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AN EXPLICIT-IMPLICIT DIFFERENCE SCHEME FOR A TWO-DIMENSIONAL LINEAR HYPERBOLIC SYSTEM WITH DYNAMIC BOUNDARY CONDITIONS

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This paper considers a two-dimensional linear hyperbolic system with dynamic boundary conditions and proposes a difference scheme for its numerical solution. An explicit–implicit directional splitting method is constructed, where discretization is performed explicitly in one direction and implicitly in the other, while preserving the dissipative structure of the boundary conditions. The stability of the scheme is established under the Courant–Friedrichs–Lewy condition and a linear matrix inequality. It is shown that the discrete energy decreases exponentially in time. Numerical experiments confirm the theoretical results, demonstrating monotonic decay of the discrete l_2-norm and good agreement with the exact solution. The proposed method is stable, dissipative, and computationally efficient, and can be effectively applied to two-dimensional hyperbolic systems with dynamic boundary conditions.

Keywords: hyperbolic system, dynamic boundary condition, difference scheme, directional splitting, explicit–implicit method, exponential stability, CFL condition.

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1 Introduction

Two-dimensional hyperbolic systems play an essential role in the mathematical modeling of wave propagation, transport phenomena, gas dynamics, and various distributed dynamical processes. In many applied problems, the interaction at the boundary is not static but evolves in time, which leads to the formulation of dynamic boundary conditions. Such conditions significantly influence the qualitative behavior of solutions, particularly their dissipative properties and stability characteristics.

The stability of hyperbolic systems with boundary feedback has been widely investigated in the literature. Dissipative boundary conditions for nonlinear hyperbolic systems were studied by Coron, Bastin, and d'Andréa-Novel [1], while robust boundary control methods were developed in [2]. Lyapunov stability analysis for hyperbolic models was further explored in [3]. Dynamic boundary stabilization problems and their control-theoretic interpretation were considered in [4, 5], where sufficient conditions for the stability of continuous systems were established.

From a numerical point of view, the construction of reliable difference schemes for multidimensional hyperbolic systems remains a challenging problem, especially in the presence of dynamic boundary interactions. Directional splitting (operator splitting) techniques are widely used for multidimensional problems since they allow reduction of a multidimensional scheme to a sequence of one-dimensional subproblems. Stability properties of splitting schemes for hyperbolic systems were investigated in [6–9], and computational models for quasilinear hyperbolic systems were constructed in [10]. However, the analysis of splitting schemes in the presence of dynamic boundary terms requires additional investigation.

Recent studies have addressed difference schemes for hyperbolic systems with dynamic boundary conditions and their stability properties [11–14]. Nevertheless, most of the available results are limited to one-dimensional models or require further development in the multidimensional case. In particular, the preservation of dissipative properties at the discrete level and the establishment of exponential stability for multidimensional splitting schemes remain open and practically important problems.

In this paper, an explicit–implicit directional splitting difference scheme is constructed for a two-dimensional linear hyperbolic system with dynamic boundary conditions. The proposed scheme combines explicit discretization in one spatial direction with implicit discretization in the other direction, including a consistent approximation of the dynamic boundary condition. Sufficient conditions ensuring exponential stability of the numerical solution are derived using the Courant–Friedrichs–Lewy condition and a linear matrix inequality framework [15]. It is shown that the discrete energy of the numerical solution decreases exponentially in time. The theoretical results are supported by numerical experiments demonstrating the dissipative behavior and accuracy of the proposed method.

The obtained results extend previously known stability results for one-dimensional hyperbolic models to the case of multidimensional operator-splitting discretizations with dynamic boundary interaction.

2 Mathematical Formulation of the Problem

In this paper, we consider the following linear hyperbolic system written in Riemann coordinates:

$$\frac{\partial V}{\partial t} + \Lambda \frac{\partial V}{\partial x} + C \frac{\partial V}{\partial y} = 0, \quad t \geqslant 0, \quad x \in (0, X), \quad y \in (0, Y). \tag{1}$$

Here X and Y are given positive constants defining the spatial domain. The matrix Λ is an $n \times n$ diagonal matrix with an existing inverse. It has the form

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

and its eigenvalues are ordered as follows:

$$\lambda_1 < \lambda_2 < \cdots < \lambda_m < 0 < \lambda_{m+1} < \lambda_{m+2} < \cdots < \lambda_n.$$

This condition guarantees the hyperbolicity of the system, that is, all eigenvalues are real and mutually distinct.

The unknown vector function V is represented in the following form:

$$V(x,t) = \begin{pmatrix} V^{I}(t,x,y) \\ V^{II}(t,x,y) \end{pmatrix}, \quad V^{I} = (v_1, v_2, \ldots, v_m)^T, \quad V^{II} = (v_{m+1}, v_{m+2}, \ldots, v_n)^T.$$

Furthermore, we introduce the notation

$$|\Lambda| = \mathrm{diag}(|\lambda_1|, |\lambda_2|, \dots, |\lambda_n|).$$

The stability problem for linear hyperbolic systems has been studied in [1–3], where static boundary conditions of the form

$$\begin{pmatrix} V^I(t,X,y) \\ V^{II}(t,0,y) \end{pmatrix} = S \begin{pmatrix} V^I(t,0,y) \\ V^{II}(t,X,y) \end{pmatrix},$$

were considered. Here S is a given $n \times n$ matrix.

Hyperbolic systems with dynamic boundary conditions are less studied in the literature. Nevertheless, finite-dimensional approximation techniques such as those proposed in [4] can be applied for stabilization purposes.

For the boundary conditions, we consider the following dynamic boundary condition:

$$\frac{\partial}{\partial t} \begin{pmatrix} V^I(t,X,y) \\ V^{II}(t,0,y) \end{pmatrix} = A \begin{pmatrix} V^I(t,X,y) \\ V^{II}(t,0,y) \end{pmatrix} + B \begin{pmatrix} V^I(t,0,y) \\ V^{II}(t,X,y) \end{pmatrix}, \quad y \in [0,Y], \tag{2}$$

$$V(t,x,0) = 0, \quad x \in [0,X], \quad t \in [0,+\infty).$$

Assume that $\Phi(x,y)$ is a continuously differentiable function satisfying the compatibility conditions for the initial and boundary conditions (not only the function itself, but also its first derivatives). Then the initial condition can be defined as

$$V(0,x,y) = \Phi(x,y), \quad x \in [0,X], \quad y \in [0,Y]. \tag{3}$$

Here

$$\Phi(x,y) = \big(\varphi_1(x,y), \varphi_2(x,y), \dots, \varphi_n(x,y)\big)^T.$$

The stability of equations (1), (3), and (2) was analyzed in [5] using a Lyapunov function approach. In that work, sufficient conditions for exponential stability of problem (1), (3), and (2) were established.

3 Difference Scheme for the Two-Dimensional Mixed Problem

We construct a difference scheme for the mixed problem defined by equations (1), (3), and (2).

Let

$$G = \{(x,y,t) : 0 \leqslant x \leqslant X,\ 0 \leqslant y \leqslant Y,\ 0 \leqslant t \leqslant T\},$$

be the space-time domain. We introduce a uniform grid with step sizes Δx in the x – direction, Δy in the y – direction, and Δt in the t – direction.

The grid nodes are defined by

$$x_j = j\Delta x, \quad y_l = l\Delta y, \quad t^k = k\Delta t,$$

where $j = 0, \dots, J$, $l = 0, \dots, L$, and $k = 0, \dots, K$.

The set of grid nodes is denoted by

$$G_h = \left\{(x_j, y_l, t^k) : j = 0, \dots, J;\ l = 0, \dots, L;\ k = 0, \dots, K\right\}.$$

The numerical solution at the grid nodes is denoted by

$$(v_i)_{jl}^k = v_i(x_j, y_l, t^k), \quad i = 1, \dots, n.$$

The step sizes Δx, Δy, and Δt are chosen such that

$$J\Delta x = X, \qquad L\Delta y = Y, \qquad K\Delta t = T.$$

To construct the numerical solution of problem (1), (3), and (2) at the grid nodes G_h, we propose the following difference scheme.

For simplicity of exposition, we assume that the matrix C is diagonal:

$$C = \mathrm{diag}(c_1, c_2, \ldots, c_n), \qquad c_i > 0, \quad i = 1, \ldots, n.$$

To construct the numerical solution of problem ((1), (3), and (2) at the grid nodes G_h, we propose the following splitting explicit–implicit difference scheme:

$$\begin{cases} (w_i)_{jl}^k = (v_i)_{jl}^k - c_i \dfrac{\Delta t}{\Delta y}\left[(v_i)_{jl}^k - (v_i)_{j\,l-1}^k\right], \\ (v_i)_{jl}^{k+1} = (w_i)_{jl}^k - \lambda_i \dfrac{\Delta t}{\Delta x}\left[(v_i)_{jl}^{k+1} - (v_i)_{j-1\,l}^{k+1}\right], \end{cases} \tag{4}$$

$$i = 1, \ldots, n, \quad j = 1, \ldots, J, \quad l = 1, \ldots, L, \quad k = 0, \ldots, K-1.$$

Alternatively, in vector form:

$$\begin{cases} W_{jl}^k = V_{jl}^k - C \dfrac{\Delta t}{\Delta y}\left[V_{jl}^k - V_{j\,l-1}^k\right], \\ V_{jl}^{k+1} = W_{jl}^k - \Lambda \dfrac{\Delta t}{\Delta x}\left[V_{jl}^{k+1} - V_{j-1\,l}^{k+1}\right], \end{cases} \tag{5}$$

$$j = 1, \ldots, J, \quad l = 1, \ldots, L, \quad k = 0, \ldots, K-1.$$

The initial condition (3) is approximated in the vector form as

$$V_{jl}^0 = \Phi_{jl} = \Phi(x_j, y_l), \quad j = 0, \ldots, J, \quad l = 0, \ldots, L. \tag{6}$$

The boundary conditions are approximated as follows:

$$\frac{V_{0l}^{k+1} - W_{0l}^k}{\Delta t} = A V_{Jl}^{k+1} + B V_{0l}^{k+1}, \quad l = 0, \ldots, L, \quad k = 0, \ldots, K-1, \tag{7}$$

$$W_{0l}^k = V_{0l}^k, \qquad V_{j0}^k = 0, \quad i = 1, \ldots, n, \quad j = 0, \ldots, J, \quad k = 0, \ldots, K.$$

4 Statement of the Problem

In this section, we consider a mixed problem for a two-dimensional linear hyperbolic equation with a dynamic boundary condition.

4.1 Main Equation

Consider the following scalar hyperbolic equation:

$$u_t + 0.2\,u_x + u_y = 0, \quad 0 < x < 1, \quad 0 < y < 1, \quad 0 < t \leqslant T.$$

Here:

- the spatial domain is $\Omega = (0,1) \times (0,1)$,
- the time interval is $0 < t \leqslant T$,

– the characteristic velocities are positive $(0.2 > 0,\ 1 > 0)$.

Since both characteristic velocities are positive, the characteristic curves are directed toward increasing values of x and y. Consequently, the information propagates into the computational domain through the boundaries $x = 0$ and $y = 0$, which therefore play the role of inflow boundaries for the considered hyperbolic problem..

4.2 Dynamic Boundary Condition

At the left boundary $x = 0$, the following dynamic boundary condition is imposed:

$$u_t(0, y, t) = -1.1\, u(0, y, t) + e^{-1} u(1, y, t), \quad 0 \leqslant y \leqslant 1, \quad 0 \leqslant t \leqslant T.$$

In condition (12):

– the term $-1.1\, u(0, y, t)$ represents a dissipative component,
– the term $e^{-1} u(1, y, t)$ describes the coupling through the opposite boundary value.

This dynamic boundary condition has a control-type character.

4.3 Boundary Condition at $y = 0$

The boundary condition at $y = 0$ is given by

$$u(x, 0, t) = e^{x-0.1t}, \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant t \leqslant T.$$

4.4 Initial Condition

The initial condition is

$$u(x, y, 0) = e^{x-0.1y}, \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 1.$$

For the given mixed problem, the following exponential function serves as the exact solution:

$$u(x, y, t) = e^{x-0.1y-0.1t}.$$

The obtained exact solution satisfies the differential equation as well as the initial and boundary conditions. Therefore, it is used as a benchmark solution for verifying the accuracy and stability of the numerical scheme.

5 Difference Scheme and Its Approximation

For the mixed problem formulated above, according to formulas (5), (6) and (7), we apply an upwind directional splitting explicit–implicit difference scheme.

The proposed scheme consists of the following two stages:

$$\begin{cases} w_{j,l}^k = u_{j,l}^k - c\dfrac{\Delta t}{\Delta y}\left(u_{j,l}^k - u_{j,l-1}^k\right), \\ u_{j,l}^{k+1} = w_{j,l}^k - \lambda\dfrac{\Delta t}{\Delta x}\left(u_{j,l}^{k+1} - u_{j-1,l}^{k+1}\right), \end{cases} \tag{8}$$

$$j = 1, \ldots, J, \quad l = 1, \ldots, L, \quad k = 0, \ldots, K-1.$$

This scheme is based on a two-stage splitting algorithm (directional operator splitting), where the first stage corresponds to the y-direction and the second stage corresponds to the x-direction. Here the parameters are

$$\lambda = 0.2, \qquad c = 1.$$

5.1 Approximation of the Initial Condition

The initial condition is discretized as

$$u_{j,l}^{0} = \varphi_{j,l} = \varphi(x_j, y_l) = e^{x_j - 0.1y_l}, \quad j = 0, \dots, J, \quad l = 0, \dots, L.$$

5.2 Approximation of the Boundary Conditions

The dynamic boundary condition at $x = 0$ is discretized as

$$\frac{u_{0,l}^{k+1} - w_{0,l}^{k}}{\Delta t} = A\, u_{0,l}^{k+1} + B\, u_{J,l}^{k+1}, \quad l = 0, \dots, L, \quad k = 0, \dots, K-1.$$

Here the intermediate function satisfies

$$w_{0,l}^{k} = u_{0,l}^{k}.$$

For the lower boundary $(y = 0)$, we set

$$u_{j,0}^{k} = e^{x_j - 0.1t_k}, \quad j = 0, \dots, J, \quad k = 0, \dots, K,$$

the parameters are

$$A = -1.1, \qquad B = e^{-1}.$$

Scientific Remarks

The proposed difference scheme is based on the directional splitting (operator splitting) technique, which significantly simplifies the computational procedure. At the first stage, an explicit scheme is applied in the y-direction, while at the second stage, an implicit scheme is employed in the x-direction.

The discrete representation of the dynamic boundary condition is of first-order accuracy in time and preserves the dissipative character of the system.

The selected parameters

$$\lambda = 0.2, \quad c = 1, \quad A = -1.1, \quad B = e^{-1},$$

satisfy the theoretical stability conditions and ensure exponential decay.

5.3 Grid Parameters

We recall the grid parameters used in the computations.
The selected values are

$$J = 100, \quad L = 15, \quad K = 51, \quad T = 1.$$

Hence,

$$\Delta x = \frac{1}{J} = 0.01, \quad \Delta y = \frac{1}{L} \approx 0.067, \quad \Delta t = \frac{T}{K} \approx 0.02.$$

5.4 Justification of Exponential Stability

The proposed difference scheme has a directional splitting (operator splitting) structure. The computational process is carried out in two stages:

1. an explicit scheme in the y-direction,
2. an implicit scheme in the x-direction (including the dynamic boundary condition).

Each stage, considered separately, corresponds to a one-dimensional difference scheme for a hyperbolic equation.

For the one-dimensional case, the exponential stability criterion has been established in [12]. Therefore, if the scheme in each direction decreases the discrete energy, then the composed (splitting) scheme also decreases the energy, that is,

$$E^{k+1} \leqslant qE^k, \quad 0 < q < 1.$$

This implies the following exponential estimate:

$$\|u^k\| \leqslant Ce^{-\gamma k \Delta t}\|u^0\|.$$

Thus, the two-dimensional scheme is exponentially stable.

According to the theorem in [12], the following linear matrix inequality ensures the stability of the one-dimensional stage:

$$\begin{pmatrix} A^T P + PA + P\Lambda & PB \\ B^T P & -P\Lambda \end{pmatrix} < 0.$$

If this condition is satisfied, the one-dimensional stage in the x-direction ensures exponential decay. The one-dimensional stage in the y-direction is stable under the CFL condition. Consequently, the overall splitting scheme is exponentially stable.

A complete formal proof of the two-dimensional case is left for future research.

5.5 Verification of the CFL (Courant–Friedrichs–Lewy) Condition

To verify stability from a theoretical point of view, we examine the CFL condition.

For the scalar case, the stability condition reads

$$c\frac{\Delta t}{\Delta y} < 1.$$

In our case,

$$c = 1.$$

Substituting the grid parameters, we obtain

$$c\frac{\Delta t}{\Delta y} = 1 \cdot \frac{0.02}{0.067} \approx 0.294 < 1.$$

Therefore, the CFL condition is satisfied.

Scientific Remark on the CFL Condition

This condition follows from the classical Courant–Friedrichs–Lewy stability criterion. It characterizes the relationship between the characteristic velocities and the mesh steps in explicit difference schemes.

5.6 Verification of the Linear Matrix Inequality

To ensure exponential stability, the following linear matrix inequality (LMI) must be satisfied:

$$\begin{pmatrix} A^T P + PA + P\lambda & PB \\ B^T P & -P\lambda \end{pmatrix} < 0.$$

The parameters are

$$A = -1.1, \quad B = e^{-1}, \quad \lambda = 0.2, \quad P = 0.1.$$

Since we consider the scalar case, we have

$$A^T = A, \quad B^T = B.$$

Computation of the matrix elements.

First diagonal element:

$$A^T P + PA + P\lambda = (-1.1)(0.1) + (0.1)(-1.1) + (0.1)(0.2).$$

$$-0.11 - 0.11 + 0.02 = -0.20.$$

Off-diagonal element:

$$PB = 0.1\, e^{-1} \approx 0.1 \cdot 0.3679 \approx 0.03679.$$

Second diagonal element:

$$-P\lambda = -(0.1)(0.2) = -0.02.$$

Therefore, the matrix takes the form

$$\begin{pmatrix} -0.20 & 0.03679 \\ 0.03679 & -0.02 \end{pmatrix}.$$

To verify negative definiteness, we apply Sylvester's criterion and check the leading principal minors.

$$-0.20 < 0,$$

and

$$\det = (-0.20)(-0.02) - (0.03679)^2 =$$

$$= 0.004 - 0.001354 \approx 0.002646 > 0.$$

Since the first leading principal minor is negative and the determinant is positive, the matrix is negative definite according to Sylvester's criterion.

Thus, the linear matrix inequality is satisfied, which guarantees exponential stability of the x-direction stage.

5.7 Scientific Discussion

The negative definiteness of the 2×2 symmetric matrix was verified according to Sylvester's criterion [15]. Therefore, the matrix is negative definite.

For the selected parameters, the stability conditions were thoroughly examined. In particular, the Courant–Friedrichs–Lewy (CFL) criterion was verified, showing that the ratio between the temporal and spatial steps satisfies the necessary restriction for stable operation of the explicit stage.

Moreover, the linear matrix inequality ensuring exponential stability was checked and its negative definiteness was established. This result guarantees the decay of the energy functional for the system with dynamic boundary conditions.

Thus, based on the theoretical analysis, the exponential stability of the constructed difference scheme is proven.

The obtained theoretical conclusions are confirmed by the numerical experiments presented in the next section, where the numerical solution approaches the exact solution and the norm decreases exponentially.

6 Computational Procedure

6.1 Initialization of the First Layer

The computational process starts from the initial condition. At all grid nodes, the values of the initial layer are determined by the formula

$$u_{j,l}^0 = e^{x_j - 0.1 y_l}, \quad j = 0, \ldots, J, \quad l = 0, \ldots, L.$$

The obtained values are presented in Table 1.

Table 1 Tabular representation of the initial values $u_{j,l}^0$.

$j \backslash l$	0	1	2	3	4	5
0	1.000	0.905	0.819	0.741	0.670	0.607
1	1.105	1.000	0.905	0.819	0.741	0.670
2	1.221	1.105	1.000	0.905	0.819	0.741
3	1.350	1.221	1.105	1.000	0.905	0.819
4	1.492	1.350	1.221	1.105	1.000	0.905
5	1.649	1.492	1.350	1.221	1.105	1.000

6.2 Computation of the First Time Layer ($k = 0$)

To compute the values $u_{j,l}^1$, we write the difference scheme at $k = 0$:

$$w_{j,l}^0 = u_{j,l}^0 - c\frac{\Delta t}{\Delta y}\left(u_{j,l}^0 - u_{j,l-1}^0\right), \quad u_{j,l}^1 = w_{j,l}^0 - \lambda\frac{\Delta t}{\Delta x}\left(u_{j,l}^1 - u_{j-1,l}^1\right),$$

$$j = 1, \ldots, J, \qquad l = 1, \ldots, L.$$

The first equation determines the intermediate values $w_{j,l}^0$.

The second equation is implicit and contains the unknown values $u_{j,l}^1$.

6.3 Rewriting and Simplification of the Scheme

Expanding the second equation, we obtain

$$u_{j,l}^1 + \lambda\frac{\Delta t}{\Delta x}\left(u_{j,l}^1 - u_{j-1,l}^1\right) = u_{j,l}^0 - c\frac{\Delta t}{\Delta y}\left(u_{j,l}^0 - u_{j,l-1}^0\right).$$

Rewriting the scheme, we obtain

$$\left(1 + \lambda\frac{\Delta t}{\Delta x}\right)u_{j,l}^1 - \lambda\frac{\Delta t}{\Delta x}u_{j-1,l}^1 = \left(1 - c\frac{\Delta t}{\Delta y}\right)u_{j,l}^0 + c\frac{\Delta t}{\Delta y}u_{j,l-1}^0.$$

6.4 Introduction of Auxiliary Notation

To simplify the computations, we introduce the following notation:

$$\lambda_1 = \lambda\frac{\Delta t}{\Delta x}, \quad p = 1 + \lambda_1, \;\; c_1 = c\frac{\Delta t}{\Delta y}, \quad q = 1 - c_1.$$

Then the equation takes the compact form

$$p\,u^1_{j,l} - \lambda_1 u^1_{j-1,l} = q\,u^0_{j,l} + c_1 u^0_{j,l-1}. \tag{9}$$

6.5 Sequential Solution with Respect to j

Equation (9) can be written sequentially in the j-direction.
For $j = 1$:

$$p\,u^1_{1,l} - \lambda_1 u^1_{0,l} = q\,u^0_{1,l} + c_1 u^0_{1,l-1}.$$

For $j = 2$:

$$p\,u^1_{2,l} - \lambda_1 u^1_{1,l} = q\,u^0_{2,l} + c_1 u^0_{2,l-1}.$$

$$\dots$$

For $j = J$:

$$p\,u^1_{J,l} - \lambda_1 u^1_{J-1,l} = q\,u^0_{J,l} + c_1 u^0_{J,l-1}.$$

Thus, for each fixed l, we obtain J equations.

6.6 Dynamic Boundary Condition at $k = 0$

At $k = 0$, the dynamic boundary condition takes the form

$$\frac{u^1_{0,l} - u^0_{0,l}}{\Delta t} = A u^1_{0,l} + B u^1_{J,l}.$$

After simplification, we obtain

$$(1 - A\Delta t) u^1_{0,l} - B\Delta t\, u^1_{J,l} = u^0_{0,l}. \tag{10}$$

Combining equations (9) and (10), for each fixed l we obtain a linear system with $(J + 1)$ unknowns and $(J + 1)$ equations.

This system has a lower-triangular structure and can be solved using a recurrent (forward substitution) method.

6.7 Recurrent Formula

Using the sequential substitution method, the following general formula is obtained:

$$u^1_{j,l} = \sum_{i=1}^{j} \frac{q\,u^0_{i,l} + c_1 u^0_{i,l-1}}{p^{j-i+1}} + \left(\frac{\lambda_1}{p}\right)^j u^1_{0,l}. \tag{11}$$

First, the value $u^1_{0,l}$ is determined from equations (10) and (11), and then the remaining values $u^1_{j,l}$ are computed using equation (9).

6.8 Algorithm of the Computational Procedure

The computational process is carried out in the following sequence:

1. The initial layer $u^0_{j,l}$ is computed.
2. For each time layer:
   - the linear system corresponding to equations (9) and (10) is formed;
   - the general recurrent formula (11) is constructed;
   - using these relations, the values $u^{k+1}_{0,l}$ and all $u^{k+1}_{j,l}$ are computed.
3. The procedure is repeated for $k = 0, \dots, K - 1$.

To automate the computations, a dedicated program was developed in the Python programming language, and a series of numerical experiments were performed.

The computational results obtained by the program are presented below.

7 Numerical Results

7.1 Comparison of Numerical and Exact Solutions

Figure 1 shows comparison between the numerical and exact solutions at $T = 20$. The numerical surface (left) closely follows the exact solution (right), confirming the accuracy of the proposed scheme.

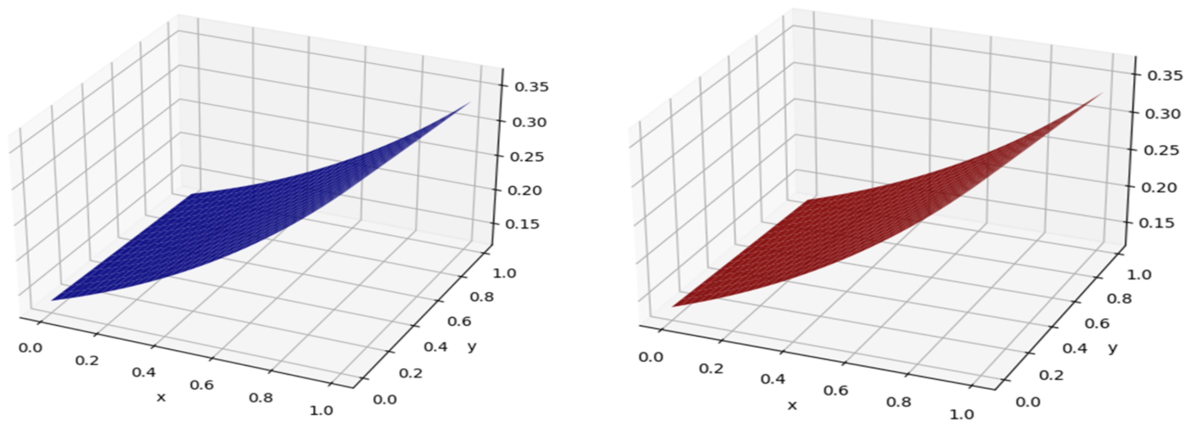

Figure 1 Comparison of numerical and exact solutions at $T = 20$

Table 2 Pointwise error values $|u_{num} - u_{\text{exact}}|$ at the grid nodes (x_i, y_j)

$x\backslash y$	0.000	0.067	0.133	0.200	0.267	0.333	0.400	0.467	0.533	0.600
0.000	0.000000	0.000055	0.000073	0.000091	0.000109	0.000127	0.000145	0.000162	0.000179	0.000195
0.010	0.000000	0.000451	0.000653	0.000748	0.000796	0.000825	0.000844	0.000860	0.000874	0.000887
0.020	0.000000	0.000266	0.000496	0.000650	0.000744	0.000800	0.000836	0.000860	0.000878	0.000893
0.030	0.000000	0.000161	0.000361	0.000536	0.000664	0.000752	0.000810	0.000849	0.000876	0.000897
0.040	0.000000	0.000100	0.000258	0.000426	0.000572	0.000685	0.000767	0.000824	0.000864	0.000893
0.050	0.000000	0.000066	0.000185	0.000332	0.000479	0.000606	0.000708	0.000785	0.000841	0.000882
0.060	0.000000	0.000046	0.000134	0.000256	0.000393	0.000524	0.000639	0.000733	0.000805	0.000859
0.070	0.000000	0.000035	0.000101	0.000199	0.000319	0.000446	0.000566	0.000672	0.000758	0.000827
0.080	0.000000	0.000029	0.000079	0.000156	0.000258	0.000375	0.000494	0.000605	0.000703	0.000784
0.090	0.000000	0.000026	0.000065	0.000126	0.000211	0.000314	0.000427	0.000539	0.000643	0.000734
0.100	0.000000	0.000024	0.000056	0.000106	0.000175	0.000264	0.000366	0.000475	0.000581	0.000679
0.110	0.000000	0.000023	0.000051	0.000091	0.000148	0.000224	0.000315	0.000416	0.000520	0.000621
0.120	0.000000	0.000022	0.000048	0.000082	0.000129	0.000193	0.000272	0.000364	0.000463	0.000563
0.130	0.000000	0.000022	0.000046	0.000076	0.000116	0.000169	0.000238	0.000319	0.000411	0.000508
0.140	0.000000	0.000022	0.000045	0.000072	0.000107	0.000152	0.000210	0.000282	0.000366	0.000457

The Table 2 demonstrates the distribution of the pointwise error between the numerical and exact solutions over the computational domain. It is observed that the error magnitude is significantly reduced compared to the previous mesh resolution, indicating the convergence of the proposed numerical scheme. The error remains small throughout the domain and increases smoothly in the direction of spatial propagation. The maximum error is of order 10^{-4}, which confirms the high accuracy of the explicit–implicit splitting scheme. These numerical findings are in agreement with the theoretical exponential stability and convergence properties established earlier.

7.2 Error Analysis

The maximum absolute error was computed as

$$E_{\max}(t_k) = \max_{j,l}$$

Table 3 Maximum error at different time levels.

Time T	$E_{\max}$
1	2.6×10^{-3}
5	1.3×10^{-3}
10	7.0×10^{-4}
20	3.0×10^{-4}

7.3 Decay of the Discrete l_2 – Norm

Figure 2 illustrates the time evolution of the discrete l_2 – norm: $\|u^k\|_{l_2}$. The norm decreases monotonically, confirming the exponential stability of the scheme.

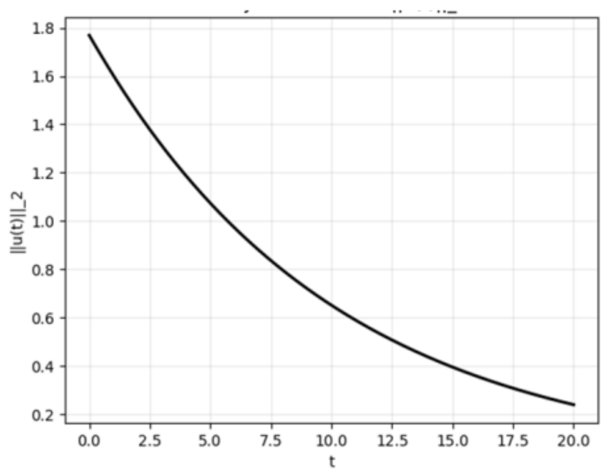

Figure 2 Time evolution of the discrete l_2 – norm

7.4 Discussion of Numerical Results

The spatial distributions of the numerical and exact solutions are illustrated for the case $T = 20$. The surfaces demonstrate that the numerical solution reproduces the qualitative behavior of the exact solution without visible oscillations or instability.

The maximum errors between the numerical and exact solutions are presented for the time moments $T = 1, 5, 10$, and 20. The obtained values remain small, which confirms the convergence of the proposed difference scheme.

In addition, the time evolution of the discrete l_2-norm of the numerical solution is shown. The norm decreases monotonically as time increases, which confirms the dissipative character of the system and supports the theoretical exponential stability of the constructed scheme.

8 Concluding Remarks

An explicit–implicit directional splitting difference scheme for a two-dimensional linear hyperbolic equation with dynamic boundary conditions was constructed and analyzed.

Sufficient conditions for exponential stability were established using the CFL condition and a linear matrix inequality approach.

Numerical experiments confirm the theoretical results: the discrete l_2 – norm decreases monotonically and exhibits exponential decay, while the numerical solution remains close to the exact solution.

Further research may focus on a complete multidimensional stability analysis and on extensions to nonlinear and higher-dimensional problems.

References

[1] Coron J.M., Bastin G., d'Andréa Novel B. Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems // SIAM Journal on Control and Optimization . – 2008. – Vol. 47. – Issue 3. – P. 1460-1498. doi: http://dx.doi.org/10.1137/06066363X.

[2] Prieur C., Winkin J., Bastin G. Robust boundary control of systems of conservation laws // Mathematics of Control, Signals, and Systems – 2008. – Vol. 20. – Issue 2. – P. 173-197. doi: http://dx.doi.org/10.1007/s00498-007-0018-0.

[3] Bastin G., Coron J.M., d'Andréa Novel B. On Lyapunov stability of linearized Saint-Venant equations for a sloping channel // Networks and Heterogeneous Media – 2009. – Vol. 4. – Issue 2. – P. 177-187. doi: http://dx.doi.org/10.3934/nhm.2009.4.177.

[4] Castillo F., Witrant E., Dugard L. Contrôle de température dans un flux de Poiseuille // Proceedings of IEEE Conférence Internationale Francophone d'Automatique – 2012. – Grenoble, France.

[5] Castillo F., Witrant E., Prieur C., Dugard L. Dynamic boundary stabilization of hyperbolic systems // Proceedings of the 51st IEEE Conference on Decision and Control (CDC) – 2012. – Maui, HI, USA. doi: http://dx.doi.org/10.1109/CDC.2012.6425820.

[6] Aloev R., Berdyshev A., Bliyeva D., Dadabayev S., Baishemirov Z. Stability analysis of an upwind difference splitting scheme for two-dimensional Saint–Venant equations // Symmetry . – 2022. – Vol. 14. – Article 1986. doi: http://dx.doi.org/10.3390/sym14101986.

[7] Aloev R.D., Dadabaev S.U. Stability of the upwind difference splitting scheme for symmetric -hyperbolic systems with constant coefficients // Results in Applied Mathematics. – 2022. – Vol. 16. – Article 100298. doi: http://dx.doi.org/10.1016/j.rinam.2022.100298.

[8] Aloev R.D., Hudayberganov M.U. A discrete analogue of the Lyapunov function for hyperbolic systems // Journal of Mathematical Sciences – 2022. – Vol. 268. – P. 640-651. doi: http://dx.doi.org/10.1007/s10958-022-06028-y.

[9] Aloev R.D., Eshkuvatov Z.K., Khudoyberganov M.U., Nematova D.E. The difference splitting scheme for -dimensional hyperbolic systems // Malaysian Journal of Mathematical Sciences – 2022. – Vol. 16. – Issue 3. – P. 421-438.

[10] Aloev R., Khasanov M., Berezovsky A. Construction and research of adequate computational models for quasilinear hyperbolic systems // Numerical Algebra, Control and Optimization. – 2018. – Vol. 8. – Issue 2. – P. 161-177. doi: http://dx.doi.org/10.3934/naco.2018017.

[11] Berdyshev A., Aloev R., Abdiramanov Z., Ovlaeva M. An explicit–implicit upwind difference splitting scheme in directions for a mixed boundary control problem for a two-dimensional symmetric -hyperbolic system // Symmetry. – 2023. – Vol. 15. – Article 1863. doi: http://dx.doi.org/10.3390/sym15101863.

[12] Aloev R.D., Ovlaeva M.Kh. Construction and study of the stability of a difference scheme for a linear hyperbolic system with dynamic boundary // Uzbek Mathematical Journal. – 2023. – Vol. 67. – Issue 2. – P. 17-24. doi: http://dx.doi.org/10.29229/uzmj.2023-2-2.

[13] Aloev R.D., Ovlaeva M.Kh., Nishonaliyeva M. Construction and stability analysis of a difference scheme for a linear hyperbolic system with dynamic boundary conditions // *Proceedings of the VII International Scientific and Technical Conference "Problems of Mechanical Engineering"*. – 2023. – Omsk, Russia.

[14] Aloev R.D., Ovlaeva M.Kh. Numerical solution of a mixed problem for a hyperbolic system with dynamic boundary conditions // *Proceedings of the International Scientific and Practical Conference "System Analysis and Modeling in Economy and International Relations"*. – 2026. – Tashkent, Uzbekistan. doi: `http://dx.doi.org/10.5281/zenodo.18217074`.

[15] Horn R.A., Johnson C.R. Matrix Analysis. – 2nd ed. – 2013. – Cambridge: Cambridge University Press.

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ЯВНО-НЕЯВНАЯ РАЗНОСТНАЯ СХЕМА ДЛЯ ДВУХМЕРНОЙ ЛИНЕЙНОЙ ГИПЕРБОЛИЧЕСКОЙ СИСТЕМЫ С ДИНАМИЧЕСКИМИ ГРАНИЧНЫМИ УСЛОВИЯМИ

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В работе рассматривается двумерная линейная гиперболическая система с динамическими граничными условиями и предлагается разностная схема для ее численного решения. Построен явно-неявный метод расщепления по направлениям, в котором аппроксимация выполняется явно по одному направлению и неявно по другому с сохранением диссипативной структуры граничных условий. Устойчивость схемы установлена при выполнении условия Куранта–Фридрихса–Леви и линейного матричного неравенства. Показано, что дискретная энергия экспоненциально убывает со временем. Численные эксперименты подтверждают теоретические результаты, демонстрируя монотонное убывание нормы l_2 и согласование с точным решением. Предложенный метод является устойчивым, диссипативным и вычислительно эффективным и может применяться для двумерных гиперболических систем с динамическими граничными условиями.

Ключевые слова: гиперболическая система, динамическое граничное условие, разностная схема, направленное расщепление, явно–неявный метод, экспоненциальная устойчивость.

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