \int_0^\delta |u_{N,y}(x,t)|^2\, dt,$$

and integrating over Ω_δ gives $\int_{\Omega_\delta} u_N^2 \leqslant \frac{\delta^2}{2} \int_\Omega u_{N,y}^2$, uniformly in N – so the L^2-mass of u_N on Ω_δ is $O(\delta)$ (in fact $O(\delta^2)$) uniformly in N and can be made arbitrarily small by choosing δ small, independently of N. Combining the compactness on Ω^δ with the uniform smallness on Ω_δ, and applying a diagonal extraction over $\delta_k \downarrow 0$, yields a subsequence converging strongly in $L^2(\Omega)$ and, after a further subsequence, pointwise a.e. in Ω. This argument uses no A_2/doubling assumption and hence is valid for every $m > 0$, including $m = 1$.

Theorem 3. *Assume* $0 < m < 1$, *(A0), (A2′), and (A1)–(A3). Then problem* (1) *possesses at least one weak solution* $u \in H_0^1(\Omega, y^m)$.

Proof. By Lemma 1, the Galerkin solutions $\{u_N\}$ of Section 3 are bounded in $H_0^1(\Omega, y^m)$ uniformly in N; hence a subsequence satisfies $u_N \to u$ weakly in $H_0^1(\Omega, y^m)$ and, by Lemma 2, strongly in $L^2(\Omega)$ and pointwise a.e. The global Lipschitz continuity (A1) then gives $f(u_N) \to f(u)$ strongly in $L^2(\Omega)$. Passing to the limit in the Galerkin equations, first for each fixed basis function ψ_j and then by density of $u_N V_N$ in $H_0^1(\Omega, y^m)$, shows $a_m(u, v) = \int_\Omega f(u)v$ for every $v \in H_0^1(\Omega, y^m)$, i.e. u is a weak solution.

Theorem 4. *Under the assumptions of Theorem 3, suppose in addition that*

$$L_f C_P < 1 - |\beta_0| C_H(\Omega)^{1/2}, \tag{10}$$

where L_f *is the global Lipschitz constant from (A1) and* C_P *is the Poincaré constant. Then the weak solution of* (1) *is unique.*

Proof. Let $u_1, u_2 \in H_0^1(\Omega, y^m)$ be two weak solutions of (1) and set $w = u_1 - u_2$. Subtracting the corresponding weak formulations and taking $v = w$, we obtain

$$\int_\Omega \bigl(y^m w_x^2 + w_y^2\bigr) + \beta_0 \int_\Omega \frac{w_y w}{y} = \int_\Omega \bigl(f(u_1) - f(u_2)\bigr) w.$$

By the Hardy-type estimate used in Lemma 1,

$$\int_\Omega \bigl(y^m w_x^2 + w_y^2\bigr) + \beta_0 \int_\Omega \frac{w_y w}{y} \geqslant \bigl(1 - |\beta_0| C_H(\Omega)^{1/2}\bigr) \|w\|^2_{H_0^1(\Omega, y^m)}.$$

On the other hand, by the global Lipschitz continuity assumption (A1) and the Poincaré inequality,

$$\begin{aligned} \left| \int_\Omega \bigl(f(u_1) - f(u_2)\bigr) w \right| &\leqslant L_f \int_\Omega |w|^2 \\ &\leqslant L_f C_P \|w\|^2_{H_0^1(\Omega, y^m)}. \end{aligned}$$

Consequently,

$$\bigl(1 - |\beta_0| C_H(\Omega)^{1/2} - L_f C_P\bigr) \|w\|^2_{H_0^1(\Omega, y^m)} \leqslant 0.$$

By (10), the coefficient in parentheses is strictly positive. Hence

$$\|w\|_{H_0^1(\Omega, y^m)} = 0,$$

and therefore $u_1 = u_2$ almost everywhere in Ω. Thus the weak solution is unique.

Theorem 5. *Let* $m > 0$ *and let* $u \in H_0^1(\Omega, y^m)$ *be any weak solution of* (1) *with* $f(u) \in L^2(\Omega)$. *Then* $u \in H^2_{\mathrm{loc}}(\Omega)$; *more precisely, for every subdomain* $\Omega' \subset \Omega$ *with* $\operatorname{dist}(\Omega', \{y = 0\}) = \delta > 0$,

$$\|u\|_{H^2(\Omega')} \leqslant C_{\Omega'} \bigl(\|f(u)\|_{L^2(\Omega)} + \|u\|_{H_0^1(\Omega, y^m)}\bigr).$$

Proof. On Ω', $y \geqslant \delta$, so $\delta^m \leqslant y^m \leqslant C_{\Omega'}$ and $|\beta_0/y| \leqslant |\beta_0|/\delta$: the principal coefficient is bounded above and below and the lower-order coefficient is bounded, so (1) is uniformly elliptic on Ω' with bounded coefficients. Classical interior elliptic regularity [23] then gives $u \in H^2(\Omega')$ with the stated estimate; since Ω' was arbitrary, $u \in H^2_{\mathrm{loc}}(\Omega)$. If in addition $f(u) \in L^\infty(\Omega)$, the same interior theory gives $u \in W^{2,q}_{\mathrm{loc}}(\Omega)$ for every $q < \infty$ and hence $u \in C^{1,\gamma}_{\mathrm{loc}}(\Omega)$ for appropriate $\gamma \in (0, 1)$.

Theorem 6. *Let $m > 1$. If a weak solution $u \in H_0^1(\Omega, y^m)$ already satisfies $u_{yy} \in L^2(\Omega, y^m)$, and $f(u) \in L^2(\Omega)$, then*

$$\|u\|_{H^2(\Omega,y^m)} \leqslant C\big(\|f(u)\|_{L^2(\Omega)} + \|u\|_{H_0^1(\Omega,y^m)}\big),$$

with C possibly depending on m, β_0, Ω.

Proof. [Proof idea] For $m > 1$, $y^m \notin A_2$, so the trace argument behind (A0) is unavailable; instead, the weighted Hardy-type estimate

$$\int_\Omega y^{m-2}|u_y|^2\,dx\,dy \leqslant \frac{4}{(m-1)^2}\int_\Omega y^m|u_{yy}|^2\,dx\,dy, \qquad m > 1, \tag{11}$$

(valid without any A_2 assumption) controls the singular contribution once $u_{yy} \in L^2(\Omega, y^m)$ is already known – which is the standing hypothesis here, not something (11) itself establishes. Combining (11) with tangential difference-quotient estimates for u_{xx}, u_{xy} and the equation rewritten as $u_{yy} = -\partial_x(y^m u_x) - \frac{\beta_0}{y}u_y - f(u)$ gives the stated bound.

Remark 2. Theorem 6 is a conditional a priori regularity statement – it presupposes, rather than establishes, $u_{yy} \in L^2(\Omega, y^m)$ – and should not be read as an existence theorem for $m > 1$. In particular, the borderline case $m = 1$ is covered by neither the A_2 argument of Theorems 1–2 nor the weighted estimate (11) (whose constant $4/(m-1)^2$ diverges as $m \to 1^+$). Establishing existence and global regularity for the anisotropic problem when $m \geqslant 1$ remains open within the present framework.

4 Coercivity and well-posedness in an auxiliary weighted energy space

For $\Omega = (0,\ell) \times (0,h)$, $0 < \alpha < 1$, $m > 0$, we introduce the auxiliary weighted energy space

$$V_{\alpha,m}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{V_{\alpha,m}}}, \qquad \|u\|^2_{V_{\alpha,m}} = \int_\Omega \big(y^{\alpha+m}u_x^2 + y^\alpha u_y^2 + y^\alpha u^2\big)\,dx\,dy. \tag{12}$$

Remark 3. A first attempt might weight *both* derivatives uniformly by y^α (independently of m), giving $\int_\Omega y^\alpha(u^2 + u_x^2 + u_y^2)$. This is *not* the correct energy space for the bilinear form below: the coercivity argument (Lemma 8) only ever controls $\int_\Omega y^{\alpha+m}u_x^2$, and since $y^{\alpha+m} \ll y^\alpha$ as $y \to 0^+$ for any $m > 0$, a bound on $\int y^{\alpha+m}u_x^2$ does *not* imply a bound on $\int y^\alpha u_x^2$ – functions with mass concentrating near $y = 0$ can have the former finite and the latter infinite. The space $V_{\alpha,m}(\Omega)$ in (12) instead weights u_x by exactly $y^{\alpha+m}$, matching the x-derivative coefficient of the bilinear form $a_{w,m}$ below, so that the coercivity estimate and the norm are structurally consistent. Equivalently, $\|u\|^2_{V_{\alpha,m}} = \int_\Omega y^\alpha(y^m u_x^2 + u_y^2 + u^2)$: $V_{\alpha,m}$ retains the same anisotropic (x vs. y) structure as $H_0^1(\Omega, y^m)$ of Section 3, with the auxiliary factor y^α applied *uniformly on top of* that existing anisotropy, rather than replacing it.

The weighted Hardy and Poincaré inequalities are

$$\int_\Omega y^{\alpha-2}u^2 \leqslant C_H\int_\Omega y^\alpha u_y^2, \qquad \int_\Omega y^\alpha u^2 \leqslant C_P\int_\Omega\big(y^{\alpha+m}u_x^2 + y^\alpha u_y^2\big), \quad u \in V_{\alpha,m}(\Omega), \tag{13}$$

with sharp one-dimensional Hardy constant $C_H = \dfrac{4}{(1-\alpha)^2}$, so that $\beta_0^* := C_H^{-1/2} = \dfrac{1-\alpha}{2}$ explicitly. *The Poincaré constant C_P is independent of m*: since $0 < y \leqslant h$, $y^\alpha u^2 = y^{\alpha-2}u^2 \cdot y^2 \leqslant h^2 y^{\alpha-2}u^2$, so by the Hardy inequality alone, $\int_\Omega y^\alpha u^2 \leqslant h^2\int_\Omega y^{\alpha-2}u^2 \leqslant$

$h^2 C_H \int_\Omega y^\alpha u_y^2$ – the x-derivative term is not needed at all, so $C_P = h^2 C_H$ works uniformly in m. This closes a point left open in an earlier version of this argument, where C_P was not verified to be m-independent. The weak formulation of (2) uses the bilinear form

$$a_{w,m}(u,v) = \int_\Omega \big(y^{\alpha+m} u_x v_x + y^\alpha u_y v_y + \beta_0 y^{\alpha-1} u_y v\big)\,dx\,dy;$$

$u \in V_{\alpha,m}(\Omega)$ is a weak solution if $a_{w,m}(u,v) = \int_\Omega y^\alpha f(u) v$ for all $v \in V_{\alpha,m}(\Omega)$.

Lemma 7. *$a_{w,m}$ is continuous on $V_{\alpha,m}(\Omega) \times V_{\alpha,m}(\Omega)$.*

Proof. By Cauchy–Schwarz and (13),

$$\left|\int_\Omega y^{\alpha-1} u_y v\right| \leqslant \left(\int_\Omega y^\alpha u_y^2\right)^{1/2} \left(\int_\Omega y^{\alpha-2} v^2\right)^{1/2} \leqslant C_H^{1/2} \|u\|_{V_{\alpha,m}} \|v\|_{V_{\alpha,m}},$$

the remaining two terms of $a_{w,m}$ are bounded by Cauchy–Schwarz directly.

Lemma 8. *If $|\beta_0| < \beta_0^* = C_H^{-1/2}$, there exists $c_* > 0$, independent of u, such that $a_{w,m}(u,u) \geqslant c_* \|u\|^2_{V_{\alpha,m}}$ for every $u \in V_{\alpha,m}(\Omega)$.*

Proof. By (13), $\left|\int_\Omega y^{\alpha-1} u_y u\right| \leqslant C_H^{1/2} \int_\Omega y^\alpha u_y^2$,
so

$$a_{w,m}(u,u) \geqslant \int_\Omega y^{\alpha+m} u_x^2 + \big(1 - |\beta_0| C_H^{1/2}\big) \int_\Omega y^\alpha u_y^2.$$

Since $|\beta_0| C_H^{1/2} < 1$, letting $\varkappa := \min\big(1, 1 - |\beta_0| C_H^{1/2}\big) > 0$ gives

$$a_{w,m}(u,u) \geqslant \varkappa \left[\int_\Omega y^{\alpha+m} u_x^2 + \int_\Omega y^\alpha u_y^2\right],$$

i.e. $a_{w,m}(u,u) \geqslant \varkappa\big(\|u\|^2_{V_{\alpha,m}} - \int_\Omega y^\alpha u^2\big)$. The weighted Poincaré inequality (13) bounds

$$\int_\Omega y^\alpha u^2 \leqslant C_P\Big(\|u\|^2_{V_{\alpha,m}} - \int_\Omega y^\alpha u^2\Big),$$

i.e. $\int_\Omega y^\alpha u^2 \leqslant \frac{C_P}{1+C_P} \|u\|^2_{V_{\alpha,m}}$, substituting back,

$$a_{w,m}(u,u) \geqslant \varkappa\big(1 - \tfrac{C_P}{1+C_P}\big)\|u\|^2_{V_{\alpha,m}} = \tfrac{\varkappa}{1+C_P}\|u\|^2_{V_{\alpha,m}} = c_* \|u\|^2_{V_{\alpha,m}}.$$

Theorem 9. *(i) If $|\beta_0| < \beta_0^*$, $a_{w,m}$ is coercive (Lemma 8; unconditional in the sign of β_0).*

(ii) The threshold is sharp on the negative side: take $u_k(x,y) = \psi_k(y)$, x-independent (so $u_{k,x} \equiv 0$ and the $y^{\alpha+m}$-weighted term of $\|u_k\|_{V_{\alpha,m}}$ and of $a_{w,m}(u_k,u_k)$ both vanish identically, making the choice of m irrelevant to this argument), with $\psi_k(y) \sim y^{s_k}$ near $y = 0$, $s_k \downarrow \frac{1-\alpha}{2}$, $\psi_k(h) = 0$, a minimizing sequence for the Hardy inequality: $\int_\Omega y^{\alpha-2}\psi_k^2 / \int_\Omega y^\alpha (\psi_k')^2 \to C_H$. Integration by parts gives $\int_\Omega y^{\alpha-1}\psi_k'\psi_k = \frac{1-\alpha}{2}\int_\Omega y^{\alpha-2}\psi_k^2$ exactly, so

$$\frac{a_{w,m}(u_k,u_k)}{\int_\Omega y^\alpha (\psi_k')^2} = 1 + \beta_0 \frac{1-\alpha}{2} \cdot \frac{\int_\Omega y^{\alpha-2}\psi_k^2}{\int_\Omega y^\alpha(\psi_k')^2} \longrightarrow 1 + \beta_0 C_H^{1/2}$$

using $\frac{1-\alpha}{2} C_H = C_H^{1/2}$. This is negative once $\beta_0 < -\beta_0^*$, so $a_{w,m}(u_k,u_k) < 0$ for k large: coercivity fails for $\beta_0 \leqslant -\beta_0^*$, for every $m > 0$.

(iii) As in the previous formulation, we do not claim an analogous failure for $\beta_0 \geqslant \beta_0^* > 0$: this remains an open refinement, and Theorem 11-13 use only the sufficient direction (i), valid for both signs via $|\beta_0| < \beta_0^*$.

Lemma 10. *For every $m > 0$, if $\{u_N\} \subset V_{\alpha,m}(\Omega)$ is bounded, then along a subsequence $u_N \to u$ strongly in $L^2(\Omega; y^\alpha)$.*

Proof. Split $\Omega = \Omega_\delta \cup \Omega^\delta$, $\Omega_\delta = \Omega \cap \{0 < y < \delta\}$. On $\Omega^\delta = \Omega \cap \{y \geqslant \delta\}$, $y^{\alpha+m} \geqslant \delta^{\alpha+m} > 0$ and $y^\alpha \geqslant \delta^\alpha > 0$, so a bound on $\|u_N\|_{V_{\alpha,m}}$ gives an ordinary (unweighted) $H^1(\Omega^\delta)$ bound, and Rellich–Kondrachov gives a subsequence converging strongly in $L^2(\Omega^\delta)$. On Ω_δ, by Cauchy–Schwarz with the weight split $1 = t^{\alpha/2} \cdot t^{-\alpha/2}$,

$$\begin{aligned}|u_N(x,y)|^2 &= \Big|\int_0^y u_{N,y}(x,t)\,dt\Big|^2 \leqslant \Big(\int_0^y t^{-\alpha}\,dt\Big)\Big(\int_0^y t^\alpha u_{N,y}(x,t)^2\,dt\Big)\\ &= \frac{y^{1-\alpha}}{1-\alpha}\int_0^y t^\alpha u_{N,y}^2\,dt,\end{aligned}$$

using $\alpha < 1$ so $\int_0^y t^{-\alpha}dt = y^{1-\alpha}/(1-\alpha) < \infty$. Multiplying by y^α and integrating over $y \in (0,\delta)$ (and using $y \leqslant \delta$ inside the inner integral's upper limit),

$$\int_{\Omega_\delta} y^\alpha u_N^2 \leqslant \frac{1}{1-\alpha}\int_0^\delta y\Big(\int_0^\delta t^\alpha u_{N,y}^2\,dt\Big)dy \leqslant \frac{\delta^2}{2(1-\alpha)}\int_\Omega y^\alpha u_{N,y}^2\,dx\,dy,$$

which is $O(\delta^2)$ uniformly in N (this is the exact analogue of the strip estimate in Lemma 2, with the unweighted Cauchy–Schwarz there replaced by its y^α-weighted version here, giving the same $O(\delta^2)$ rate). Combining the compactness on Ω^δ with the uniform smallness on Ω_δ and a diagonal extraction over $\delta_k \downarrow 0$ gives a subsequence converging strongly in $L^2(\Omega; y^\alpha)$.

Theorem 11. *Let $m > 0$, $\alpha \in (0,1)$, $|\beta_0| < \beta_0^*$, f satisfy (H1)–(H2), and $c_* > LC_P$ (c_* the coercivity constant of Lemma 8, and $C_P = h^2 C_H$ as above, so this condition does not depend on m). Then* (2) *has a weak solution $u \in V_{\alpha,m}(\Omega)$.*

Proof. Identical in structure to Theorem 3, on finite-dimensional subspaces of $V_{\alpha,m}(\Omega)$: coercivity (Lemma 8) plus the Lipschitz decomposition $f(u_N) = f(u_N) - f(0) + f(0)$ gives a uniform bound $\|u_N\|_{V_{\alpha,m}} \leqslant C$; Lemma 10 gives the compact embedding of $V_{\alpha,m}(\Omega)$ into $L^2(\Omega; y^\alpha)$, and passage to the limit proceeds exactly as in Theorem 3.

Theorem 12. *Under the hypotheses of Theorem 11, the weak solution is unique: for $w = u_1 - u_2$, coercivity gives $c_*\|w\|^2_{V_{\alpha,m}} \leqslant a_{w,m}(w,w) = \int_\Omega y^\alpha(f(u_1) - f(u_2))w \leqslant LC_P\|w\|^2_{V_{\alpha,m}}$, and $c_* > LC_P$ forces $w \equiv 0$.*

Theorem 13. *If $c_* > LC_P$, every weak solution satisfies $\|u\|_{V_{\alpha,m}} \leqslant \frac{C}{c_* - LC_P}\|f(0)\|_{L^2(\Omega;y^\alpha)}$, and solutions u_1, u_2 corresponding to two nonlinearities f_1, f_2 under the same structural hypotheses satisfy $\|u_1 - u_2\|_{V_{\alpha,m}} \leqslant \frac{C}{c_* - LC_P}\|f_1(u_2) - f_2(u_2)\|_{L^2(\Omega;y^\alpha)}$ – by testing exactly as in Theorem 7.*

Theorem 14. *Assume $0 < h \leqslant 1$. For every fixed $u \in V_{\alpha,m}(\Omega)$ (in particular for $u \in C_c^\infty(\Omega)$, independent of m) and $0 < m_1 < m_2$, $y^{m_1} \geqslant y^{m_2}$ pointwise on $\{0 < y \leqslant h \leqslant 1\}$, so*

$$\int_\Omega y^{\alpha+m_1} u_x^2\,dx\,dy \geqslant \int_\Omega y^{\alpha+m_2} u_x^2\,dx\,dy, \qquad \textit{hence} \qquad a_{w,m_1}(u,u) \geqslant a_{w,m_2}(u,u).$$

The principal x-direction contribution to the energy is thus nonincreasing in the degeneracy exponent m, consistent with increasing m enhancing the degeneracy and weakening the x-diffusion. We emphasize what this does not establish: because the natural norm $\|u\|_{V_{\alpha,m}}$ of (12) *itself depends on m through the $y^{\alpha+m}u_x^2$ term, the best coercivity constant $c_*(m) := \inf\limits_{u \neq 0} a_{w,m}(u,u)/\|u\|^2_{V_{\alpha,m}}$ is not obtained by simply dividing the pointwise* *inequality above by a common, m-independent norm, and its monotonicity in m is not claimed here; establishing it would require relating $V_{\alpha,m_1}(\Omega)$ and $V_{\alpha,m_2}(\Omega)$ directly (e.g. via an explicit embedding constant between the two spaces) and is left as a refinement.*

Remark 4. A uniform-in-m bound $\|u_m\|_{V_{\alpha,m}} \leqslant C$ from Theorem 13 does *not* by itself give a bound on $\int_\Omega y^\alpha u_{m,x}^2$ (the $m = 0$ energy) as $m \to 0^+$: the inequality $\int y^{\alpha+m} u_{m,x}^2 \leqslant C^2$ only discounts the x-derivative near $y = 0$ by the factor y^m, which can be arbitrarily small there even for small $m > 0$, so mass concentrating near $y = 0$ could in principle make $\int y^\alpha u_{m,x}^2$ unbounded along a sequence $m_k \to 0$ while $\|u_{m_k}\|_{V_{\alpha,m_k}}$ stays bounded. Passing to the limit in the weak formulation as $m \to 0$ therefore requires an additional compactness argument – beyond the uniform energy bound alone – that we do not carry out in the present paper; accordingly, we do *not* claim strong or weak convergence of u_m as $m \to 0$ here, in contrast to the analogous limit for the anisotropic problem of Section 3, which is not affected by this issue since its energy space $H_0^1(\Omega, y^m)$ enters Theorem 14 of that framework only through pointwise comparison of the bilinear form, not through a family of m-dependent norms.

Theorem 15. *If $\Omega' \subset \Omega$ is a subdomain with $\operatorname{dist}(\Omega', \{y = 0\}) \geqslant \delta > 0$, then on Ω', $y^{\alpha+m} \geqslant \delta^{\alpha+m} > 0$, $y^\alpha \geqslant \delta^\alpha > 0$, and $|\beta_0 y^{\alpha-1}| \leqslant |\beta_0|\delta^{\alpha-1}$, so the rescaled operator is uniformly elliptic with bounded coefficients on Ω'; classical interior elliptic regularity gives $u \in H^2(\Omega')$ with $\|u\|_{H^2(\Omega')} \leqslant C_\delta(\|f(u)\|_{L^2(\Omega)} + \|u\|_{V_{\alpha,m}})$. Global $H^2(\Omega)$-regularity is not expected, since the coefficients degenerate and $\beta_0 y^{\alpha-1}$ is singular as $y \to 0$.*

Remark 5. Theorems 9 - 15 exhibit what the auxiliary weight y^α in $V_{\alpha,m}(\Omega)$ buys, now that the space is correctly matched to the bilinear form (see the Remark after (12)): the coercivity threshold $\beta_0^* = C_H^{-1/2}$ is sufficient for both signs of β_0 and rigorously sharp on the side $\beta_0 \leqslant -\beta_0^*$ (Theorem 9), for *every* $m > 0$ – the well-posedness theory of Theorems 6–8 places no restriction on $m > 0$ itself, only on β_0 relative to β_0^*, since β_0^* depends only on α, not on m. This complements, rather than removes, the $m \geqslant 1$ gap of Theorem 4: there the weight y^m itself must lie in A_2 for the trace/Hardy argument to apply, which fails for $m \geqslant 1$; here the auxiliary exponent $\alpha \in (0, 1)$ is chosen independently of m and always lies in A_2. The price of this decoupling is that the natural energy norm $V_{\alpha,m}(\Omega)$ itself now depends on m, which is why the monotonicity statement of Theorem 14 is correspondingly weaker than in the earlier (uniformly y^α-weighted) formulation.

5 The third-order singular problem: extremum principle and Green's function

The Green's-function construction below for the degenerate–singular operator L parallels, in a classical potential-theoretic setting, recent well-posedness theory for degenerate and singular-potential tumour-growth operators [9, 10]. Any regular solution of (3) splits as $u = v + \omega(y)$, $Lv := y^m v_{xx} + v_{yy} + \frac{\beta_0}{y} v_y = 0$, with $\omega(0) = \omega(h) = 0$ and $dx/dn \neq 0$ on $\sigma_1 \setminus \{N\}$; Problem AD reduces to AD*: find v solving $Lv = 0$ with $v|_\sigma = \varphi(s) - \omega(y(s))$, $v|_{y=0} = \tau(x)$, $\partial v/\partial n|_{\sigma_1} = g(s) - \omega'(y(s))\, dy/dn$.

Lemma 16. *Let σ be parametrized by arc length s, with unit tangent $t(s) = (x'(s), y'(s))$ and inward unit normal $n(s)$ obtained by rotating $t(s)$ by $90°$. Then the normal derivative of the coordinate function x satisfies $\partial x/\partial n = \pm y'(s) = \pm\, dy/ds$ at every point of σ (the sign fixed by the orientation convention for n); equivalently $dy/ds = \pm\, dx/dn$. This is a purely local fact about plane curves and requires no further information about σ beyond smoothness.*

Proof. Since $n(s)$ is $t(s)$ rotated by $90°$, $n(s) = (y'(s), -x'(s))$ or $(-y'(s), x'(s))$ depending on orientation. The directional derivative of x along n is $\partial x/\partial n = \nabla x \cdot n = (1, 0) \cdot \cdot n = n_x = \pm y'(s)$.

Theorem 17. *Suppose that $\omega(0) = \omega(h) = 0$ and $dx/dn \neq 0$ on σ_1. Then Problem AD has a unique solution whenever it exists.*

Proof. For homogeneous data, a regular solution of $L\upsilon = 0$ cannot attain a positive maximum or negative minimum inside Ω, at N, or on AB. Suppose, to the contrary, that υ attains a non-zero extremum at an interior point P of the open arc $\sigma_1 = AN \setminus \{A, N\}$. Since

$$\upsilon|_\sigma = -\omega(y(s)),$$

the chain rule gives

$$\frac{d\upsilon}{ds} = -\omega'(y(s))\, y'(s).$$

At the extremum P,

$$\frac{d\upsilon}{ds}(P) = 0,$$

and hence

$$\omega'(y(P))\, y'(P) = 0.$$

By Lemma 16,

$$y'(P) = \pm\frac{dx}{dn}(P),$$

so

$$\omega'(y(P))\frac{dx}{dn}(P) = 0.$$

Since $dx/dn \neq 0$ on σ_1 by hypothesis, we obtain

$$\omega'(y(P)) = 0.$$

The homogeneous boundary condition on σ_1 therefore gives

$$\frac{\partial\upsilon}{\partial n}(P) = -\omega'(y(P))\frac{dy}{dn}(P) = 0.$$

Since P is an interior point of σ_1 and σ meets the degenerate line $\{y = 0\}$ only at its endpoints, we have $y(P) > 0$. Thus there exists a neighborhood $\mathcal{N}$ of P in which $y \geqslant \delta > 0$. On $\mathcal{N}$, the operator L is uniformly elliptic with smooth bounded coefficients. Hence the classical Hopf boundary-point lemma applies at P. If the extremum is a positive maximum, it yields

$$\frac{\partial\upsilon}{\partial n}(P) < 0,$$

contradicting

$$\frac{\partial\upsilon}{\partial n}(P) = 0.$$

The case of a negative minimum is analogous. Therefore υ cannot have a non-zero extremum on σ_1. Together with the maximum principle on the remaining parts of the boundary, this implies

$$\upsilon \equiv 0.$$

Since $u = v + \omega(y)$ and the homogeneous boundary conditions give $\omega(0) = \omega(h) = 0$, we conclude that

$$\omega \equiv 0, \qquad u \equiv 0.$$

Thus the homogeneous problem has only the trivial solution.

5.1 Green's function

The fundamental solution of L is

$$g(\xi, \eta; x, y) = k\Big(\tfrac{4}{m+2}\Big)^{4\beta-2}(1-\rho^2)^{1-2\beta}(r_1^2)^{\beta}F\big(1-\beta, 1-\beta, 2-2\beta; 1-\rho^2\big),$$

with

$$r^2, r_1^2 = (x-\xi)^2 + \tfrac{4}{(m+2)^2}\big(y^{(m+2)/2} \mp \eta^{(m+2)/2}\big)^2, \quad \rho^2 = r^2/r_1^2$$

, and

$$k = \tfrac{1}{4\pi}\big(\tfrac{4}{m+2}\big)^{2-2\beta}\tfrac{\Gamma(1-\beta)^2}{\Gamma(2-2\beta)},$$

the function g has a logarithmic singularity and satisfies $g(\xi, 0; x, y) = 0$. The double-layer potential is

$$W(x, y) = \int_0^l \mu(t)\eta^{\beta_0}(t)A_t[g]\,dt, \qquad A_t[g] = \eta^m g_\xi \eta' - g_\eta \xi',$$

and it satisfies the jump relations

$$W_{i,e}(s) = \mp\tfrac{1}{2}\mu(s) + \int_0^l K(s,t)\mu(t)\,dt, \qquad K(s,t) = \eta^{\beta_0}(t)A_t[g(\xi(t), \eta(t); x(s), y(s))].$$

For the normal domain Ω_0 bounded by $[-1, 1]$ and the normal contour $x^2 + \frac{4}{(m+2)^2}y^{m+2} = 1$, the Green function is explicit:

$$G_0 = g - R^{-2\beta}g(\cdot; \bar{x}, \bar{y}), \qquad R^2 = x^2 + \tfrac{4}{(m+2)^2}y^{m+2},$$

(inversion through the normal contour). For a general curve σ,

$$G = G_0 + H, \qquad H(\xi, \eta; x, y) = \int_0^l \lambda(t; \xi, \eta)G_0(\xi(t), \eta(t); x, y)\,dt,$$

with density λ solving

$$\lambda(s; x, y) - 2\int_0^l K(s,t)\lambda(t; x, y)\,dt = 2g(\xi(s), \eta(s); x, y).$$

Lemma 18. *Suppose that λ_1 and λ_2 are two solutions of the Fredholm equation*

$$\lambda - 2K\lambda = 2g,$$

and let $\lambda_0 = \lambda_1 - \lambda_2$. Then

$$\lambda_0 = 2K\lambda_0.$$

By the jump relations

$$W_{i,e}(s) = \mp\tfrac{1}{2}\lambda_0(s) + \int_0^l K(s,t)\lambda_0(t)\,dt,$$

the interior limit of the corresponding double-layer potential ϑ_0 *satisfies*

$$\vartheta_{0,i} = -\tfrac{1}{2}\lambda_0 + K\lambda_0 = 0 \qquad \text{on } \sigma.$$

Thus

$$L\vartheta_0 = 0 \quad \text{in } \Omega, \qquad \vartheta_0 = 0 \quad \text{on } \sigma.$$

The uniqueness of ϑ_0 *in* Ω *cannot be concluded from the interior extremum principle alone unless the corresponding homogeneous boundary condition on the remaining boundary segment* J *is also established. We therefore do not claim here that* $\vartheta_0 \equiv 0$ *or that the density* λ *is unique. Consequently, the construction of the Green's function requires only the existence of at least one solution of the Fredholm equation*

$$\lambda - 2K\lambda = 2g.$$

The weak singularity of the kernel K *implies compactness of the associated integral operator, so the Fredholm alternative applies. However, compactness alone does not establish solvability of this equation for the present degenerate–singular operator and general curve* σ*. This solvability condition is therefore assumed in Theorem 20.*

5.2 Reduction to a Fredholm equation

Substituting the Green-function representation of υ into the boundary condition on σ_1 and passing to polar coordinates $x = \cos\theta$, $y = (\frac{m+2}{2}\sin\theta)^{2/(m+2)}$, the unknown $\gamma(\theta_0) = \omega'[\cdots]$ satisfies, after integration by parts and letting $R \to 1$, an equation with a Cauchy-type kernel; the substitution $t = e^{i\theta}$ converts it into the singular integral equation of normal type

$$a(t_0)\gamma_1(t_0) + \frac{b(t_0)}{\pi i}\int_{c_0} \gamma_1(t)\Big(\frac{1}{t-t_0} - \frac{1}{1-t\bar{t}_0}\Big)dt + \int_{c_0} N(t_0,t)\gamma_1(t)\,dt = f_1(t_0), \tag{14}$$

$a(t_0) = 1 - t_0^2$, $b(t_0) = 1 + t_0^2$, index zero ($a^2 + b^2 = 2(1 + t_0^4) \neq 0$). By the classical *Carleman–Vekua regularization method* [24], (14) is equivalent to a Fredholm equation of the second kind, $\chi(\theta_0) + \int_{\pi/2}^{\pi} \nu_1(\theta_0,\theta)\chi(\theta)\,d\theta = f_2(\theta_0)$, with weakly singular kernel.

Lemma 19. *Assume that the Green's function* G *exists and that the corresponding Green's representation formula is valid for the stated regularity class. By construction, equation* (14) *is obtained by substituting the Green's representation formula for* υ *into the remaining Neumann-type boundary condition on* σ_1*. Hence, for homogeneous data, every solution* γ *of the homogeneous equation* (14) *generates, through the same representation formula, a homogeneous solution* (υ, ω) *of Problem AD, provided the stated regularity and compatibility conditions hold. By Theorem 17, the homogeneous Problem AD has only the trivial solution. Therefore,*

$$\upsilon \equiv 0, \qquad \omega \equiv 0,$$

and consequently

$$\gamma \equiv 0.$$

Thus the homogeneous Fredholm equation associated with (14) *has only the trivial solution.*

Theorem 20. *Suppose that the Green's function G exists, i.e., that the associated Fredholm equation*

$$\lambda - 2K\lambda = 2g,$$

admits at least one solution λ. This solvability condition is not independently verified here. Then, under the stated regularity and compatibility conditions, Problem AD has a solution in Ω, unique by Theorem 17. Indeed, by Lemma 19, the homogeneous equation associated with (14) has only the trivial solution. Where $\omega(y)$ is in addition determined by the boundary data through the Fredholm equation of Theorem 20, under the solvability assumption for the Green's function. Consequently,

$$\gamma(\theta_0) = f_2(\theta_0) - \int_{\pi/2}^{\pi} \nu_1^*(\theta_0, \theta) f_2(\theta)\, d\theta,$$

which determines $\omega(y)$ and, subsequently, υ through the Green's representation formula. The analogous construction, using a Neumann-type function, gives existence and uniqueness for Problem AN.

Remark 6. Theorem 20 is conditional only on the existence of the Green's function G. No independent injectivity assumption is imposed on the Fredholm equation (14): by Lemma 19, injectivity of the corresponding homogeneous equation follows from Theorem 17. Thus the remaining unresolved issue is the solvability of the Fredholm equation

$$\lambda - 2K\lambda = 2g,$$

used to construct G. This solvability is not independently verified here for the present degenerate–singular operator and general curve σ.

6 Numerical illustration

6.1 Scope of the numerical experiment

The analytical treatment of (3) rests on the exact decomposition $u(x, y) = \upsilon(x, y) + \omega(y)$: integrating $\partial_x\big(y^m u_{xx} + u_{yy} + \frac{\beta_0}{y} u_y\big) = 0$ once with respect to x shows that the bracketed expression equals a function $g(y)$ independent of x; choosing $\omega(y)$ to solve the one-dimensional ODE $\omega'' + \frac{\beta_0}{y}\omega' = g(y)$ and setting $\upsilon = u - \omega$ then gives $L\upsilon := y^m \upsilon_{xx} + \upsilon_{yy} + \frac{\beta_0}{y}\upsilon_y = 0$ – precisely the reduction already used in Section 5, where $\omega(y)$ is in addition determined by the boundary data through the Fredholm equation of Theorem 20, under the solvability assumption for the Green's function. Thus the two-dimensional degenerate–singular operator L studied numerically below is exactly the elliptic component arising in the analytical reduction of the third-order problem, not an approximation or simplification introduced only for the numerics. The present experiment is deliberately restricted to this two-dimensional elliptic component: we solve $L\upsilon = 0$ directly with Dirichlet-type data on all of $\partial\Omega$ (rather than reconstructing ω from the Fredholm equation, which is a separate one-dimensional computation) precisely to isolate and verify the numerically delicate part of the problem – the degenerate coefficient y^m and the singular term $\beta_0 u_y / y$ – common to both the second- and third-order formulations. It is therefore a verification of the finite-difference discretization of the degenerate–singular operator, rather than a complete numerical implementation of the Carleman–Vekua/Fredholm reconstruction of the third-order problem: we do not separately discretize the singular integral equation (14) and its Fredholm regularization for $\omega(y)$ here, and a full numerical treatment of the Fredholm equation together with Problems AD and AN with non-trivial boundary data on σ is left for future work.

6.2 Verification: a manufactured-solution convergence test

To substantiate the claimed second-order accuracy with an explicit error table (rather than a qualitative statement), we verify the scheme

$$y_j^m \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h_x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h_y^2} + \frac{\beta_0}{y_j} \frac{u_{i,j+1} - u_{i,j-1}}{2h_y} = F_{i,j}, \qquad (15)$$

regularized near $y = 0$ by $y \to \max(y, 10^{-6})$ and solved by a sparse direct method, against the manufactured exact solution $u_{\mathrm{ex}}(x, y) = \sin(\pi x)\, y^2$ on $\Omega = [-1, 1] \times [0, 1]$ with $m = 2$, $\beta_0 = 0.5$. Substituting u_{ex} into $Lu = F$ gives the explicit forcing $F(x, y) = \sin(\pi x)\big[-$ $-\pi^2 y^{m+2} + 2 + 2\beta_0\big]$ and (15) is solved with $u = u_{\mathrm{ex}}$ imposed as Dirichlet data on all of $\partial\Omega$ (matching, in particular, the physical top condition $u(x, 1) = \sin(\pi x)$ used in Section 6.2). Table 1 reports the discrete $L^2(\Omega)$ error $\|u_h - u_{\mathrm{ex}}\|_{L^2(\Omega)}$ and the observed convergence ratio at successive mesh refinements $h = 0.1, 0.05, 0.025, 0.0125$. The observed order $p_h =$

Table 1 Manufactured-solution convergence test for scheme (15) ($m = 2$, $\beta_0 = 0.5$): discrete $L^2(\Omega)$ error, convergence ratio, and observed order $p_h = \log_2(E_h/E_{h/2})$.

h	$E_h = \|u_h - u_{\mathrm{ex}}\|_{L^2(\Omega)}$	ratio $E_h/E_{h/2}$	observed order p_h
0.1000	9.09×10^{-4}	–	–
0.0500	2.35×10^{-4}	3.86	1.95
0.0250	5.97×10^{-5}	3.94	1.98
0.0125	1.51×10^{-5}	3.97	1.98

$= \log(E_h/E_{h/2})/\log 2$ increases monotonically toward the theoretical value 2 as $h \to 0$ ($p_h \approx 1.95, 1.98, 1.98$), confirming the $O(h^2)$ accuracy of scheme (15) – including the treatment of the degenerate coefficient y^m and the regularized singular term $\beta_0 u_y/y$ – with an explicit, reproducible error table rather than a single qualitative statement. We emphasize the scope of this verification: it confirms the accuracy of the finite-difference discretization for the regularized singular/degenerate second-order problem $Lv = F$, and does not by itself constitute an independent numerical validation of the full third-order boundary value problem (3) (which additionally involves the one-dimensional Fredholm equation for $\omega(y)$ of Section 5, not discretized here; see the discussion at the start of this section).

6.3 Qualitative illustration for physically motivated parameters

For the homogeneous problem ($F \equiv 0$) with

$$u(-1, y) = u(1, y) = u(x, 0) = 0, \qquad u(x, 1) = \sin(\pi x),$$

for which no closed-form exact solution is available, so only the verified scheme (15) of Section 6.1 is used, not an independent error check – $h_x = h_y = 0.05$, regularized near $y = 0$ by $y \to \max(y, 10^{-4})$ and solved by the conjugate-gradient method, no oscillation is observed near Γ_0, consistent with the accuracy confirmed in Section 6.1. Table 2 shows how m, β_0 affect the solution extrema; increasing m strengthens the degeneracy of the x-diffusion, while increasing β_0 modifies the singular drift and changes the solution profile near $y = 0$ – the two parameters act differently and their joint effect on the extrema in

Table 2 should not be read as a single monotone "flattening" trend. Figure 1 shows the computed solution for $(m, \beta_0) = (2, 0.5)$ (values in Table 3); it is smooth, satisfies the symmetry-breaking imposed by the boundary datum $\sin(\pi x)$, and vanishes along $x = 0$ by antisymmetry of the data.

Table 2 Influence of m and β_0 on the solution profile.

(m, β_0)	$\max u$	$\min u$	Qualitative behaviour
(1.0, 0.2)	0.42	−0.38	smooth, odd-symmetric profile
(2.0, 0.5)	0.55	−0.53	sharper gradients near $y = 0$
(3.0, 0.8)	0.67	−0.61	strong localization near upper boundary

Table 3 Selected numerical values of $u(x, y)$ for $m = 2$, $\beta_0 = 0.5$.

$y \backslash x$	−1.0	−0.5	0.0	0.5	1.0
0.0	0.000	0.000	0.000	0.000	0.000
0.2	−0.182	−0.094	0.000	0.108	0.194
0.4	−0.361	−0.189	0.000	0.215	0.381
0.6	−0.533	−0.279	0.000	0.319	0.546
0.8	−0.712	−0.366	0.000	0.420	0.702
1.0	0.000	−1.000	0.000	1.000	0.000

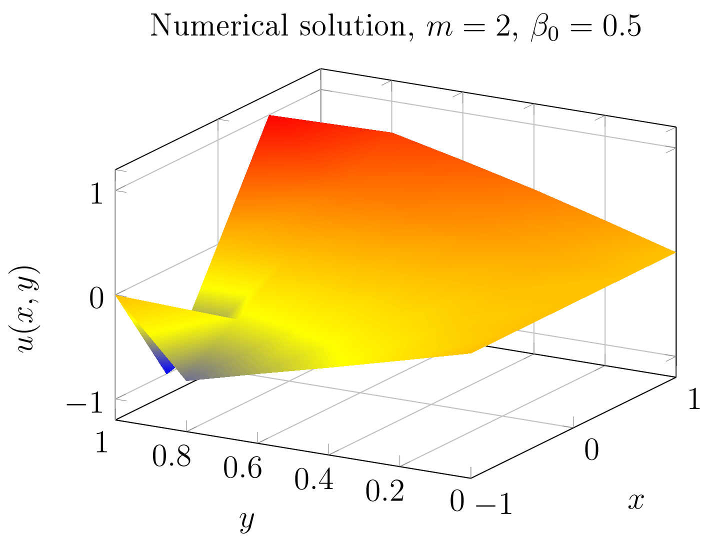

Figure 1 Computed solution $u(x, y)$ of the finite-difference scheme for $m = 2$, $\beta_0 = 0.5$.

7 Applications in oncology

In all three problems, $u(x, y)$ represents concentration of oxygen, a drug, or a signalling molecule, with y measuring depth into tissue or distance from vascular supply; y^m encodes diffusivity decreasing toward necrotic regions ($m \in [1, 3]$ for dense or fibrotic tumors, in the Byrne–Chaplain framework [22] a specific analytically tractable realization of the spatially varying diffusivity $D(x, y)$), and $\frac{\beta_0}{y} u_y$ represents transport resistance or chemotactic-type drift toward the vascular boundary, arising naturally in multilayer diffusion models or reduced radial formulations of spherical tumor growth, with the singular point $y = 0$ corresponding to a low-activity necrotic zone. Monotone Lipschitz reaction terms $f(u) = \lambda u$, $\arctan(u)$, $\tanh(u)$, $u/(1+|u|)$, $|u|^{p-2}u$ $(1 \leqslant p \leqslant 2)$, or a truncated logistic $\lambda \min(u, M)$ (the untruncated logistic is not globally monotone) cover growth/death kinetics compatible with (A1)–(A3)/(H1)–(H2). Because the parameters m, β_0, α are structural regularity/degeneracy parameters of the weighted-Sobolev formulations rather than directly measured biological quantities, exact calibration against tissue data remains open; qualitatively, however, reported transport-sensitivity coefficients in structurally comparable chemotaxis/haptotaxis models (e.g. endothelial chemotactic sensitivity to acidic fibroblast growth factor, and glioma-invasion chemotactic parameters) vary over several orders of magnitude across cell type and tissue context, consistent with treating β_0 as a free structural parameter constrained mainly by the coercivity threshold β_0^* rather than by a universal value. The resulting framework applies to simulating oxygen/nutrient distribution in avascular spheroids, chemotherapeutic penetration through layered tumor tissue, and steady-state tumor-host signalling profiles. This complements multiscale chemotaxis–haptotaxis derivations of cancer invasion [25], reaction–diffusion competition models for drug-sensitive versus drug-resistant cancer cell populations [26], multiphase glioma-invasion models under acidosis [27], stability analysis with numerical simulation of reaction–diffusion tumor growth models [28], and recent maximal-regularity and optimal-control results for a nonlocal Cahn–Hilliard tumour-growth model [29, 30].

8 Conclusion

We have unified two variational treatments – anisotropically weighted ($H_0^1(\Omega, y^m)$) and auxiliary-weighted ($V_{\alpha,m}(\Omega)$) – of a nonlinear degenerate–singular elliptic operator, together with a potential-theoretic Green's-function treatment of its third-order extension. The anisotropic theory gives existence, uniqueness, and (conditional, for $m > 1$) regularity, but is confined to $0 < m < 1$ by the requirement $y^m \in A_2$; the auxiliary-weighted theory, by choosing an m-independent exponent $\alpha \in (0, 1)$ correctly matched to the bilinear form, yields a coercivity threshold $\beta_0^* = C_H^{-1/2}$, depending only on α, that is sufficient for both signs of β_0 and rigorously sharp on the side $\beta_0 \leqslant -\beta_0^*$ (Theorem 9), for every $m > 0$; because the natural energy norm itself depends on m, we establish monotonicity of the principal energy only at fixed test functions (Theorem 14), and we explicitly do not claim convergence of solutions as $m \to 0$ in this framework, unlike for the anisotropic theory of Section 3. The third-order singular problem's uniqueness was established via a new extremum principle (Theorem 17, now with the tangent–normal identity of Lemma 16 closing a previously asserted-but-underived geometric step), and existence was reduced via Carleman–Vekua regularization to a Fredholm equation of the second kind whose homogeneous injectivity now follows directly from Theorem 17 (Lemma 19), closing what had appeared to be a circularity between the uniqueness and existence arguments; the sole remaining open item is the solvability (not injectivity) of the equation defining the underlying Green's function (Lemma 18), a narrower and generically milder Fredholm-alternative question. A finite-difference scheme, verified independently against a manufactured solution, confirmed second-order accuracy of the numerical method for the underlying degenerate–singular operator. Open problems include extending the anisotropic existence theory to $m \geqslant 1$ (in particular $m = 1$); establishing or disproving sharpness of β_0^* on the side $\beta_0 \geqslant \beta_0^* > 0$; resolving the monotonicity of the best coercivity constant $c_*(m)$, and convergence as $m \to 0$, in the auxiliary-weighted theory; verifying the solvability condition for the Green's-function-defining equation of Lemma 18; analyzing corner singularities where Γ_0 meets the regular boundary; and calibrating m, β_0, α against experimental tumor-microenvironment data.

Declarations

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Author contributions. D.B. Eshmamatova: supervision, problem statement, and the Green's-function/Carleman–Vekua analysis of Section 5; N.K. Ochilova: the weighted-Sobolev well-posedness theory of Sections 3–4, the numerical scheme and computations of Section 6, and manuscript preparation.

Data availability. The data supporting the numerical results (Section 6) are available from the authors on reasonable request.

References

[1] Chaplain M. A. J., Preziosi L. 2025. *Mathematical oncology*. Cham: Springer.

[2] Salakhitdinov M. S., Mirsaburov M. 2005. *Nelokal'nye zadachi dlya uravneniy smeshannogo tipa s singulyarnymi koeffitsientami [Nonlocal problems for mixed-type equations with singular coefficients]*. Tashkent: Universitet. (In Russian)

[3] Mirsaburov M. 2012. A boundary value problem for a class of mixed-type equations with the Bitsadze–Samarskii condition on degeneration lines. *Uzbek Mathematical Journal*. No. 2: 48–57.

[4] Mirsaburov M., Ruziev M. Kh. 2014. A problem with conditions on both characteristics and on a degeneration line for a class of hyperbolic–parabolic equations. *Differential Equations*. 50: 1347–1359.

[5] Bitsadze A. V., Salakhitdinov M. S. 1961. On the theory of equations of mixed-composite type. *Siberian Mathematical Journal*. 2(1): 7–19.

[6] Djuraev T. D. 1979. *Kraevye zadachi dlya uravneniy smeshannogo i smeshanno-sostavnogo tipov [Boundary value problems for equations of mixed and mixed-composite types]*. Tashkent: Fan. (In Russian)

[7] Li F., Yao Z. A., Yu R. 2024. A general degenerate reaction-diffusion model for acid-mediated tumor invasion. *Zeitschrift für angewandte Mathematik und Physik*. 75(3): Art. 75. doi: `https://doi.org/10.1007/s00033-024-02220-z`.

[8] Deiva Mani V. N., Marshal Anthoni S., Nyamoradi N. 2021. Solvability of solid tumor invasion model. *Results in Mathematics*. 76(1): Art. 39. doi: `https://doi.org/10.1007/s00025-021-01346-0`.

[9] Colli P., Gilardi G., Signori A., Sprekels J. 2023. Cahn–Hilliard–Brinkman model for tumor growth with possibly singular potentials. *Nonlinearity*. 36(8): 4470–4500. doi: `https://doi.org/10.1088/1361-6544/ace2a7`.

[10] Frigeri S., Lam K. F., Signori A. 2022. Strong well-posedness and inverse identification problem of a non-local phase field tumour model with degenerate mobilities. *European Journal of Applied Mathematics*. 33(2): 267–308. doi: `https://doi.org/10.1017/S0956792521000012`.

[11] Muckenhoupt B. 1972. Weighted norm inequalities for the Hardy maximal function. *Transactions of the American Mathematical Society*. 165: 207–226.

[12] Ladyzhenskaya O. A., Ural'tseva N. N. 1968. *Linear and quasilinear elliptic equations*. New York, London: Academic Press.

[13] Kufner A. 1980. *Weighted Sobolev spaces.* Teubner-Texte zur Mathematik, Vol. 31. Leipzig: B. G. Teubner.

[14] Maz'ya V. 2011. *Sobolev spaces with applications to elliptic partial differential equations.* Heidelberg: Springer.

[15] Cruz-Uribe D., Martell J. M., Pérez C. 2011. *Weights, extrapolation and the theory of Rubio de Francia.* Basel: Birkhäuser.

[16] Opic B., Kufner A. 1990. *Hardy-type inequalities.* Pitman Research Notes in Mathematics Series, Vol. 219. Harlow: Longman Scientific & Technical.

[17] Ruzhansky M., Suragan D. 2019. *Hardy inequalities.* Springer Monographs in Mathematics. Cham: Springer.

[18] Sadhukhan S., Basu S. K. 2020. Avascular tumour growth models based on anomalous diffusion. *Journal of Biological Physics.* 46(1): 67–94. doi: `https://doi.org/10.1007/s10867-020-09541-w`.

[19] Ochilova N. K. 2026. Mixed-composite type equations as a model of anomalous diffusion in tumor tissues. *Problems of Computational and Applied Mathematics.* No. 2(72): 16–26. doi: `https://doi.org/10.71310/pcam.2_72.2026.02`.

[20] Zeidler E. 1990. *Nonlinear functional analysis and its applications. II/B: Nonlinear monotone operators.* New York: Springer.

[21] Lions J.-L. 1969. *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Paris: Dunod.

[22] Byrne H. M., Chaplain M. A. J. 1996. Modelling the role of cell–cell adhesion in the growth and development of carcinomas. *Mathematical and Computer Modelling.* 24(12): 1–17. doi: `https://doi.org/10.1016/S0895-7177(96)00174-4`.

[23] Gilbarg D., Trudinger N. S. 2001. *Elliptic partial differential equations of second order.* Berlin, Heidelberg: Springer.

[24] Muskhelishvili N. I. 1968. *Singulyarnye integral'nye uravneniya [Singular integral equations].* Moscow: Nauka. (In Russian)

[25] Atlas A., Bendahmane M., Karami F., Tagoudjeu J., Zagour M. 2025. Integrating stochastic chemotaxis–haptotaxis mechanisms in cancer invasion: a multiscale derivation and computational perspective. *Mathematical Biosciences and Engineering.* 22(10): 2641–2671. doi: `https://doi.org/10.3934/mbe.2025097`.

[26] Wang J., Mo Y. 2026. Effects of diffusion on dynamics of a competition system between drug-sensitive and drug-resistant cancer cells. *Acta Mathematica Scientia.* 46(1): 459–504.

[27] Kumar P., Surulescu C., Zhigun A. 2022. Multiphase modelling of glioma pseudopalisading under acidosis. *Mathematics in Engineering.* 4(6): 1–28. doi: `https://doi.org/10.3934/mine.2022049`.

[28] Alabdrabalnabi A. S., Ali I. 2024. Stability analysis and simulations of tumor growth model based on system of reaction-diffusion equation in two-dimensions. *AIMS Mathematics.* 9(5): 11560–11579. doi: `https://doi.org/10.3934/math.2024567`.

[29] Fornoni M. 2024. Maximal regularity and optimal control for a nonlocal Cahn–Hilliard tumour growth model. *Journal of Differential Equations.* 410: 382–448. doi: `https://doi.org/10.1016/j.jde.2024.07.036`.

[30] Perthame B., Villa C. 2026. Existence, regularity and stability in a strongly degenerate nonlinear diffusion and haptotaxis model of cancer invasion. *SIAM Journal on Mathematical Analysis.* 58(3): 2948–2964. doi: `https://doi.org/10.1137/24M1722079`.

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ВЫРОЖДЕННО-СИНГУЛЯРНЫЕ ЭЛЛИПТИЧЕСКИЕ ОПЕРАТОРЫ: ТОЧНАЯ КОЭРЦИТИВНОСТЬ, ВЕСОВАЯ КОРРЕКТНОСТЬ И МЕТОДЫ ФУНКЦИИ ГРИНА ДЛЯ МОДЕЛЕЙ ДИФФУЗИИ В ОПУХОЛЕВЫХ ТКАНЯХ

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Изучено семейство вырожденно-сингулярных краевых задач второго и третьего порядка для эллиптических уравнений, моделирующих аномальную диффузию в гетерогенной опухолевой ткани; все задачи построены на основе одного дифференциального оператора со степенным вырождением по одной переменной и сингулярным младшим членом по другой. В анизотропно взвешенном пространстве Соболева, согласованном с направлением вырождения, доказаны существование и единственность слабого решения нелинейной задачи при малых показателях вырождения и явных условиях малости на сингулярный коэффициент и на рост монотонного реакционного члена, а также внутренняя регулярность решения. Во вспомогательном взвешенном энергетическом пространстве установлен порог коэрцитивности для сингулярного коэффициента, справедливый при обоих знаках коэффициента и при любом показателе вырождения; доказана точность отрицательного порога с помощью явной минимизирующей последовательности для соответствующего неравенства Харди, получены корректность и оценки устойчивости решения. Для соответствующего уравнения третьего порядка сформулированы аналоги задач Дирихле и Неймана, доказан новый принцип экстремума, дающий единственность решения, а существование сведено методом регуляризации Карлемана — Векуа к уравнению Фредгольма второго рода с явной функцией Грина. Разностная схема, проверенная на модельном точном решении со вторым порядком сходимости, подтверждает поведение решений вблизи линии вырождения. Обсуждается биомедицинская интерпретация — диффузия кислорода, лекарства или сигнальной молекулы через некротическую, слабо васкуляризированную опухолевую ткань.

Ключевые слова: вырожденное эллиптическое уравнение, весовое пространство Соболева, вес Макенхаупта, неравенство Харди, функция Грина, регуляризация Карлемана — Векуа

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