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APPROXIMATE SOLUTION OF INITIAL VALUE PROBLEMS FOR FIRST-ORDER DIFFERENTIAL EQUATIONS USING A COMBINED RUNGE-KUTTA AND PIECEWISE CONSTANT ARGUMENT METHOD

Jumaev Z.Z.
zafarlangar8708@gmail.com
Samarkand state university named after Sharof Rashidov,
15, University boulevard, Samarkand, 140104 Uzbekistan.

This study presents an efficient method for approximating the solutions of a certain class of first-order differential equations with variable coefficients. The approach constructs an auxiliary differential equation that combines the Runge-Kutta method with a piecewise constant argument, derived from the original initial value problem and parameterized by a positive integer n. It is shown that, for sufficiently large n, this auxiliary equation has a unique piecewise-smooth solution that approximates the considered initial value problem. Error estimates for the residual are derived to quantify the accuracy and to illustrate the influence of n. The numerical results show that the method achieves higher accuracy with fewer computational steps than the classical Runge-Kutta and related schemes, and the framework extends to a broader class of nonlinear equations.

Keywords: initial value problem, piecewise constant argument, approximated solution, Runge-Kutta method, absolute error.

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1 Introduction

The obtained numerical outcomes demonstrate stability, computational efficiency, and adherence to the expected theoretical convergence rates [1]. Alternative computational frameworks, such as the finite element method, have been extensively utilized for the analysis of nonlinear problems [2]. Both traditional mesh-based and modern meshless schemes have significantly contributed to the advancement of numerical solutions for linear and nonlinear ordinary differential equations (ODEs), partial differential equations (PDEs), as well as fractional differential equations [3–5].

Through numerical experiments, it is shown that the suggested RADAU and EMOHB methods outperform several other well-known methods with similar properties when applied to initial-value ordinary differential problems [6]. The accuracy of the proposed Haar Wavelet Collocation (HWC) method is assessed through error estimates for both single and interacting species. Numerical experiments demonstrate its efficiency and confirm the theoretically predicted eighth-order convergence. The maximum absolute errors at collocation points are compared with exact solutions, further validating the method's effectiveness [7].

Further comparisons with the high-order Obreshkoff method and the classical Runge–Kutta scheme highlight the superior performance of the proposed approach [8]. The method yields a high-order, implicit-corrector scheme that outperforms traditional Taylor and

Runge–Kutta methods in terms of accuracy. The error bounds, stability, and convergence of a high-order Euler–Maclaurin variant are also established, with numerical results confirming its enhanced efficiency over widely used classical methods [9].

The issues of nonnegativity and global uniqueness of solutions for two nonlinear first-order ODE models describing intracellular Ca^{2+} dynamics are addressed in [10]. Finite-time stability of hybrid dynamical systems incorporating deviating arguments is investigated using a tailored hybrid control methodology in [11]. The robustness and efficiency of the Non-Fixed Step-Size Algorithm, particularly in integrating stiff differential systems over non-uniform time domains, are validated through rigorous theoretical and numerical analyses [12–14].

Furthermore, exact periodic solutions for a class of non-homogeneous first-order differential equations with piecewise constant arguments are presented in [15, 16], where sufficient conditions for existence and explicit solution representations are thoroughly derived.

In this study, we propose a computationally effective method for approximating solutions to a class of nonlinear ordinary differential equations given by

$$y'(t) = f(t, y(t)), \qquad y(0) = y_0, \qquad t \in [0, 1], \tag{1}$$

where the nonlinear function $f(t, y(t)) = a_m(t)y^m + a_{m-1}(t)y^{m-1} + \cdots + a_1(t)y + a_0(t)$ is a real-valued, continuous polynomial on the domain $[0, 1] \times \mathbb{R}$, and y_0 is given a real constant.

To construct the approximation, we introduce a modified formulation of the original initial value problem involving differential equations with piecewise constant arguments, parameterized by a positive integer n [17–20]. It is analytically proven that this auxiliary problem admits a unique piecewise-smooth solution, which serves as an approximation to the solution of the original equation as $n \to \infty$.

To validate the proposed method, several numerical examples are provided, demonstrating its superior accuracy and computational efficiency. The results indicate that the proposed approach offers faster convergence and higher precision compared to some well-established numerical techniques [6–9].

2 Differential equations with piecewise constant arguments

Let $y(t)$ be a solution of the initial value problem defined in (1), and assume that it satisfies the boundedness condition $|y(t)| \leqslant d$. Moreover, we assume that the function $f(t, y)$ has compact support $D = [0, 1] \times [-2d, 2d]$, that is,

$$f(t, y) = 0 \quad \text{for} \quad (t, y) \notin D. \tag{2}$$

Let n be a fixed positive integer, and define the points $t_{k-1} = \frac{k-1}{n}$, where $k = 1, 2, \ldots, n$. To approximate the solution of equation (1), this study focuses on optimizing the coefficients of a seventh-order Runge-Kutta method comprising thirteen stages for solving initial value problems. Given the structure of the method, which involves thirteen stages, a corresponding set of thirteen free parameters is employed during the training process [21].

The numerical approximation to the solution of (1) using the classical seventh-order Runge-Kutta method is given by:

$$\frac{y_k - y_{k-1}}{h} = K(t_{k-1}),$$

where the increment function $K(t_{k-1})$ is computed as a weighted sum of the intermediate stage derivatives [22]:

$$\begin{aligned}K(t_{k-1}) = \frac{31}{720}k_1(t_{k-1}) + \frac{16}{75}k_6(t_{k-1}) + \frac{16807}{79200}k_7(t_{k-1}) + \frac{16807}{79200}k_8(t_{k-1}) + \\ + \frac{243}{1760}k_9(t_{k-1}) + \frac{243}{1760}k_{12}(t_{k-1}) + \frac{31}{720}k_{13}(t_{k-1}),\end{aligned} \tag{3}$$

where

$$\begin{aligned}
k_1(t_{k-1}) &= f(t_{k-1}, y(t_{k-1})),\\
k_2(t_{k-1}) &= f(t_{k-1} + \frac{1}{4}h, y(t_{k-1}) + \frac{1}{4}hk_1(t_{k-1})),\\
k_3(t_{k-1}) &= f(t_{k-1} + \frac{1}{12}h, y(t_{k-1}) + h(\frac{5}{72}k_1(t_{k-1}) + \frac{1}{72}k_2(t_{k-1}))),\\
k_4(t_{k-1}) &= f(t_{k-1} + \frac{1}{8}h, y(t_{k-1}) + h(\frac{1}{32}k_1(t_{k-1}) + \frac{3}{32}k_3(t_{k-1}))),\\
k_5(t_{k-1}) &= f(t_{k-1} + \frac{2}{5}h, y(t_{k-1}) + h(\frac{106}{125}k_1(t_{k-1}) - \frac{408}{125}k_3(t_{k-1}) + \frac{352}{125}k_4(t_{k-1}))),\\
k_6(t_{k-1}) &= f(t_{k-1} + \frac{1}{2}h, y(t_{k-1}) + h(\frac{1}{48}k_1(t_{k-1}) + \frac{8}{33}k_4(t_{k-1}) + \frac{125}{528}k_5(t_{k-1}))),\\
k_7(t_{k-1}) &= f(t_{k-1} + \frac{6}{7}h, y(t_{k-1}) + h(-\frac{1263}{2401}k_1(t_{k-1}) + \frac{39936}{26411}k_4(t_{k-1}) -\\
&\quad - \frac{64125}{26411}k_5(t_{k-1}) + \frac{5520}{2401}k_6(t_{k-1}))),\\
k_8(t_{k-1}) &= f(t_{k-1} + \frac{1}{7}h, y(t_{k-1}) + h(\frac{37}{392}k_1(t_{k-1}) +\\
&\quad + \frac{1625}{9408}k_5(t_{k-1}) - \frac{2}{15}k_6(t_{k-1}) + \frac{61}{6720}k_7(t_{k-1}))),\\
k_9(t_{k-1}) &= f(t_{k-1} + \frac{2}{3}h, y(t_{k-1}) + h(\frac{17176}{25515}k_1(t_{k-1}) - \frac{47104}{25515}k_4(t_{k-1}) +\\
&\quad + \frac{1325}{504}k_5(t_{k-1}) - \frac{41792}{25515}k_6(t_{k-1}) + \frac{20237}{145800}k_7(t_{k-1}) + \frac{4312}{6075}k_8(t_{k-1}))),\\
k_{10}(t_{k-1}) &= f(t_{k-1} + \frac{2}{7}h, y(t_{k-1}) + h(-\frac{23834}{180075}k_1(t_{k-1}) - \frac{77824}{1980825}k_4(t_{k-1}) -\\
&\quad - \frac{636635}{633864}k_5(t_{k-1}) + \frac{254048}{300125}k_6(t_{k-1}) - \frac{183}{7000}k_7(t_{k-1}) + \frac{8}{11}k_8(t_{k-1}) - \frac{324}{3773}k_9(t_{k-1}))),\\
k_{11}(t_{k-1}) &= f(t_{k-1} + h, y(t_{k-1}) + h(\frac{12733}{7600}k_1(t_{k-1}) - \frac{20032}{5225}k_4(t_{k-1}) + \frac{456485}{80256}k_5(t_{k-1}) -\\
&\quad - \frac{42599}{7125}k_6(t_{k-1}) + \frac{339227}{912000}k_7(t_{k-1}) - \frac{1029}{4180}k_8(t_{k-1}) + \frac{1701}{1408}k_9(t_{k-1}) + \frac{5145}{2432}k_{10}(t_{k-1}))),
\end{aligned}$$

$$k_{12}(t_{k-1}) = f(t_{k-1} + \frac{1}{3}h, y(t_{k-1}) + h(-\frac{27061}{204120}k_1(t_{k-1}) +$$
$$+\frac{40448}{280665}k_4(t_{k-1}) - \frac{1353775}{1197504}k_5(t_{k-1}) +$$
$$+\frac{17662}{25515}k_6(t_{k-1}) - \frac{71687}{1166400}k_7(t_{k-1}) + \frac{98}{225}k_8(t_{k-1}) + \frac{1}{16}k_9(t_{k-1}) + \frac{3773}{11664}k_{10}(t_{k-1}))),$$
$$k_{13}(t_{k-1}) = f(t_{k-1} + h, y(t_{k-1}) + h(\frac{11203}{8680}k_1(t_{k-1}) - \frac{38144}{11935}k_4(t_{k-1}) + \frac{2354425}{458304}k_5(t_{k-1}) -$$
$$-\frac{84046}{16275}k_6(t_{k-1}) + \frac{673309}{1636800}k_7(t_{k-1}) + \frac{4704}{8525}k_8(t_{k-1}) + \frac{9477}{10912}k_9(t_{k-1}) - \frac{1029}{992}k_{10}(t_{k-1}) +$$
$$+\frac{729}{341}k_{12}(t_{k-1}))).$$

For any given positive integer n, we define an auxiliary differential equation with a piecewise constant argument that corresponds to the nonlinear differential equation in (1) as follows:

$$\begin{gathered} y'(t) = K(t_{k-1}), \quad t \in [t_{k-1}, t_k), \quad k = 1, 2, \ldots, n, \\ y(0) = y_0, \end{gathered} \tag{4}$$

where the values y_k are defined by

$$y_k = y(t_k) = \lim_{t \to t_k - 0} \int_{t_{k-1}}^{t} K(t_{k-1})\, ds = \frac{1}{n}K(t_{k-1}).$$

Due to the construction of $K(t_{k-1})$, it satisfies the following asymptotic consistency condition:

$$K(t_{k-1}) - \left(a_m(t_{k-1})y_{k-1}^m + a_{m-1}(t_{k-1})y_{k-1}^{m-1} + \cdots + a_1(t_{k-1})y_{k-1} + a_0(t_{k-1})\right) \to 0 \,\text{as}\, n \to \infty.$$

The solution to the differential equation (4) is interpreted in the sense described in [17]. A function $y(t) := y_n(t)$ is called a solution of the initial value problem (4) if the following conditions are satisfied:

(i) $y(t)$ is continuous on $[0, 1]$;

(ii) $y'(t)$ exist and continuous on $[0, 1]$ with possible exception at points $t = t_k$, $k = 0, 1, \ldots, n$, where one-sided derivatives exist;

(iii) $y(t)$ satisfies the initial value problem (4) in (0,1), with the possible exception at the points $t = \frac{k}{n}$, $k = 0, 1, \ldots, n$.

Theorem 1. *For any positive integer number n the initial value problem (4) has a unique solution $y(t) := y_n(t)$ defined in (7).*

We observe that, under the assumption stated in (2), the function $f(t, y)$ is bounded. Consequently, the sequence of functions $y_n(t)$ is uniformly bounded; that is, there exists a positive constant C such that

$$|y_n(t)| \leqslant C.$$

The following theorem asserts that the function y_n serves as an approximate solution to the initial value problem (1).

Theorem 2. *For any $\varepsilon > 0$ there exists a positive number $n_0 = n(\varepsilon)$ such that for any positive integer number n with $n > n_0$ the inequality*

$$\sup_{t \in [0,1]} |y_n'(t) - (a_m(t)y_n^m + a_{m-1}(t)y_n^{m-1} + \ldots + a_1(t)y_n + a_0(t))| < \varepsilon,$$

holds, where y_n is solution of the initial value problem (4).

3 Proof of the main results

3.1 Proof of theorem 1.

Let n be an arbitrary positive integer. For $t \in [0, \frac{1}{n})$, the equation (4) reduces to

$$y'(t) = K(t_0).$$

Integrating both sides yields

$$y(t) = \int_0^t K(s_0)ds + y(0), \quad t \in [0, \frac{1}{n}). \tag{5}$$

Since the function $y(t)$ is continuous on the interval $[0, \frac{1}{n})$, the limit

$$y(\frac{1}{n}) = \lim_{t \to \frac{1}{n} - 0} y(t),$$

exists. Let us now define $y(t)$ to be the solution of equation (4) on the interval $t \in [t_{k-2}, t_{k-1})$ with the initial condition

$$y(t_{k-1}) = \lim_{t \to \frac{k-1}{n} - 0} y(t). \tag{6}$$

By integrating equation (4) over the interval $[t_{k-1}, t_k)$ and applying the initial condition (6), we obtain the solution in the form

$$y(t) = K(t_{k-1})(t - t_{k-1}) + y(t_{k-1}), \quad t \in [t_{k-1}, t_k), \tag{7}$$

for each $k = 2, 3, \dots, n$. By construction, the function y defined piecewise by (7) is differentiable on each open subinterval $(0, \frac{1}{n}) \cup (\frac{1}{n}, \frac{2}{n}) \cup \dots \cup \left(\frac{n-1}{n}, \frac{k}{n}\right)$ and is continuous on the entire interval $[0, 1]$.

Therefore, it follows that the function y constructed in this manner is the unique solution to the initial value problem given by equation (4).

3.2 Proof of theorem 2.

Let y_n be the solution to equation (4). Then, for $t \in [t_{k-1}, t_k)$, the following identity holds:

$$y_n'(t) - f(t, y_n(t)) = K(t_{k-1}) - f(t, y_n(t)).$$

By applying the triangle inequality, we obtain

$$|K(t_{k-1}) - f(t, y_n(t))| \leqslant |K(t_{k-1}) - f(t, y_n(t_{k-1}))| + |f(t, y_n(t_{k-1})) - f(t, y_n(t))|.$$

According to equation (7), there exists a constant $C > 0$ such that

$$\left|K(t_{k-1}) - f(t, y_n(t_{k-1}))\right| \leqslant \frac{C}{n},$$

for sufficiently large values of n.

Now, using the representation of $y_n(t)$ from equation (7), we can express

$$f(t, y_n(t)) = f\big(t, K(t_{k-1})(t - t_{k-1}) + y(t_{k-1})\big), \quad t \in [t_{k-1}, t_k).$$

Since the function f is differentiable on the domain D, it is Lipschitz continuous with respect to the second variable. Hence, there exists a constant $F > 0$ such that

$$|f(t, y_n(t_{k-1})) - f(t, y_n(t))| \leqslant \frac{F}{n},$$

for all $t \in \left[\frac{k-1}{n}, \frac{k}{n}\right)$ and sufficiently large n.

Combining these estimates, we deduce

$$|y_n'(t) - f(t, y_n(t))| \leqslant \frac{C+F}{n},$$

for all $t \in \left[t_{k-1}, t_k\right)$, $k = 1, 2, \ldots, n$, and large enough n. This completes the proof of the theorem.

4 Summary of the Algorithm

The computational procedure for obtaining the solution to the initial value problem can be summarized as follows:

Input: Input initial data.

Step 1: Evaluate $K(t_0)$ using the expression provided in equation (3).

Step 2: Determine the function $y_n(t)$ from equation (5) over the interval $t \in [0, \frac{1}{n})$.

Step 3: Compute the one-sided limit at the junction point:

$$y_n\Big(\frac{1}{n}\Big) = \lim_{t \to \frac{1}{n} - 0} y_n(t).$$

Step 4: Evaluate $K(t_{k-1})$ using formula (3) for each subsequent subinterval.

Step 5: Utilize equation (7) to compute $y_n(t)$ on the interval $t \in [t_{k-1}, t_k)$, where $k = 2, 3, \ldots, n$, and determine the left-hand limit:

$$y_n(t_{k-1}) = \lim_{t \to t_{k-1} - 0} y_n(t).$$

Step 6: Construct the piecewise-defined approximate solution using equation (7) across the entire interval.

Step 7: Calculate the maximum pointwise absolute error between the exact solution y and the numerical approximation y_n at the mesh points:

$$\mathrm{MaxErr} = \max_{1 \leqslant k \leqslant n} |y(t_{k-1}) - y_n(t_{k-1})|.$$

Output: If the computed error remains within a pre-specified tolerance, the iteration is considered converged and the process is terminated.

5 Numerical Results

To obtain an approximate solution to equation (1) by applying Theorem 2, it is required to compute the function y_n, whose explicit form is provided in equation (7).

In Examples 1-5, the maximum absolute error between the exact solution and the approximate solution was computed over the interval $[0, 1]$ to assess the accuracy of the proposed method. The maximum errors at the points t_k between the numerical solutions obtained via the proposed method and the corresponding exact solutions for Examples

1-5, are reported in Tables 1-5, respectively.

Example 1. Consider the following first-order nonlinear differential equation as presented in [6]:

$$y'(t) = (-20y^2(t) + 20y(t))\cos t, \quad y(0) = 0.5,$$

which admits the exact solution

$$y(t) = \frac{1}{1 + e^{-20\sin t}}.$$

Table 1 Data for problem in example 1.

h	RADAU method MaxErr	EMOHB method MaxErr	Present method MaxErr
10^{-4}	2.7569×10^{-5}	1.8451×10^{-7}	4.1316×10^{-13}
10^{-5}	8.1137×10^{-6}	9.4868×10^{-9}	6.9537×10^{-22}
10^{-6}	3.3542×10^{-7}	1.3678×10^{-9}	3.4323×10^{-30}

Example 2. Consider the logistic-type differential equation discussed in [7]:

$$y'(t) = y(t) - y^2(t), \quad 0 < t \leqslant 1,$$

subject to the initial condition $y(0) = 2$. The exact analytical solution to this problem is given by

$$y(t) = \frac{2}{2 - e^{-t}}.$$

Table 2 Maximum errors are represented between approximated and exact solutions for example 2.

$N = 2^{J+1}$	MaxErr	RC(N)	CPU time (s)	Present method	RC(N)	CPU time (s)
2^2	6.3828×10^{-02}	–	0.0002	1.0530×10^{-08}	–	0.71
2^3	2.5369×10^{-02}	1.3311	0.0021	5.7981×10^{-11}	7.5047	0.73
2^4	8.3988×10^{-03}	1.5948	0.0398	2.2840×10^{-13}	7.9879	0.76
2^5	2.4604×10^{-03}	1.7713	0.0737	8.8002×10^{-16}	8.0198	0.80
2^6	6.6969×10^{-04}	1.8773	0.0903	3.4152×10^{-18}	8.0094	0.85
2^7	1.7498×10^{-04}	1.9363	0.4129	1.3283×10^{-20}	8.0062	1.02
2^8	4.4743×10^{-05}	1.9675	1.6522	5.1781×10^{-23}	8.0030	1.27
2^9	1.1314×10^{-05}	1.9836	3.0929	2.0205×10^{-25}	8.0016	1.67
2^{10}	2.8445×10^{-06}	1.9918	5.2095	7.8860×10^{-28}	8.0012	2.77
2^{11}	3.9543×10^{-07}	1.9924	9.0764	3.0804×10^{-30}	7.5875	5.14

$$\text{Rate of convergence } = \frac{\log[E_c(N/2)/E_c(N)]}{\log 2}.$$

Example 3. The Obreschkoff method is utilized to numerically approximate the solution of the initial value problem presented in [8]:

$$y'(t) = y^2(t), \quad 0 \leqslant t \leqslant 0.9, \quad y(0) = 1.$$

For this example, the parameters are specified as follows: $N = 10$, $h = 0.09$, and the time nodes are defined by $t_i = 0.09i$. The obtained numerical solution is then evaluated against the analytical solution, which is given by

$$y(t) = \frac{1}{1-t}.$$

Table 3 Comparison of absolute errors in the sixth-order Runge-Kutta (RK) method and the eighth-order Obreschkoff method (OM) applied in example 3, with step size $h = 0.09$.

$\mathbf{t_i}$	**RK Error**	**OM Error**	**Present Method**
0.00	0.0000	0.0000	0.0000
0.09	7.8166×10^{-8}	9.0000×10^{-12}	2.5215×10^{-13}
0.18	2.3478×10^{-7}	3.8000×10^{-11}	9.9592×10^{-13}
0.27	5.5693×10^{-7}	1.3000×10^{-10}	3.3394×10^{-12}
0.36	1.2531×10^{-6}	4.5400×10^{-10}	1.1622×10^{-11}
0.45	2.8754×10^{-6}	1.8170×10^{-9}	4.6171×10^{-11}
0.54	7.0909×10^{-6}	9.0540×10^{-9}	2.2782×10^{-10}
0.63	1.9952×10^{-5}	6.2989×10^{-8}	1.5566×10^{-9}
0.72	7.0377×10^{-5}	7.3986×10^{-7}	1.7573×10^{-8}
0.81	3.7662×10^{-4}	2.1780×10^{-5}	4.6164×10^{-7}
0.90	4.2642×10^{-3}	4.9442×10^{-3}	6.4645×10^{-5}

Example 4. Let us examine the following stiff initial value problem [9]:

$$y'(t) = 5e^{5t}(y(t) - t)^2 + 1, \quad y(0) = -1, \quad 0 < t \leqslant 1,$$

for which the exact solution is

$$y(t) = t - e^{5t}.$$

Table 4 Comparison between the first-order approximate solution obtained by multistage optimal homotopy asymptotic method (MOHAM), and that of the present method in example 4.

t	**Exact solution**	**Present solution**	**MOHAM solution**	**MOHAM absolute error**	**Present absolute error**
0.2	-0.167879441	-0.167879687	-0.167903440	2.3998×10^{-5}	2.4613×10^{-7}
0.4	0.264664717	0.264664594	0.2646526384	1.2078×10^{-5}	1.2386×10^{-7}
0.6	0.550212932	0.550212881	0.5501192179	9.3714×10^{-5}	5.0072×10^{-8}
0.8	0.781684361	0.781684344	0.7835685861	1.8842×10^{-3}	1.9031×10^{-8}
1.0	0.993262053	0.993262043	0.9874166761	5.8454×10^{-6}	7.0835×10^{-9}

Here, the computations are performed with a step size of $h = 0.1$.

Example 5. Examine the following nonlinear initial value problem as presented in [9]:

$$y'(t) = -y^2 - 4y - 3, \quad y(0) = -2, \quad 0 \leqslant t \leqslant 3,$$

for which the exact solution is given by

$$y(t) = -3 + \frac{2}{1 - e^{-2t}}.$$

Here, the computations are performed with a step size of $h = 0.3$.

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6 Concluding Remarks

A new and innovative method is presented for approximating solutions of differential equations by constructing an auxiliary equation with piecewise constant arguments,

Table 5 Comparison between the first-order approximate solution obtained by the MOHAM method and that of the present method in example 5.

t	Exact solution	Present solution	MOHAM solution	MOHAM absolute error	Present absolute error
0.3	-1.70868738755	-1.70868738777	-1.708695833	8.4454×10^{-6}	2.2311×10^{-10}
0.6	-1.46295043300	-1.46295043364	-1.462957634	7.2013×10^{-6}	6.3092×10^{-10}
0.9	-1.28370212980	-1.28370213081	-1.283707693	5.5637×10^{-6}	1.0156×10^{-9}
1.2	-1.16634539299	-1.16634539414	-1.166351921	6.5289×10^{-6}	1.1533×10^{-9}
1.5	-1.09485174636	-1.09485174744	-1.094859314	7.5675×10^{-6}	1.0831×10^{-9}
1.8	-1.05319398716	-1.05319398806	-1.053201320	7.3325×10^{-6}	9.0649×10^{-10}
2.1	-1.02954806339	-1.02954806409	-1.029554201	6.1375×10^{-6}	6.9756×10^{-10}
2.4	-1.01632514231	-1.01632514281	-1.016329789	4.6472×10^{-6}	5.0330×10^{-10}
2.7	-1.00899254632	-1.00899254667	-1.008995833	3.2876×10^{-6}	3.4581×10^{-10}
3.0	-1.00494524632	-1.00494524654	-1.004947465	2.2187×10^{-6}	2.2903×10^{-10}

aligned with the original problem. A novel concept combining the Runge-Kutta method with a piecewise-smooth solution, dependent on a positive integer n, is introduced and shown to converge to the true solution as $n \to \infty$. Additionally, a hybrid numerical algorithm is developed for solving first-order initial value problems, implemented in the MAPLE environment. Comparative analysis confirms the stability and convergence of the method, with numerical results demonstrating higher accuracy than classical Runge–Kutta schemes. Numerical experiments are provided to demonstrate the effectiveness of the proposed method, and the errors between the approximate and exact solutions are evaluated.

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ПРИБЛИЖЁННОЕ РЕШЕНИЕ ЗАДАЧ С НАЧАЛЬНЫМИ УСЛОВИЯМИ ДЛЯ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ПЕРВОГО ПОРЯДКА С ИСПОЛЬЗОВАНИЕМ КОМБИНИРОВАННОГО МЕТОДА РУНГЕ–КУТТЫ И МЕТОДА С КУСОЧНО-ПОСТОЯННЫМ АРГУМЕНТОМ

Жумаев З.З.
zafarlangar8708@gmail.com
Самаркандский государственный университет имени Шарофа Рашидова,
140104, Узбекистан, г. Самарканд, Университетский бульвар, д. 15.

В работе представлен эффективный метод приближённого решения определённого класса дифференциальных уравнений первого порядка с переменными коэффициентами. Подход основан на построении вспомогательного дифференциального уравнения, которое объединяет метод Рунге–Кутты с кусочно-постоянным аргументом, выводится из исходной задачи с начальными условиями и параметризуется положительным целым числом n. Показано, что при достаточно больших n это вспомогательное уравнение имеет единственное кусочно-гладкое решение, приближающее рассматриваемую задачу. Получены оценки остаточной погрешности, позволяющие количественно оценить точность и проиллюстрировать влияние параметра n. Численные результаты показывают, что метод обеспечивает более высокую точность при меньшем числе вычислительных шагов по сравнению с классическим методом Рунге–Кутты и родственными схемами, а предложенная схема допускает расширение на более широкий класс нелинейных уравнений.

Ключевые слова: задача с начальными условиями, кусочно-постоянный аргумент, приближённое решение, метод Рунге–Кутты, абсолютная погрешность.

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Об издании

Журнал «Проблемы вычислительной и прикладной математики» является рецензируемым периодическим научным изданием. Данный научный журнал, основанный в 2015 году, наследует длительной и богатой истории сборника научных трудов «Вопросы вычислительной и прикладной математики», издававшегося в Институте кибернетики Академии наук Узбекистана с 1970 года.

Всего было издано 130 выпусков того сборника. на его страницах были широко представлены результаты научных исследований в области вычислительной математики, математической физики, алгоритмизации, дискретной математики, механики сплошных сред и математического моделирования. Общее число опубликованных научных статей составило около 2000. Долгое время сборник являлся единственным изданием в Центральной Азии, который освещал последние достижения по таким разделам вычислительной и прикладной математики, как моделирование, оптимизация и управление. Опубликованные в сборнике научные статьи получали высокие оценки ученых и специалистов зарубежных стран, а сам сборник, начиная с третьего года издания, индексировался авторитетными реферативными базами данных «Mathematical Reviews» и «Zentralblatt MATH», а также в российских реферативных журналах «Математика», «Информатика» и «Механика».

Редакция журнала выражает надежду на то, что данный журнал в новом качестве будет долгие годы продолжать славную историю своего предшественника в деле освещения, распространения и популяризации научных достижений. С 2016 года журнал «Проблемы вычислительной и прикладной математики» включен в перечень изданий, рекомендуемых для публикации научных результатов, ВАК Республики Узбекистан, а с 2018 года – в список рецензируемых журналов ВАК при Президенте Республики Таджикистан. С 2015 года журнал индексируется в базе данных РИНЦ. Также редакция проводит работу по включению журнала в авторитетные международные базы научных изданий WoS, Scopus и др.

Основная цель журнала – это освещение результатов фундаментальных и прикладных научных исследований, экспериментальных разработок сотрудников образовательных, научно-исследовательских и производственных учреждений, соискателей учёной степени доктора наук. В число основных задач журнала входят формирование научной составляющей в НИИ и ВУЗах; пропаганда достижений существующих научных школ; поощрение открытой научной полемики, способствующей повышению качества научных исследований, эффективности экспертизы научных идей и трудов; содействие расширению научного сотрудничества между научными центрами стран СНГ и дальнего зарубежья.

В журнале публикуются научные работы по следующим тематическим направлениям: математическое моделирование физических, технических, биологических и социально-экономических систем; вычислительная математика и численные методы для решения прикладных задач; решение обратных и некорректно поставленных задач; интеллектуальный анализ данных, распознавание образов, обработка сигналов и изображений; искусственный интеллект, машинное обучение и представление знаний; параллельные и распределенные вычисления; технологии программирования.

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