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# MODELING OF THE PROCESS OF UNSTEADY FILTRATION OF GROUNDWATER WITH A FREE SURFACE TAKING INTO ACCOUNT INFILTRATION FLOWS

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A nonlinear mathematical model with boundary, initial and internal conditions is proposed for monitoring and forecasting unsteady groundwater seepage with a free surface, taking into account infiltration flows and the hydrodynamic parameters of the soil. To determine the main factors directly influencing groundwater level changes over time, a minimizing functional is presented, which allows one to determine the ranges of the significant parameters in which they substantially influence the infiltration process. The problem is solved numerically by the locally one-dimensional scheme of A. A. Samarskii; numerical experiments on model problems showed that the best results are achieved by applying the principle of symmetrization. A model problem for a symmetric domain with a square boundary and a square infiltration region was solved. Analysis of the results showed that, for a constant seepage coefficient, the infiltration front propagates absolutely symmetrically over time within the seepage domain, as expected from physical considerations. It was established that the locally one-dimensional scheme is effective for this class of problems, can be successfully applied in combination with the symmetrization method and can be recommended for solving problems of modeling and forecasting the groundwater regime.

**Keywords:** mathematical model, numerical algorithm, groundwater, computational experiments, hydrodynamic parameters, infiltration front

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## 1 Introduction

In recent years, the issue of water resource scarcity and protection from pollution to ensure high-quality drinking water has become particularly pressing. Statistical data show that only 2.5–3% of the Earth's water resources are freshwater, of which only about 1% is readily accessible for use, with the rest being glaciers and deep groundwater. Research has shown that approximately 2.2 billion people lack access to drinking water, and 2 billion live in countries with water shortages. Most of these people live in Africa, Asia, northeastern Mexico, the western United States, Argentina, Chile, and China. The majority of freshwater resources (over 60%) are concentrated in a few countries, including Brazil, Russia, Canada, the United States, and China. The vast majority of these global freshwater reserves are hidden in the glaciers of Antarctica, Greenland, and the Arctic, and only a small fraction (1%) is accessible for human use — in lakes, rivers, and springs. According to scientists, in a quarter of a century we could face a threatening situation: water reserves will be halved, and half of them will be polluted and unusable [1].

The search for new sources of fresh water has, paradoxically, led scientists to the deep sea. Drilling in various parts of the planet has revealed that vast reserves of fresh water are hidden deep within the seafloor. The formation of these deposits was most likely caused by the Ice Age, or rather its end. During that cold period, part of the Earth's territory now occupied by the World Ocean was dry land. In these areas, water gradually accumulated, and when the glaciers rapidly melted, the accumulated resource simply "sank to the bottom", with fresh water separated from salt water by layers of clay and silt.

From the analysis of long-term statistical data it is known that the water reserves on the globe amount to 1.4 billion km$^3$; of this, salt water (oceans) accounts for 1 365 000 000 km$^3$ (97.5%) and fresh water for 35 000 000 km$^3$ (2.5%). The replenishment of groundwater reserves occurs due to precipitation, and the largest part of precipitation falls in Asia (30%) and South America (27% of the world's total). The world's population is growing rapidly, and estimates show that with current water use methods, by 2030 there will be a 40% gap between the projected demand and the available fresh water reserves [2].

As we know, the causes of the water crisis are related to the problem of water resource shortage; this is not a natural process but the result of human activity, and the main reasons are as follows:

- the main sources of fresh water are rivers, lakes, and swamps; however, the natural distribution of resources is, unfortunately, uneven across the globe: for example, Europe accounts for 20% of the planet's population, but only 7% of its reserves;
- the number of people on Earth is growing daily, and with them the demand for drinking water; therefore, if the annual population growth is 84 million, the required increase in water resources must be at least 60 million cubic meters;
- inappropriate use of natural resources, i.e., pollution of water sources (industrial waste, emissions, fertilizer runoff from fields); for example, in America 37% of rivers and lakes are so polluted that they cannot even be used for irrigation or domestic needs;
- the positive factor of agricultural development worldwide also contributes negatively to this problem: this sector's water requirements account for 85% of the total, and therefore the price of artificially irrigated products is significantly higher;
- one of the global causes is the greenhouse effect, as more and more gases are emitted into the atmosphere; the Earth's climate is changing every year, and snowfall in hot climates and unnatural frosts in countries like Italy and Spain are all consequences of precipitation redistribution [3].

Based on the above, effective groundwater protection and wastewater treatment require the development of efficient methods and tools based on comprehensive studies of groundwater filtration processes in multilayer porous media. The inadequacy of traditional irrigation methods and the use of salinized lands in agriculture have resulted in significant amounts of brackish return water resources in Uzbekistan. Experience has shown that surface and groundwater sources continue to be polluted by mineral fertilizers, pesticides, and highly mineralized and polluted drainage water used in agriculture.

Mathematical modeling and computational experiments are effective tools for research, monitoring, forecasting, and management decision-making. Mathematical models of groundwater filtration in multilayer porous media are typically described by systems of partial differential equations with corresponding boundary and internal conditions of various types. Significant applied and fundamental results have already been obtained in the development of mathematical tools for solving such problems.

In particular, work [4] addressed the problem of unsteady groundwater filtration in the clay core of a high rock-and-earth dam, using the Nurek Hydroelectric Power Plant (HPP) dam as an example. Numerical calculations were performed on a computing system using the developed mathematical and software tools. A 30-year operational period of the Nurek HPP dam was studied. Calculations during this time showed that only the lower portion of the dam core was fully saturated with water. This confirmed the conclusions previously drawn using other methods regarding the relatively slow establishment of a quasi-static filtration regime in similar structures. According to the authors, the difference in results is explained by previously adopted assumptions and a more sophisticated filtration model in the Plaxis 2D software suite. The model used in this suite allows for the accounting of changes in soil transmissivity depending on its water saturation.

The study [5] examines the process of modeling transient fluid flow in a reservoir system. A mathematical model of groundwater flow in porous media is developed based on parabolic partial differential equations with various initial, boundary, and internal conditions, and an analytical solution is obtained using the Laplace transform. Computational experiments were conducted to determine the pressure change along the length of the filter layers without considering the elastic regime, and it was found that the pressure in both layers increases exponentially. The computational experiment established that water flow through the interface between the filter layers depends significantly on the piezoconductivity of the highly permeable layer. The authors obtained a new generalized formula for well adit control and developed a mathematical framework that allows for the design of vertical drainage well layouts and capacity plans to protect irrigated and non-irrigated areas from flooding, as well as to protect groundwater from pollution sources and isolate already contaminated areas.

Modeling of nonlinear groundwater filtration processes in multilayer heterogeneous porous media is considered in [6]. The key hydrodynamic parameters affecting groundwater filtration in multilayer heterogeneous porous media are analyzed, particularly the nonlinear nature of fluid flow and fluid transfer between different layers.

In [7], a mathematical model is proposed for monitoring and predicting nonlinear groundwater filtration processes in multilayer heterogeneous porous media. This model represents a complex domain that integrates various mathematical and physical principles, the nonlinear nature of fluid flow, and the interaction between different layers of porous materials. The paper provides a detailed analysis of fundamental and applied research in the field of groundwater filtration in heterogeneous and multilayer porous media, based on the laws of H. Darcy, J. Dupuit, N. E. Zhukovsky, P. Forchheimer, and others.

A numerical simulation model is proposed in [8] to assess the impact of the key hydrodynamic parameters on groundwater levels over a specified period. For this purpose, a structured model was created for the Beijing Plain region and run using the GMS 10.6 software with a finer grid around the reservoirs. The model's accuracy agrees well with real data, and the overall average relative error is less than 20%. Comparison of the calculated water balance with the results of numerous studies shows that the reliability of the equilibrium analysis results is relatively high. The numerical groundwater model is used to predict water levels over a 15-year period. The simulation results also demonstrate the impact of several projects related to the South-North Water Diversion Project, implemented to restore overexploited groundwater resources. The model predicts stabilization and a significant rise in groundwater levels in central Beijing.

In [9], an analytical approach for mathematical modeling of groundwater flows in a vertically inclined multilayer porous medium with an unconfined aquifer was proposed. A mathematical toolkit was built for this approach. The analytical solution of the problem was obtained using the eigenfunction expansion method, which can be applied to any vertically multilayered soil stratification with time-varying rainfall. The authors validated the obtained solution using previous analytical solutions and numerical results simulated using the Modflow software.

The authors of [10] proposed an effective physics-based deep learning model for solving multiphase flow problems in three-dimensional heterogeneous porous media, fully utilizing the capabilities of convolutional neural networks. The model utilizes input data, including porous medium properties, fluid properties, and well control parameters, and predicts spatiotemporal changes in pressure and saturation. While maintaining continuous fluid flow, the three-dimensional spatial domain is decomposed into 2D images. The model performs forecasting with a speedup of approximately 1400 times compared to physically based modeling, and the average time errors of predicted pressure and saturation values are 0.27% and 0.099%, respectively. Furthermore, a separate simulation model is created as a postprocessor for calculating well flow rates based on state variable predictions from the deep learning model, where water flow rates are effectively predicted with an average error of less than 5%. Based on the authors' unique design for ensuring accurate fluid flow in porous media, the deep learning model with physically based constraints can become an effective predictive model for computationally complex inverse problems or other related processes.

A new method for constructing hybrid macropore-Darcy models based on micro-CT images of microporous rocks is presented in [11]. Darcy cells effectively capture microscale connectivity variations absent in macroscopic networks. In this approach, the authors extract macropore networks using conventional methods, while microporosity is characterized using segmented phase classification and incorporated into the model as Darcy cells. This two-component model therefore incorporates both the traditional macroscopic pore structure and critical flow paths in underresolved microporous regions. The proposed workflow is thoroughly validated by comparing permeability estimates with direct numerical simulation (DNS) and experimental measurements. The results demonstrate that this hybrid approach reliably reproduces fluid flow behavior in complex porous media while significantly reducing computational requirements.

In [12], the process of two-dimensional water movement under the influence of gravity in a heterogeneous unsaturated porous medium is modeled at various infiltration rates. The authors confirm that, depending on the saturation rate, the nature of moisture movement can transition from stable (diffusion) to unstable (formation of "fingers" — finger flow) and back. To describe these phenomena, a semi-continuum model is proposed that accurately reproduces the transitions between flow types and characterizes the width and distance between fluid flow channels. This model takes into account the phenomenon of "saturation overshoot" and allows for more accurate modeling of unstable water distribution in a porous medium.

In [13], mathematical modeling of the problem of nonlinear groundwater filtration in a heterogeneous porous medium is considered, taking into account evaporation and infiltration from the daylight surface. The authors of this article focus on the influence of nonlinear effects on the distribution and dynamics of seepage flows in porous media, where the proposed model is described by a system of nonlinear partial differential equations. Numerical methods, including difference schemes and iterative algorithms, were used to integrate the problem, ensuring the stability and convergence of calculations.

The developed numerical algorithm is implemented as software designed to solve various hydrogeological and land reclamation forecasting problems.

As noted in [14], studying the hydrodynamic regime of groundwater, in particular the formation of new freshwater reserves and monitoring their quantitative and qualitative indicators, is of significant scientific and practical importance. Consequently, the authors of this article analyzed the main causes of groundwater level fluctuations in two-layer environments, including precipitation and evaporation, sources of water abstraction and recharge, the geological structure and permeability characteristics between layers, hydraulic gradient, flow directions, irrigation infiltration, permeability coefficient, porosity, layer thickness, drainage, and other factors. To accurately model water level changes in open and confined aquifers, a mathematical model was developed that takes into account the physical, geological, and hydrogeological parameters of the study area. The problem was described using mathematical and numerical modeling of geofiltration and geomigration processes. The model is represented by nonlinear differential equations that have no analytical solution due to the presence of free variables. Therefore, a fully robust numerical solution scheme based on high-precision approximation was proposed. Solutions were obtained using iterative calculations and forward and backward substitution methods. The study also considered parameters such as soil density, effective porosity, and third-order open boundary conditions. As a result, the model provides reliable predictions for determining groundwater flow and changes in its quality, and this method offers practical solutions that can be applied in water resources management and planning.

The development of a nonlinear mathematical model for studying, predicting, and making decisions on transient groundwater flow in a three-layer heterogeneous aquifer system, including a confined aquifer, a capping layer, and a low-permeability barrier, is discussed in [15], where the mathematical model incorporates infiltration and evaporation, governed by the law of M. M. Krylov and S. F. Averyanov. In the mathematical formulation, the evaporation rate depends on the critical groundwater level and takes into account periodic fluctuations in precipitation and evaporation, as well as the changing permeability of the boundaries. This approach enables realistic modeling of the dynamics of a multilayered aquifer under various climatic and hydrogeological conditions, offering a reliable tool for sustainable groundwater management, drought risk assessment, and aquifer protection strategies. The authors used an implicit difference scheme for the numerical integration of the problem.

As emphasized in the work [16], the method of mathematical modeling is an important tool for solving complex problems related to the analysis of groundwater dynamics and heat and mass transfer processes in them when studying their pollution from various anthropogenic sources in the event of emergency spills of chemically hazardous substances, etc. Based on the above, the work is devoted to the development of a set of mathematical tools for monitoring and forecasting the filtration process of unconfined groundwater, mass transfer of impurities and the heat transfer process in groundwater, where the two-dimensional Boussinesq filtration equation was used for their solution. In the work, the freezing process of individual sections of the groundwater flow is modeled using the Laplace equation, and to calculate the flow velocity field in time-varying geometry a two-dimensional heat transfer equation in groundwater is used. Finite difference schemes were used to solve the model equations of groundwater dynamics and heat and mass transfer. The experiment confirmed the adequacy of the constructed numerical model of filtration of unconfined groundwater flow. An effective mathematical model has been developed for determining temperature fields in groundwater during the operation of a well used to freeze specific sections of the flow. Computer simulation results demonstrate the effectiveness of the developed mathematical models.

Based on the analysis of the literature, this paper proposes a mathematical model for monitoring and forecasting groundwater levels, taking into account factors such as infiltration and evaporation and changes in the hydrodynamic parameters of the soil depending on the groundwater pressure.

## 2 Problem statement

The work focuses on the process of groundwater filtration in a porous medium, where the infiltration factor is taken into account, including an assessment of the inflow of water into underground horizons due to precipitation, through the use of methods for their balance over time and the determination of quantitative indicators of water intake and consumption over a certain period and in a certain area, as well as the use of geophysical, hydrogeological and mathematical methods for modeling the object as a whole.

As is known, infiltration recharge is formed under the influence of natural factors that determine the intensity of atmospheric precipitation in the region under consideration, the time of year and the recharge of surface watercourses and reservoirs (Fig. 1).

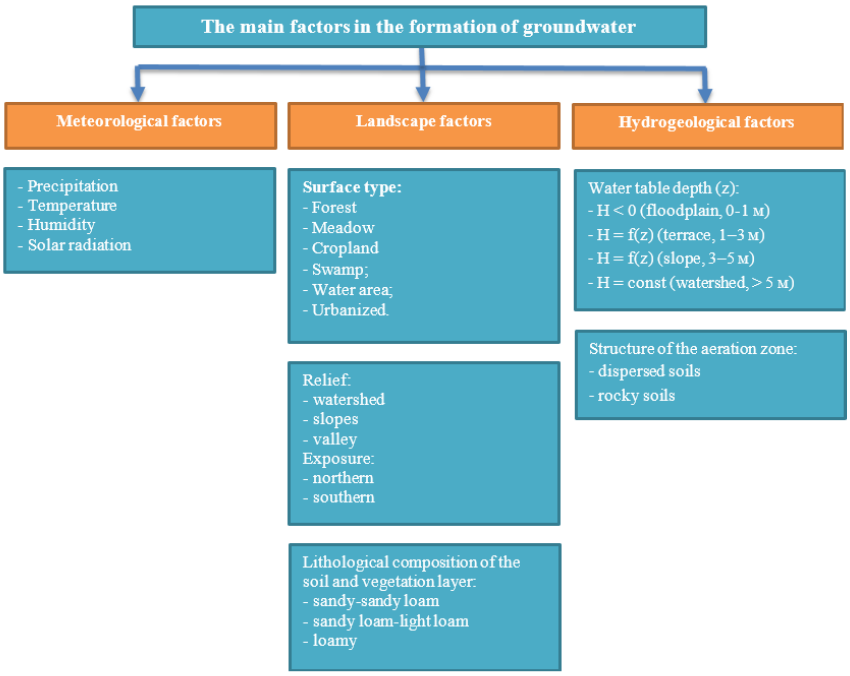


**Figure 1** Sources of infiltration recharge.

Taking into account the above, in order to conduct a comprehensive study of unsteady filtration of groundwater with a free surface in a heterogeneous porous medium with a variable aquifer area, we consider the following problem, which is described by a nonlinear partial differential equation:

$$
\begin{aligned}
m\frac{\partial H(x,y,t)}{\partial t} &= \frac{\partial}{\partial x}\left\{\varkappa(x,y,H)\left[b(x,y)-H(x,y,t)\right]\frac{\partial H(x,y,t)}{\partial x}\right\}+ \\
&+\frac{\partial}{\partial y}\left\{\varkappa(x,y,H)\left[b(x,y)-H(x,y,t)\right]\frac{\partial H(x,y,t)}{\partial y}\right\}\pm\varphi(x,y,t),
\end{aligned} \tag{1}
$$

where $H$ is the desired height of the free surface with respect to a certain horizontal plane taken as the reference plane; $\varkappa$ is an isotropic function corresponding to the filtration coefficient of the heterogeneous porous medium; $b$ is the height of the aquifer bed, which is a function of the spatial coordinates; $\varphi$ is the magnitude of infiltration or evaporation, variable over the aquifer area; $m$ is the porosity coefficient.

To determine infiltration or evaporation, the following relationship can be used:

$$\varphi(x,y,t) = q_0\left(1 - \frac{h-H}{h-H_{\mathrm{cr}}}\right)^n .$$

Here $q_0$ is the intensity of evaporation (infiltration) at the daylight surface, which depends on weather and climate factors and the time of soil irrigation; $H_{\mathrm{cr}}$ is the critical depth of groundwater; $n$ is an exponent; $h$ is the thickness of the aquifer.

An analysis of the literature has shown that the filtration coefficient, which is the main hydrodynamic parameter of the process, varies depending on the groundwater pressure. As indicated in [17], for small changes in groundwater pressure the filtration coefficient can be taken as a linear relationship

$$\varkappa(x,y,t) = \varkappa_0(x,y)\big(1 - a_k\big(H_0(x,y) - H(x,y,t)\big)\big),$$

and for large changes in pressure — in exponential form

$$\varkappa(x,y,t) = \varkappa_0(x,y)\, e^{a_k(H_0(x,y)-H(x,y,t))},$$

where $\varkappa_0(x,y)$ is the initial value of the filtration coefficient at $t=0$, $a_k$ is the coefficient of change in permeability.

As noted in the work [18], for a realistic description of the mathematical model of the process, one can also use an empirical dependence of the form

$$\varkappa(x,y,t) = \varkappa_0(x,y)\left(\frac{m}{m_0}\right)^d ,$$

where $m_0$ is the initial value of the porosity of the inhomogeneous medium, $d$ is a constant determined from experimental data.

In [19], several analytical representations of the law of change in permeability due to changes in groundwater pressure are proposed in the form of a power law

$$\varkappa(x,y,t) = \varkappa_0(x,y)\big(1 - a_k\big(H_0(x,y) - H(x,y,t)\big)\big)^{n-1},$$

where the index $n$ takes the values 2, 3 and 4, respectively, and in the work [20] a dependence of the following type was used to calculate the filtration coefficient:

$$\varkappa(x,y,t) = \varkappa_0(x,y)\left(\frac{H}{H_0}\right)^{a_k} .$$

To complete the mathematical formulation of the problem, the internal boundary conditions on the well contours can correspond to the regime of specified withdrawals or groundwater levels as functions of time. As is known, another significant hydrodynamic parameter that influences groundwater filtration is the porosity of the aquifer, which varies depending on the aquifer's internal pressure head. As the effective stress changes, the forces compressing each rock grain decrease as the groundwater pressure increases.

Therefore, as the pressure head falls, the grain volume increases and the pore volume decreases.

To integrate equation (1), in the first case the boundary condition is represented by the relationship

$$Q_j(t) = \int\limits_{\gamma_j} \varkappa(x,y,t)\, \frac{\partial H(x,y,t)}{\partial n}\, H(x,y,t)\, ds. \tag{2}$$

Here $Q_j$ is the flow rate of the $j$-th well, $\partial H(x,y,t)/\partial n$ is the derivative along the external normal to the circular contour $\gamma_j$ of the $j$-th well, $ds$ is the length element of the contour.

In the second case, the height of the free surface at the well is known, which, for design reasons, is constant along the entire supply contour:

$$H\left(x_{c_k}, y_{c_k}\right) = \psi_k(t). \tag{3}$$

Here $x_{c_k}$, $y_{c_k}$ are the coordinates of the $k$-th well, $k = \nu_1, \nu_2$, $\nu_1 + \nu_2 = n$, where $\nu_1$ or $\nu_2$ can be equal to zero, which corresponds to the conditions of given withdrawals or given groundwater levels at all wells.

External boundary conditions can be of the most general form; in particular, on the contour $G$ the height of the free surface is known as a function of time:

$$H(x,y,t)|_G = \psi(x,y,t). \tag{4}$$

The initial condition determines the distribution of the groundwater level in the region $G$ up to the boundary:

$$H(x,y,t)|_{t=0} = H_0(x,y). \tag{5}$$

## 3 Method of solution

The most general boundary value problem under consideration in a multiply connected domain is complex, and therefore obtaining analytical solutions is practically impossible. From the formulation of the boundary value problem (1)–(5), it is clear that the mathematical difficulties encountered in solving it are due to the following circumstances:

- the differential equation describing unsteady groundwater filtration with a free surface in a heterogeneous porous medium is a nonlinear, inhomogeneous partial differential equation with variable coefficients;
- the integration domain of the problem has an arbitrary outer boundary with a given finite number of internal contours, and the external boundary conditions and the initial condition are represented by functional dependencies;
- the internal conditions are non-identical functions of time (corresponding to specified functions or their derivatives on the contour);
- the dimensions of the internal contours are extremely small compared to the dimensions of the outer boundary of the domain.

Based on the above, we solve the problem using the finite difference method on a computer system, employing the technology proposed by A. A. Samarskii (the splitting method). To do this, we first pass to dimensionless variables, convenient for performing numerical calculations on a computer system, using the relations

$$H_\delta = \frac{H}{H_x}, \quad \varkappa_\delta = \frac{\varkappa}{\varkappa_x}, \quad b_\delta = \frac{b}{H_x}, \quad \varphi_\delta = \frac{L^2}{\varkappa_x H_x^2}\,\varphi, \quad x_\delta = \frac{x}{L}, \quad y_\delta = \frac{y}{L}, \quad t_\delta = \frac{\varkappa_x H_x}{mL^2}\, t.$$

Here the index "$\delta$" denotes dimensionless quantities, $H_x$ and $\varkappa_x$ are certain characteristic quantities, and $L$ is the greatest distance from the origin to the outer boundary. Then the differential equation under consideration takes the form

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left[\varkappa\,(b-H)\,\frac{\partial H}{\partial x}\right] + \frac{\partial}{\partial y}\left[\varkappa\,(b-H)\,\frac{\partial H}{\partial y}\right] \pm \varphi(x,y,t), \tag{6}$$

where $\varkappa$, $m$, $b$, $x$, $y$ are positive dimensionless variables (the index $\delta$ is omitted).

Approximating the derivatives in (6) by their grid analogs according to the three-point implicit finite-difference scheme, we obtain a system of two locally one-dimensional difference equations:

$$\Delta_x\left[\varkappa_{i,j}\left(\tilde{H}_{i,j} - b_{i,j}\right)\nabla_x \tilde{H}_{i,j}\right] = \chi\left(\frac{\Delta\xi^2}{\Delta t}\cdot\Delta t\cdot H_{i,j} - \varphi\cdot\Delta\xi^2\right); \tag{7}$$

$$\Delta_y\left[\varkappa_{i,j}\left(\tilde{\tilde{H}}_{i,j} - b_{i,j}\right)\nabla_y \tilde{\tilde{H}}_{i,j}\right] = (1-\chi)\left(\frac{\Delta\xi^2}{\Delta t}\cdot\Delta t\cdot \tilde{H}_{i,j} - \varphi\cdot\Delta\xi^2\right). \tag{8}$$

Here $\Delta_x$, $\Delta_y$, $\nabla_x$, $\nabla_y$ are the operators of the right and left differences in the directions of the $x$- and $y$-axes, respectively (a right-handed coordinate system is used); $\Delta\xi$, $\Delta t$ are the steps of the spatial and temporal grids, respectively; the grid points $(i,j)$ have the coordinates $x_i = i\cdot\Delta\xi$, $y_j = j\cdot\Delta\xi$; $\chi$ is a parameter less than one; $\tilde{H}_{i,j}$ and $\tilde{\tilde{H}}_{i,j}$ are the desired solutions associated with $H_{i,j}$, corresponding to the time layer $t_{n+1} = (1+\chi)\cdot\Delta t$.

Let us first consider the method for solving the system of finite-difference equations (7) and (8) for a simply connected region and then generalize it to the case where there are internal conditions. Let the solution of the boundary value problem at the previous time level be known. By considering the locally one-dimensional difference equation (7) for each row of the spatial grid under the given boundary conditions, an intermediate solution $\tilde{H}_{i,j}$ can be found. Considering the resulting solution as the initial condition and sequentially solving the locally one-dimensional difference equation (8) for each column of the spatial grid, one obtains the solution $\tilde{\tilde{H}}_{i,j}$. With a well-chosen value of the parameter $\chi$, for some regions the final solution can be considered the desired result at the time level $t_{n+1} = (1+\chi)\cdot\Delta t$.

However, as calculations on a high-speed computer show, the best results are obtained by applying the principle of symmetrization. This method involves performing calculations not only in the order specified above, but also in the reverse order. If we denote the solution obtained by successively solving equation (7) and then equation (8) by $\tilde{\tilde{H}}_{i,j}$, and the solution obtained by successively solving equation (8) and then (7) by $\tilde{H}_{i,j}$, then, according to the symmetrization principle, the solution on the $(n+1)$-th time layer should be taken as the combination

$$H_{i,j}^{n+1} = \mu\cdot\tilde{\tilde{H}}_{i,j} + (1-\mu)\cdot\tilde{H}_{i,j}. \tag{9}$$

Experience shows that the values of the parameters $\chi$ and $\mu$ can be set to 0.5 when using the symmetrization principle. The actual solution of equations (7) and (8) can be obtained by successive application of the sweep and iteration methods. If we assume that the nonlinear expressions in them are known from initial approximations, then finding the first iteration is equivalent to solving a linear parabolic equation with given boundary conditions. Using the sweep method, we can refine successive approximations in the nonlinear expressions and find a solution of the nonlinear parabolic equation with any desired accuracy.

We now turn to solving the boundary value problem with internal conditions. While the issues of rational use of RAM, efficient numerical approximation of the initial operators, and saving computer time are significant for a boundary value problem in a simply connected domain, in a problem with internal conditions, in addition to these, the fundamental problem of their approximation in the grid domain arises.

Research on approximating internal boundary conditions is of independent interest. This is explained by the fact that wellbore contours cannot be approximated with any practically applicable grid. Therefore, research was aimed at exploring the feasibility of other approaches, in particular, replacing the wellbore with a point coinciding with a grid node. It turned out that when approximating the wellbore contour with a point, the desired solution at that particular node is fictitious, although the results at neighboring nodes are correct. Moreover, the fictitious results correspond to a certain real wellbore with a radius $R_\varphi$ commensurate with the grid step and satisfying the relation

$$R_\varphi = 0.2 \cdot \Delta\xi.$$

Thus, only the grid spacing $\Delta\xi$ determines the radius of the enlarged fictitious well. Knowing the values of the desired function on the contour of the enlarged well and considering them as time-varying boundary conditions of the first kind, we can further consider the problem of determining the desired quantities on the contour of a well with radius $R_c$ and in its vicinity. The problem is reduced to integrating differential equation (1) in the ring

$$R_c < \zeta < R_\varphi, \qquad \zeta = \sqrt{x^2 + y^2},$$

where the wellbore conditions are described by equations (7) and (8). The latter problem is obviously equivalent to the problem of plane-radial, i.e., one-dimensional, flow toward the wellbore in a circular domain.

To approximate the internal conditions using finite-difference relations, we proceed as follows. We isolate a fluid volume element with an axis of symmetry passing through the singular point with coordinates $x_i$ and $y_i$ and consider the continuity equation for this volume, whose projection onto the horizontal plane is a square with side length $\Delta\xi$. Summing the inflow volumes through the lateral faces in the $x$- and $y$-axis directions, we obtain the expression

$$\begin{aligned}&(H-b)_{i-0.5,j} \cdot V_{i-0.5,j} \cdot \Delta\xi - (H-b)_{i+0.5,j} \cdot V_{i+0.5,j} \cdot \Delta\xi + \\ &+(H-b)_{i,j-0.5} \cdot V_{i,j-0.5} \cdot \Delta\xi - (H-b)_{i,j+0.5} \cdot V_{i,j+0.5} \cdot \Delta\xi.\end{aligned}$$

For an ordinary node, this expression should be equal to the change in the volume under consideration over the time $\Delta t$ taken with the opposite sign:

$$-\frac{\Delta H_{\text{av}}}{\Delta t} \cdot (\Delta\xi)^2,$$

where $H_{\text{av}}$ is the average value of the desired function in the domain

$$x_{i-0.5} \leqslant x \leqslant x_{i+0.5}, \qquad y_{j-0.5} \leqslant y \leqslant y_{j+0.5}.$$

For a singular node, the flow rate $Q_{i,j}$ at the point $(x_i, y_i)$ is added to this value. Under conditions of infiltration or evaporation, it is necessary to add the value $\varphi_{i,j}$, where $\varphi_{i,j} > 0$

for infiltration and $\varphi_{i,j} < 0$ for evaporation. Thus, taking into account that the filtration rates obey Darcy's law, we obtain

$$\begin{aligned}\Delta_x\left[\varkappa_{i,j}\left(H_{i,j}-b_{i,j}\right)\nabla_x H_{i,j}\right]+\Delta_y\left[\varkappa_{i,j}\left(H_{i,j}-b_{i,j}\right)\nabla_y H_{i,j}\right]=\\=\left(\frac{\Delta\xi}{\Delta t}\cdot\Delta t\cdot H_{i,j}+\varphi_{i,j}\cdot\Delta\xi^2\right)-Q_{i,j},\end{aligned}\tag{10}$$

where we accept that

$$\frac{\Delta H_{\text{av}}}{\Delta t}=\frac{\Delta H_{i,j}}{\Delta t}.$$

In relation (10) all quantities are dimensionless, and the dimensionless flow rate of the well is equal to

$$Q_\delta=\frac{Q}{\varkappa_x\cdot H_x^2}.$$

According to the splitting method, equation (10) must be represented in the form of two locally one-dimensional equations:

$$\Delta_x\left[\varkappa_{i,j}\left(H_{i,j}-b_{i,j}\right)\nabla_x H_{i,j}\right]=\chi\left[\frac{\Delta\xi^2}{\Delta t}\cdot\Delta t\cdot H_{i,j}-\varepsilon\,\Delta\xi^2-Q_{i,j}\right];\tag{11}$$

$$\Delta_y\left[\varkappa_{i,j}\left(H_{i,j}-b_{i,j}\right)\nabla_y H_{i,j}\right]=(1-\chi)\left[\frac{\Delta\xi^2}{\Delta t}\cdot\Delta t\cdot H_{i,j}-\varepsilon\,\Delta\xi^2-Q_{i,j}\right].\tag{12}$$

Comparing (11) and (12) with equations (7) and (8), it is easy to see that they coincide when $D_{i,j}=0$, i.e., equations (11) and (12) are generalizations of the approximation for an internal ordinary node to the case of a singular node.

Thus, we find that the method for solving the problem under consideration with internal conditions represented by equation (10) is not fundamentally different from the method described above for a simply connected domain. In the case where the conditions on the internal contours are described by equations (11), the following iterative process can be applied. According to Dupuit's formula, the fluid flow rate in a well under steady-state filtration conditions (at $\psi_{i,j}=\text{const}$) is

$$Q_{i,j}=\left(H_{i,j}^2-\psi_{i,j}^2\right)\cdot\frac{\varkappa_{i-0.5,j}+\varkappa_{i+0.5,j}+\varkappa_{i,j-0.5}+\varkappa_{i,j+0.5}}{-\dfrac{1}{2\pi}\left(\ln\dfrac{R_c}{\Delta\xi\cdot\varepsilon}+1.61\right)}.\tag{13}$$

Assuming $Q_{i,j}=\bar{Q}_{i,j}$ in the first approximation, one can find successively all the remaining approximations with a given degree of accuracy.

# 4 Model problems and computational experiments

## 4.1 The problem of spreading of an infiltration mound

As a result of artificial irrigation of land or natural phenomena such as precipitation, a so-called infiltration mound develops. It is necessary to predict the persistent change in groundwater level over time under the influence of the growth of this mound. This problem is part of the more general problem of locating relief wells in areas likely to be affected by infiltration mounds, reservoirs, or natural bodies of water.

Let us present the calculation formulas for this problem according to the algorithm described above. For simplicity of presentation, in (7) we set $b_{i,j}=0$ and write it in the form

$$a_i H_{i-1,j}^2-e_i H_{i,j}^2+c_i H_{i+1,j}^2=-d_i,\tag{14}$$

where the coefficients are defined as

$$a_i = \varkappa_{i+0.5,j}; \qquad e_i = \varkappa_{i+0.5,j} + \varkappa_{i-0.5,j} + \chi\frac{\Delta x^2}{2H^{(2)}_{i,j}\Delta t};$$

$$c_i = \varkappa_{i-0.5,j}; \qquad d_i = \chi\frac{\Delta x^2}{2H^{(2)}_{i,j,k}}H^2_{i,j,k+1} + \varepsilon\,\Delta x^2.$$

We seek the solution of system (14) in the form

$$H^{2(s+1)}_{i,j,n+1} = \alpha_i H^{2(s)}_{i+1,j,k+1} + \beta_i. \tag{15}$$

From equation (14), using (15), we obtain the recurrence relations

$$\alpha_i = \frac{c_i}{e_i - a_i\alpha_{i-1}}, \qquad \beta_i = \frac{d_i + a_i\beta_{i-1}}{e_i - a_i\alpha_{i-1}}.$$

Moreover, from the boundary conditions we have

$$\alpha_0 = 0, \qquad \beta_0 = H_0^2.$$

A model problem was considered for a symmetric domain with a square boundary and a central square infiltration region (Figs. 1, 2). The filtration coefficient was assumed to be constant, and numerical experiments were conducted on a computing cluster. Analysis of the numerical simulations showed that the infiltration front propagates absolutely symmetrically over time within the considered infiltration region, as expected from physical considerations. Computational experiments were conducted for $\varkappa_0 = 0.05$ m/day, $\varphi = 0.0025$ m/day, $H_0 = 100$ m (Fig. 4). Based on the obtained results, it follows that A. A. Samarskii's locally one-dimensional scheme is effective for this class of problems and can be recommended for solving modeling, forecasting, and other problems for real hydrogeological objects. In the future, the presented numerical algorithm will be applied to predicting changes in groundwater levels.

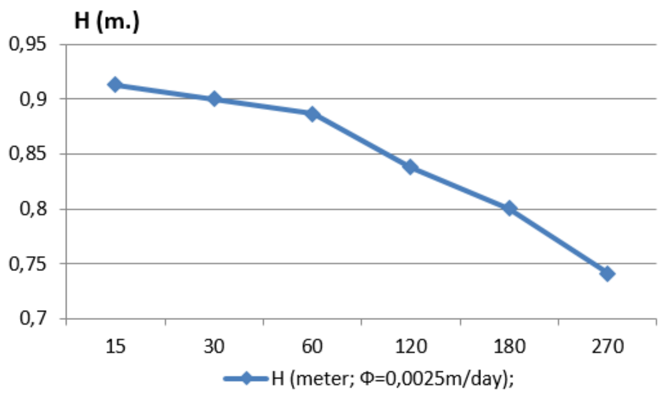


**Figure 2** Change in groundwater level over time at $\varphi = 0.0025$ m/day.

As can be seen from the curve in Fig. 2, the groundwater level decreases over time depending on the change in the infiltration parameter. A significant change in the groundwater level is observed at $t > 60$ days, after which it decreases linearly over time.

### 4.2 Unsteady groundwater inflow to a well

The objective of this problem is to conduct experimental calculations related to finding an optimal formula for approximating the well conditions in finite differences. Two methods for approximating the well boundary conditions were considered.

We assume that the movement toward the enlarged well occurs according to Darcy's law. Then, for each moment in time, we have the following formula for the well flow rate:

$$\begin{gathered}\left(H_{i+1,j}^2 - H_{i,j}^2\right) \varkappa_{i+0.5,j} + \left(H_{i-1,j}^2 - H_{i,j}^2\right) \varkappa_{i-0.5,j} + \\ + \left(H_{i,j+1}^2 - H_{i,j}^2\right) \varkappa_{i,j+0.5} + \left(H_{i,j-1}^2 - H_{i,j}^2\right) \varkappa_{i,j-0.5} = Q_\ell,\end{gathered} \tag{16}$$

where $Q_\ell$ is the water flow rate at the well. If a given production rate is maintained at the well, then $Q_\ell$ is a known function of time. In the case where the well pressure is known, the internal pressure conditions at the well can be approximated in finite differences by the equation

$$\begin{gathered}\left(H_{i+1,j}^2 - H_{i,j}^2\right) \varkappa_{i+0.5,j} + \left(H_{i-1,j}^2 - H_{i,j}^2\right) \varkappa_{i-0.5,j} + \\ + \left(H_{i,j+1}^2 - H_{i,j}^2\right) \varkappa_{i,j+0.5} + \left(H_{i,j-1}^2 - H_{i,j}^2\right) \varkappa_{i,j-0.5} = \\ = \left(H_{i,j}^2 - H_c^2\right) D_{i,j,y} + \left(H_{i,j}^2 - H_c^2\right) D_{i,j,x},\end{gathered} \tag{17}$$

where

$$D_{i,j,x} = \frac{\varkappa_{i+0.5,j} + \varkappa_{i-0.5,j}}{-\dfrac{1}{2\pi}\left(\ln\dfrac{R_c}{\Delta x} + 1.61\right)}, \qquad D_{i,j,y} = \frac{\varkappa_{i,j+0.5} + \varkappa_{i,j-0.5}}{-\dfrac{1}{2\pi}\left(\ln\dfrac{R_c}{\Delta y} + 1.61\right)}.$$

Formula (17) follows from (16), where the quantity $Q_\ell$, in accordance with the above assumption, is represented by the formula for steady plane-radial filtration. In dimensionless form, (16) and (17) retain their original form with the notation

$$Q_\ell^* = \frac{Q}{H_x^2 \cdot \varkappa_x},$$

where $Q_\ell^*$ is the dimensionless value of the withdrawal function.

For definiteness, we assume that the conditions at the well are of the form (16). According to A. A. Samarskii's locally one-dimensional method, we obtain the following equations for each direction:

$$\begin{cases}\left(H_{i+1,j}^2 - H_{i,j}^2\right) \cdot \varkappa_{i+0.5,j} + \left(H_{i-1,j}^2 - H_{i,j}^2\right) \cdot \varkappa_{i-0.5,j} = \dfrac{Q}{2}; \\ \left(H_{i,j+1}^2 - H_{i,j}^2\right) \cdot \varkappa_{i,j+0.5} + \left(H_{i,j-1}^2 - H_{i,j}^2\right) \cdot \varkappa_{i,j-0.5} = \dfrac{Q}{2}.\end{cases} \tag{18}$$

From (18) it is easy to obtain the sweep coefficients for a singular node:

$$\alpha_i = \frac{\varkappa_{i+0.5,j}}{\left(\varkappa_{i+0.5,j} + \varkappa_{i-0.5,j}\right) - \varkappa_{i-0.5,j} \cdot \alpha_{i-1}};$$

$$\beta_i = \frac{\varkappa_{i-0.5,j} \cdot \beta_{i-1} - \frac{1}{2}Q}{\left(\varkappa_{i+0.5,j} + \varkappa_{i-0.5,j}\right) - \varkappa_{i-0.5,j} \cdot \alpha_{i-1}}.$$

For the case of a given pressure, using similar reasoning, we obtain

$$\alpha_i = \frac{\varkappa_{i+0.5,j}}{\left(\varkappa_{i+0.5,j} + \varkappa_{i-0.5,j}\right) - \varkappa_{i-0.5,j} \cdot \alpha_{i-1}}; \qquad \beta_i = \frac{\varkappa_{i-0.5,j} \cdot \beta_{i-1} - \frac{1}{2}Q}{\left(\varkappa_{i+0.5,j} + \varkappa_{i-0.5,j}\right) - \varkappa_{i-0.5,j} \cdot \alpha_{i-1}}.$$

A method for approximating the well conditions for a singular node was also investigated:

$$a_i = \varkappa_{i-0.5,j}; \qquad b_i = \varkappa_{i+0.5,j} + \varkappa_{i-0.5,j} + \chi\frac{\Delta x^2}{2H_{i,j}^{(s)}\Delta t};$$

$$c_i = \varkappa_{i-0.5,j}; \qquad d_i = \chi\frac{\Delta x^2}{2H_{i,j,k}^{(2)}} H_{i,j,k+1}^{(2)} + \varepsilon\,\Delta x^2 - Q.$$

Using the approximation methods discussed, a model problem was solved for a domain with a square boundary and a single well located in the center. Based on the proposed numerical algorithm, computational experiments were conducted on a computing cluster (Figs. 3–5). Preliminary calculations showed that for this class of problems, A. A. Samarskii's locally one-dimensional scheme can be successfully applied in combination with the symmetrization method. Approximating the boundary conditions at the well using formula (18) ensures higher accuracy of the results.

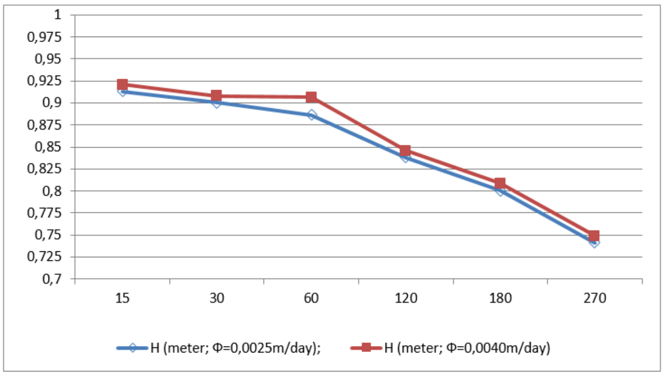


**Figure 3** Change in groundwater level depending on the value of the infiltration parameter.

The curves in Fig. 3 show that the groundwater level decreases exponentially over time depending on the infiltration coefficient. It is also observed that a significant decrease in the groundwater level occurs at $t > 65$ days, depending on the infiltration parameter.

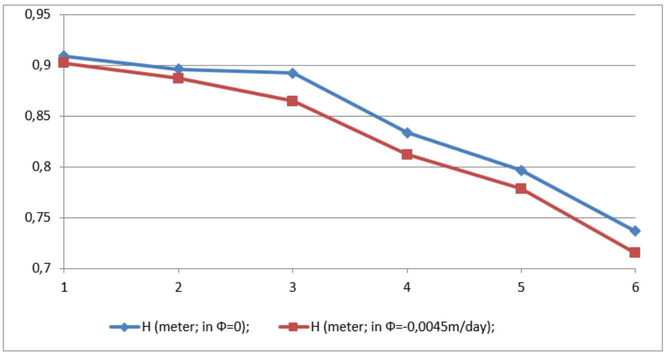


**Figure 4** Change in groundwater level depending on the value of the evaporation parameter.

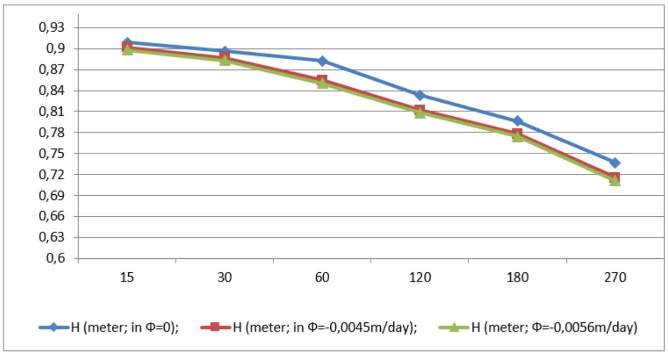


**Figure 5** Change in groundwater level for different values of the evaporation parameter.

Numerical calculations were also performed for various values of the evaporation coefficient from the daylight surface of the groundwater (Figs. 4, 5). The calculated curves show that the groundwater level decreases as the evaporation coefficient increases. This is especially noticeable at $t > 35$ days.

## 5 Conclusion

In this paper, a minimizing functional is proposed for determining the values of the parameters $q_0$, $n$, $H_{\mathrm{cr}}$ and the main factors directly affecting groundwater level changes (evaporation or infiltration).

Numerical calculations performed for model problems revealed that the best results are obtained using the symmetrization principle.

A model problem for a symmetric domain with a square boundary and a square infiltration region located at the center was considered. Analysis of the results showed that, for a constant infiltration coefficient, the infiltration front propagates absolutely symmetrically in time within the considered infiltration region, as expected from physical considerations.

It was established that A. A. Samarskii's locally one-dimensional scheme is effective for this class of problems and can be recommended for solving modeling, forecasting, and other problems for real hydrogeological objects.

Numerical calculations also showed that for this class of problems, A. A. Samarskii's locally one-dimensional scheme can be successfully applied in combination with the symmetrization method.

### Declarations

**Conflicts of interest.** The authors declare no conflict of interest.

**Data availability.** The data are available from the authors on reasonable request.

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# МОДЕЛИРОВАНИЕ ПРОЦЕССА НЕСТАЦИОНАРНОЙ ФИЛЬТРАЦИИ ГРУНТОВЫХ ВОД СО СВОБОДНОЙ ПОВЕРХНОСТЬЮ С УЧЁТОМ ИНФИЛЬТРАЦИОННЫХ ПОТОКОВ

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Предложена нелинейная математическая модель с граничными, начальными и внутренними условиями для мониторинга и прогнозирования нестационарной фильтрации грунтовых вод со свободной поверхностью с учётом инфильтрационных потоков и гидродинамических параметров грунта. Для определения основных факторов, непосредственно влияющих на изменение уровня грунтовых вод во времени, представлен минимизируемый функционал, позволяющий определить диапазоны изменения существенных параметров, в которых они оказывают значительное влияние на процесс инфильтрации. Задача решается численно по локально-одномерной схеме А. А. Самарского; численные эксперименты на модельных задачах показали, что наилучшие результаты достигаются при применении принципа симметризации. Решена модельная задача для симметричной области с квадратной границей и квадратной областью инфильтрации. Анализ результатов показал, что при постоянном коэффициенте фильтрации фронт инфильтрации распространяется во времени абсолютно симметрично в пределах рассматриваемой области фильтрации, что соответствует физическим представлениям. Установлено, что локально-одномерная схема эффективна для данного класса задач, может успешно применяться в сочетании с методом симметризации и может быть рекомендована для решения задач моделирования и прогнозирования режима грунтовых вод.

**Ключевые слова:** математическая модель, численный алгоритм, грунтовые воды, вычислительные эксперименты, гидродинамические параметры, фронт инфильтрации

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