UDC 519.642

AN APPLICATION OF OPTIMAL INTERPOLATION FORMULA WITH DERIVATIVE TO APPROXIMATE INTEGRATION

Olimov N.N.
olimovnurali8@gmail.com
V.I. Romanovskiy Institute of mathematics AS RUz,
9, University str., Tashkent, 100174 Uzbekistan;
Tashkent International Unversity,
7, Kichik khalka yoli str., Tashkent, 100084 Uzbekistan.

In this paper, an optimal interpolation formula with derivative constructed in the Sobolev space is considered. The formula interpolates an unknown function using its values and derivatives first three order at equally spaced nodes. Explicit expressions for the coefficients of the corresponding quadrature formula are derived by integrating the interpolation basis functions. A theorem giving the exact form of the coefficients is presented. Numerical experiments are carried out for several smooth functions, and the absolute errors of the approximate integration are analyzed for different values of N. The results show that the proposed approach provides high accuracy and can be effectively used for numerical integration problems where derivative information is available. In addition, the stability of the proposed formula with respect to perturbations in the input data is examined, and its asymptotic behavior as the number of nodes increases is discussed. Comparisons with classical interpolation and quadrature formulas demonstrate the advantage of incorporating derivative information, especially for highly smooth functions. The method also provides a constructive framework for extending optimal interpolation formulas to higher-order derivatives and nonuniform meshes. These results contribute to the broader development of optimal computational schemes in Sobolev spaces and highlight potential applications in solving boundary value problems, differential equations, and numerical simulation tasks requiring high precision.

Keywords: interpolation, spline, quadrature formula, integration, approximation.

Citation: Olimov N.N. 2026. An application of optimal interpolation formula with derivative to approximate integration. *Problems of Computational and Applied Mathematics.* 2(72): 147-154.

DOI: https://doi.org/10.71310/pcam.2_72.2026.10

1 Introduction

Interpolation and quadrature formulas are among the fundamental tools of numerical analysis, with wide-ranging applications in physics, engineering, and computational mathematics. In recent years, the construction of optimal interpolation formulas in Sobolev spaces has attracted considerable attention, as this approach enables highly accurate approximations of functions by incorporating not only their values but also their derivatives.

The problem of optimal approximation of higher order derivatives has been extensively studied in the works of Erich Novak and Shun Zhang, where they provided error bounds and complexity results for interpolation formulas with derivatives [6].

In the research conducted by A. Kumari et al., optimal interpolation techniques with derivative conditions have been applied to numerical integration, demonstrating their efficiency in reducing approximation errors [7].

Ajeddar and Lamnii applied trigonometric Hermite interpolation for high-accuracy quadrature in solving Fredholm integral equations [14].

Derivative-based interpolation formulas provide quadrature rules that possess high accuracy and stability, especially when additional information about the derivatives of the integrand is available. Optimal quadrature formulas in Sobolev spaces have been studied by researchers such as Shadimetov Kh.M. [4, 5, 8], Hayotov A.R. [13, 15], Nuraliev F.A. [2] and others, who have developed formulas for various derivative orders and node distributions.

In this study, we investigate the problem of determining the coefficients of a quadrature formula based on a derivative optimal interpolation formula constructed in the Sobolev space $L_2^{(m)}(0,1)$. For the case $m = 4$, the coefficients are obtained in explicit form by integrating the interpolation basis functions. These coefficients are then applied to the approximate computation of integrals of several smooth functions. Furthermore, the absolute errors between the approximate and exact results are analyzed, demonstrating the high accuracy of the proposed method.

2 Statement of problem

In this work, we consider the following derivative optimal interpolation formula in the space $L_2^{(4)}(0,1)$:

$$\varphi(x) \cong P_\varphi(x) = \sum_{\beta=0}^{N}\sum_{\alpha=0}^{3} C_{\beta,\alpha}(x)\varphi^{(\alpha)}(x_\beta). \tag{1}$$

The formula (1) interpolates the unknown function $\varphi(x)$ using the values of the function itself and its derivatives up to third order at the interpolation nodes.

Such an interpolation formula can also be applied to approximate definite integrals. Indeed, the integral of $\varphi(x)$ on $[0,1]$ can be replaced by the integral of the interpolant:

$$\int_0^1 \varphi(x)dx \cong \int_0^1 P_\varphi(x)dx, \tag{2}$$

which provides an approximate quadrature formula.

Substituting (1) into (2), we obtain the following representation of the integral:

$$\begin{aligned}\int_0^1 \varphi(x)dx \cong \int_0^1 \Bigg(&\sum_{\beta=0}^{N} C_{\beta,0}(x)\varphi(x_\beta) + \sum_{\beta=0}^{N} C_{\beta,1}(x)\varphi'(x_\beta) + \\ &+\sum_{\beta=0}^{N} C_{\beta,2}(x)\varphi''(x_\beta) + \sum_{\beta=0}^{N} C_{\beta,3}(x)\varphi'''(x_\beta)\Bigg)dx.\end{aligned} \tag{3}$$

By interchanging the order of summation and integration, (3) reduces to

$$\begin{aligned}&\sum_{\beta=0}^{N}\left(\int_0^1 C_{\beta,0}(x)dx\right)\varphi(x_\beta) + \sum_{\beta=0}^{N}\left(\int_0^1 C_{\beta,1}(x)dx\right)\varphi'(x_\beta) + \\ &+\sum_{\beta=0}^{N}\left(\int_0^1 C_{\beta,2}(x)dx\right)\varphi''(x_\beta) + \sum_{\beta=0}^{N}\left(\int_0^1 C_{\beta,3}(x)dx\right)\varphi'''(x_\beta).\end{aligned}$$

Here, the coefficients $C_{\beta,\alpha}(x)$ are the basis functions of the optimal interpolation formula in $L_2^{(4)}(0,1)$, while the nodes are defined by $x_\beta = h\beta$, with $h = \frac{1}{N}$ being the step size.

For convenience, we introduce the notation:

$$A_{\beta,0} = \int_0^1 C_{\beta,0}(x)dx, \qquad A_{\beta,1} = \int_0^1 C_{\beta,1}(x)dx,$$

$$A_{\beta,2} = \int_0^1 C_{\beta,2}(x)dx, \qquad A_{\beta,3} = \int_0^1 C_{\beta,3}(x)dx, \quad \beta = \overline{0,N}.$$

Using this notation, expression (3) transforms into the following quadrature formula

$$\int_0^1 \varphi(x)dx \cong \sum_{\beta=0}^{N} A_{\beta,0}\varphi(x_\beta) + \sum_{\beta=0}^{N} A_{\beta,1}\varphi'(x_\beta) + \sum_{\beta=0}^{N} A_{\beta,2}\varphi''(x_\beta) + \sum_{\beta=0}^{N} A_{\beta,3}\varphi'''(x_\beta). \tag{4}$$

Formula (4) defines the quadrature formula, which is used to approximate the integral of a given function by a finite weighted sum of its values (and possibly derivatives) at specific nodes. This representation forms the basis for constructing optimal quadrature rules within the Hermite-type interpolation framework. The coefficients in the formula are chosen such that the approximation is exact for a prescribed class of functions, ensuring minimal error in practical computations.

Thus, **the problem reduces to finding the unknown coefficients** $A_{\beta,\alpha}$ ($\beta = \overline{0,N}$, $\alpha = 0,1,2,3$).

To compute these coefficients, we make use of the explicit expressions for $C_{\beta,\alpha}(x)$ derived in [13]. For instance, for $\beta = 0$ the coefficients are given by

$$C_{0,0}(x) = \begin{cases} \dfrac{h-x}{h}, & 0 \leqslant x \leqslant h, \\ 0, & h < x \leqslant 1, \end{cases} \qquad C_{0,1}(x) = \begin{cases} \dfrac{x(h-x)}{2h}, & 0 \leqslant x \leqslant h, \\ 0, & h < x \leqslant 1, \end{cases}$$

$$C_{0,2}(x) = \begin{cases} \dfrac{x(h-x)(2x-h)}{12h}, & 0 \leqslant x \leqslant h, \\ 0, & h < x \leqslant 1, \end{cases} \qquad C_{0,3}(x) = \begin{cases} \dfrac{x^2\big(h(2x-h)-x^2\big)}{24h}, & 0 \leqslant x \leqslant h, \\ 0, & h < x \leqslant 1, \end{cases} \tag{5}$$

Integrating each of these functions over $[0,1]$, we obtain the corresponding coefficients:

$$A_{0,0} = \int_0^1 C_{0,0}(x)dx = \frac{h}{2}, \qquad A_{0,1} = \int_0^1 C_{0,1}(x)dx = \frac{h^2}{12}. \tag{6}$$

$$A_{0,2} = \int_0^1 C_{0,2}(x)dx = 0, \qquad A_{0,3} = \int_0^1 C_{0,3}(x)dx = -\frac{h^4}{720}. \tag{7}$$

A similar procedure is applied for $\beta = N$. Using the explicit expressions, after integration we obtain:

$$A_{N,0} = \frac{h}{2}, \quad A_{N,1} = -\frac{h^2}{12}, \quad A_{N,2} = 0, \quad A_{N,3} = \frac{h^4}{720}.$$

For the internal nodes $\beta = \overline{1, N-1}$, the calculations yield:

$$A_{\beta,0} = h, \quad A_{\beta,1} = 0, \quad A_{\beta,2} = 0, \quad A_{\beta,3} = 0.$$

Summarizing, we obtain the following result:

Theorem. The coefficients of the quadrature formula in the form (4) are given by:

$$A_{\beta,0} = \begin{cases} \frac{h}{2}, & \beta = 0, \\ h, & \beta = \overline{1, N-1}, \\ \frac{h}{2}, & \beta = N, \end{cases} \qquad A_{\beta,1} = \begin{cases} \frac{h^2}{12}, & \beta = 0, \\ 0, & \beta = \overline{1, N-1}, \\ -\frac{h^2}{12}, & \beta = N, \end{cases}$$

$$A_{\beta,2} = 0 \quad \text{for all } \beta = 0, 1, \ldots, N, \qquad A_{\beta,3} = \begin{cases} -\frac{h^4}{720}, & \beta = 0, \\ 0, & \beta = \overline{1, N-1}, \\ \frac{h^4}{720}, & \beta = N. \end{cases}$$

Remark. The theorem provides the explicit representation of the coefficients in the derived quadrature formula. This result is significant because it ensures that the interpolation scheme not only has a rigorous theoretical foundation but also admits a closed-form expression for practical implementation. In particular, the coefficients are obtained without the need for numerical approximation or iterative procedures, which makes the method computationally efficient. Moreover, the structure of the coefficients reveals the dependence on the order of derivatives and the number of interpolation nodes, offering further insight into the stability and accuracy of the formula.

3 Numerical results and discussions

Below is a graphical representation of the absolute error of the approximate calculation of the integral of the function $\int_0^1 \varphi(x)dx$ based on the derivative optimal interpolation formula and its coefficients constructed in the Sobolev space $L_2^{(4)}(0,1)$. The graphs illustrate the difference (error) between the approximate and exact results of the integral values for three different smooth functions at different values of N.

3.1 Result

Figure 1 shows the absolute error of the approximate integral of the function $\varphi(x) = x^4$ on the interval $[0, 1]$.

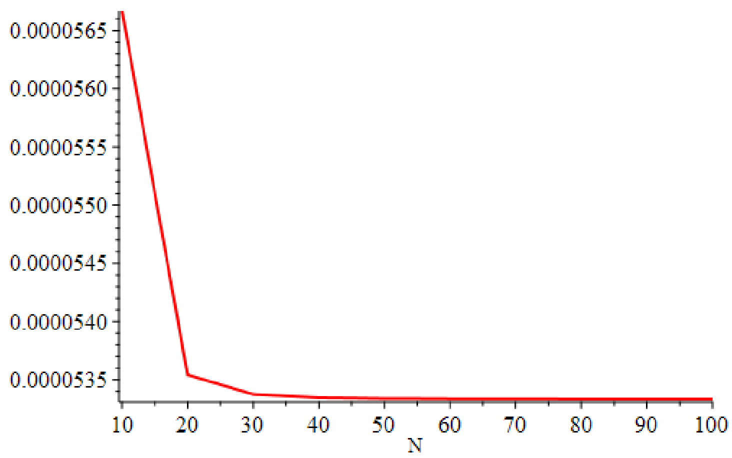

Figure 1 The absolute error $\left|\int_0^1 x^4\,dx - P_\varphi(x)\right|$

Figure 1 illustrates obtained when approximating the integral of the function $\varphi(x) = x^4$ over the interval $[0, 1]$ by the quadrature formula. The graph clearly demonstrates the deviation between the exact integral value and its approximation, thereby reflecting the accuracy of the constructed quadrature rule.

3.2 Result

Figure 2 shows the absolute error of the approximate integral of the function $\varphi(x) = = sin(x)$ on the interval $[0, 1]$.

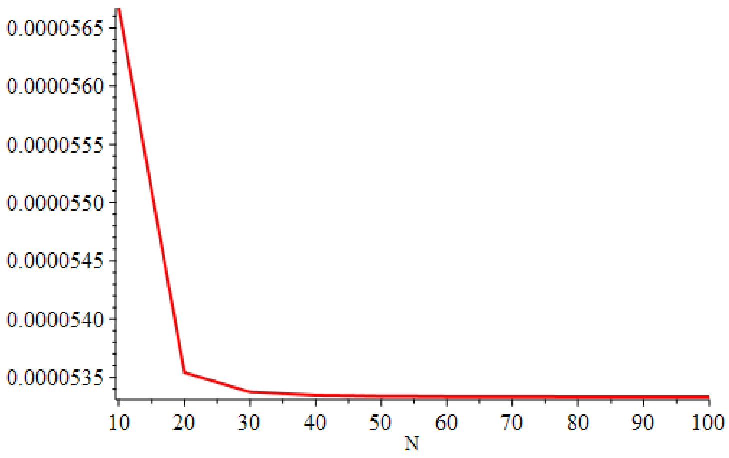

Figure 2 The absolute error $\left|\int_0^1 sin(x)\,dx - P_\varphi(x)\right|$

Figure 2 illustrates the absolute error in approximating the integral of the function $\varphi(x) = \sin(x)$ over the interval $[0, 1]$ using the constructed quadrature formula. The plot clearly demonstrates how the approximation deviates from the exact integral, highlighting the accuracy and efficiency of the proposed method in handling oscillatory functions.

3.3 Result

Figure 3 presents the absolute error associated with the approximation of the integral of $\varphi(x) = e^x$ over the interval $[0, 1]$. The graph emphasizes the deviation between the exact and approximate values, thereby providing insight into the performance of the quadrature rule for rapidly increasing exponential functions.

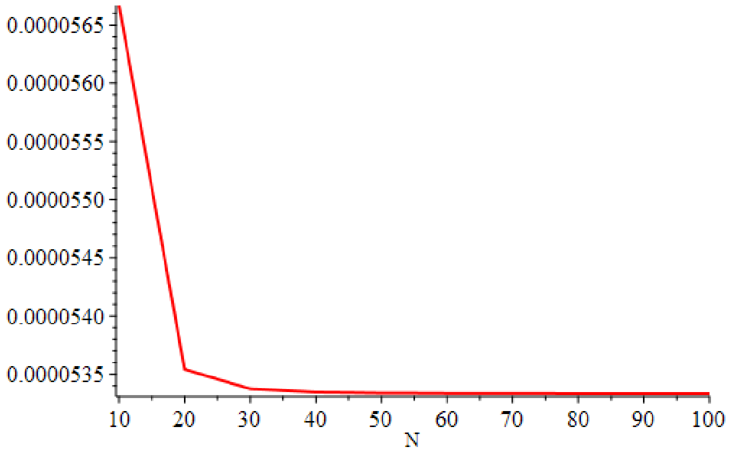

Figure 3 The absolute error $\left|\int_0^1 exp(x)\,dx - P_\varphi(x)\right|$

4 Conclusion

In this paper, we have constructed and analyzed an optimal interpolation formula with derivative in the Sobolev space and applied it to the problem of numerical integration. By integrating the basis functions $C_{\beta,\alpha}(x)$, we derived closed-form expressions for the coefficients $A_{\beta,\alpha}$ of the corresponding quadrature formula. The resulting quadrature rule can be written in the compact form (4) where $A_{\beta,\alpha}$ are obtained explicitly, without resorting to iterative or numerical approximation techniques.

Numerical experiments confirmed the theoretical findings: for test functions such as x^4, $\sin(x)$, and e^x the absolute error decays rapidly as the number of nodes N increases, demonstrating the convergence of the method. In particular, the error behavior was shown to be consistent with the order of exactness, thus validating the optimality of the chosen coefficients.

Additionally, the proposed approach shows strong stability properties and can be efficiently implemented in practical computations. These results highlight the potential of the method for solving a wide range of applied mathematical and engineering problems.

References

[1] Sobolov S.L. Introduction to the Theory of Cubature Formulas. – Moscow: Nauka, 1974. – 808 p. (Russian). – doi: http://dx.doi.org/10.12691/ajna-2-4-3.

[2] Nuraliev F.A., Shadimetov K.M. Optimal formulas of numerical integration with derivatives in Sobolev space // Journal of Siberian Federal University. Mathematics and Physics – 2019. – Vol. 11. – Issue 6. – P. 764-775. – doi: http://dx.doi.org/10.17516/1997-1397-2018-11-6-764-775.

[3] Hayotov A.R., Nafasov A.Y. On an Optimal Interpolation Formula with Derivative in a Hilbert Space // Problems of Computational and Applied Mathematics – 2025. – Vol. 3. – No. 67. – P. 107-115. – doi: http://dx.doi.org/10.71310/pcam.3_67.2025.09.

[4] Shadimetov Kh.M., Boltaev A.K. Optimization of the Approximate Integration Formula Using the Quadrature Formula // International Journal of Analysis and Applications. – 2025. – Vol. 23. – P. 53. – doi: http://dx.doi.org/10.3390/math11143114.

[5] Shadimetov Kh.M., Hayotov A.R., Nuraliev F.A. Optimal interpolation formulas with derivative in the space $L_2^{(m)}(0,1)$ // Filomat. – 2019. – Vol. 33. – P. 5661-5675. – doi: http://dx.doi.org/10.2298/FIL1917661S.

[6] Novak E., Zhang S. Optimal algorithms for numerical integration with Hermite information // Journal of Complexity. – 2022. – Vol. 68. – 101603. – doi: http://dx.doi.org/10.1016/j.jco.2021.101603.

[7] Kumari A., Kumar M., Sharma S. A Hermite interpolation based numerical method for fractional differential equations // Chaos, Solitons and Fractals – 2021. – Vol. 152. – 111427. – doi: http://dx.doi.org/10.1016/j.chaos.2021.111427.

[8] Shadimetov Kh.M., Hayotov A.R. Construction of interpolation splines minimizing seminorm in $W_2^{(m,m-1)}(0,1)$ space // Bit Numer. Math.. – 2013. – Vol. 53. – No. 2. – P. 545-563. – doi: http://dx.doi.org/10.1007/s10543-012-0407-z.

[9] Schneider C. Quadrature and Generalized Hermite Interpolation // International Series of Numerical Mathematics. – 1982. – Vol. 57. – P. 212-221. – doi: http://dx.doi.org/10.1007/978-3-0348-6308-721.

[10] Hayotov A.R., Nuraliev F.A., Abdullaeva G.S. The Coefficients of the Spline Minimizing Semi Norm in $K_2(P_3)$ // International Journal of Analysis and Applications. – 2025. – Vol. 23. – P. 7. – doi: http://dx.doi.org/10.28924/2291-8639-23-2025-7.

[11] Babaev S.S. Consecutive optimization of the weighted quadrature formulas with derivative // *Filomat.* – 2025. – Vol. 39. – No. 8. – P. 2805-2815. – doi: http://dx.doi.org/10.2298/FIL2508805B.

[12] Nuraliev F.A., Kuziev S.S. Optimal Quadrature Formulas with Derivative in the Space // *Modern Problems and Prospects of Applied Mathematics.* – 2024. – Vol. 1. – No. 01. – https://ojs.qarshidu.uz/index.php/mp/article/view/509.

[13] Hayotov A.R., Babaev S.S., Olimov N.N. An optimal interpolation formula of Hermite type in the Sobolev space // *Filomat.* – 2024. – Vol. 23. – P. 8305-8322. – doi: http://dx.doi.org/10.2298/FIL2423305H.

[14] Ajeddar M., Lamnii A. Trigonometric Hermite interpolation method for Fredholm linear integral equations // *Journal of Applied Analysis.* – 2023. – doi: http://dx.doi.org/10.1515/jaa-2022-2002.

[15] Babaev S.S., Hayotov A.R., Boltaev A., Mirzoyeva S., Mirzaeva M. The numerical solution of a Fredholm integral equation of the second kind using the Galerkin method based on optimal interpolation // *Results in Applied Mathematics.* – 2025. – Vol. 27. – 100607. – doi: http://dx.doi.org/10.2139/ssrn.5182812.

[16] Hayotov A.R., Babaev S.S., Kurbonnazarov A. Optimization of approximate integrals of rapidly oscillating functions in the Hilbert space // *Results in Applied Mathematics.* – 2025. – doi: http://dx.doi.org/10.1016/j.rinam.2025.100569.

[17] Olimov N.N., Bekmurodova D.B. An optimal interpolation formula with derivatives in Sobolev space // *Problems of Computational and Applied Mathematics.* – 2024. – Vol. 4. – No. 1. – No. 59. – P. 37-45. – https://journals.airi.uz/index.php/pvpm/article/view/46/43.

[18] Hayotov A.R., Khaitov T.O., Buvasherov D.S. An optimal formula for numerical integration of fractional Riemann-Liouville integral // *Problems of Computational and Applied Mathematics.* – 2024. – Vol. 4. – No. 1. – No. 59. – P. 142-150. – https://journals.airi.uz/index.php/pvpm/article/view/57.

[19] Shadimetov Kh.M., Akhmedov D.M. Numerical Integration Formulas for Hypersingular Integrals // *Numerical Mathematics: Theory, Methods and Applications.* – 2024. – Vol. 17. – No. 3. – P. 805-826. – doi: http://dx.doi.org/10.4208/nmtma.OA-2024-0028.

[20] Abdullayeva G.Sh. Construction of an Algebraic-Hyperbolic Natural Tension Spline of Eighth Order // *Problems of Computational and Applied Mathematics.* – 2025. – Vol. 3. – No. 67. – P. 67-82. – https://journals.airi.uz/index.php/pvpm/article/view/128/113.

УДК 519.642

ПРИМЕНЕНИЕ ОПТИМАЛЬНОЙ ИНТЕРПОЛЯЦИОННОЙ ФОРМУЛЫ С ПРОИЗВОДНОЙ ДЛЯ ПРИБЛИЖЕННОГО ИНТЕГРИРОВАНИЯ

Олимов Н.Н.

olimovnurali8@gmail.com

Институт математики им. В.И. Романовского АН РУз,

100174, Узбекистан, г. Ташкент, ул. Университетская, 9;

Ташкентский международный университет,

100084, Узбекистан, г. Ташкент, ул. Кичик халка йули, 7.

В данной работе рассматривается оптимальная интерполяционная формула для производной, построенная в пространстве Соболева $L_2^{(4)}(0,1)$. Формула интерполирует неизвестную функцию, используя её значения и производные до порядка в равноотстоящих узлах. Явные выражения для коэффициентов соответствующей квадратурной формулы выводятся интегрированием интерполяционных базисных функций. Представлена теорема, дающая точный вид коэффициентов. Проведены численные эксперименты для нескольких гладких функций, и проанализированы абсолютные погрешности приближённого интегрирования при различных значениях N. Результаты показывают, что предлагаемый подход обеспечивает высокую точность и может быть эффективно использован для задач численного интегрирования, где доступна информация о производных. Кроме того, исследуется устойчивость предложенной формулы к возмущениям входных данных и обсуждается её асимптотическое поведение при увеличении числа узлов. Сравнение с классическими интерполяционными и квадратурными формулами демонстрирует преимущество включения информации о производных, особенно для функций высокой гладкости. Метод также предоставляет конструктивную основу для расширения оптимальных интерполяционных формул на производные высших порядков и неравномерные сетки. Эти результаты способствуют более широкому развитию оптимальных вычислительных схем в пространствах Соболева и указывают на потенциальные возможности применения при решении краевых задач, дифференциальных уравнений и задач численного моделирования, требующих высокой точности.

Ключевые слова: интерполяция, сплайн, квадратурная формула, интегрирование, аппроксимация.

Цитирование: *Олимов Н.Н.* Применение оптимальной интерполяционной формулы с производной для приближенного интегрирования // Проблемы вычислительной и прикладной математики. – 2026. – № 2(72). – С. 147-154.

DOI: https://doi.org/10.71310/pcam.2_72.2026.10

№ 2(72) 2026

ISSN 2181-8460

HISOBLASH VA AMALIY MATEMATIKA MUAMMOLARI

ПРОБЛЕМЫ ВЫЧИСЛИТЕЛЬНОЙ И ПРИКЛАДНОЙ МАТЕМАТИКИ

PROBLEMS OF COMPUTATIONAL AND APPLIED MATHEMATICS

AiRI
RAQAMLI TEXNOLOGIYALAR VA SUN'IY INTELLEKTNI
RIVOJLANTIRISH ILMIY-TADQIQOT INSTITUTI

ПРОБЛЕМЫ ВЫЧИСЛИТЕЛЬНОЙ И ПРИКЛАДНОЙ МАТЕМАТИКИ

№ 2(72) 2026

Журнал основан в 2015 году.
Издается 6 раз в год.

Учредитель:
Научно-исследовательский институт развития цифровых технологий и искусственного интеллекта.

Журнал зарегистрирован в Агентстве информации и массовых коммуникаций при Администрации Президента Республики Узбекистан.
Свидетельство №0856 от 5 августа 2015 года.

ISSN 2181-8460, eISSN 2181-046X

Адрес редакции:
100125, г. Ташкент, м-в. Буз-2, 17А.
Тел.: +(998) 71 263-41-98.
Э-почта: journals@airi.uz.
Веб-сайт: https://journals.airi.uz.

Дизайн и вёрстка:
Шарипов Х.Д.

Отпечатано в типографии НИИ РЦТИИ.
Подписано в печать 22.04.2026 г.
Формат 60x84 1/8. Заказ №2. Тираж 100 экз.

PROBLEMS OF COMPUTATIONAL AND APPLIED MATHEMATICS

No. 2(72) 2026

The journal was established in 2015.
6 issues are published per year.

Founder:
Digital Technologies and Artificial Intelligence Development Research Institute.

The journal is registered by Agency of Information and Mass Communications under the Administration of the President of the Republic of Uzbekistan.
Certificate of Registration No. 0856 of 5 August 2015.

ISSN 2181-8460, eISSN 2181-046X

Address:
100125, Tashkent, Buz-2, 17A.
Tel.: +(998) 71 263-41-98.
E-mail: journals@airi.uz.
Web-site: https://journals.airi.uz.

Layout design:
Sharipov Kh.D.

DTAIRI printing office.
Signed for print 22.04.2026
Format 60x84 1/8. Order No. 2. Print run of 100 copies.

Содержание

Contents