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NUMERICAL MODELING OF THE BOUNDARY VALUE PROBLEM FOR A TWO-PARAMETER SINGULARLY PERTURBED DIFFERENTIAL EQUATION USING THE SPECTRAL-GRID METHOD

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The solutions of boundary value problems associated with two-parameter singularly perturbed differential equations are well known to exhibit the formation of two distinct boundary layers, typically occurring near the endpoints of the domain. The presence of these narrow regions, characterized by steep gradients in the solution, poses significant challenges for classical numerical techniques. As a result, standard finite difference, finite element, or low-order discretization methods often fail to capture the layer behavior accurately unless excessively fine meshes are employed, leading to increased computational cost and reduced efficiency. To address these difficulties, the present work proposes an efficient spectral-grid method for the numerical solution of boundary value problems involving two-parameter singular perturbations. The core idea of the proposed approach is to combine the high accuracy of spectral methods with a carefully designed grid that adapts to the boundary layer structure of the solution. By employing this spectral-grid framework, the original boundary value problem is transformed into an equivalent system of algebraic equations, which can be solved efficiently using standard linear algebra techniques. Extensive numerical experiments are carried out to assess the performance of the proposed method. The computed results are systematically compared with those available in the existing literature. These comparisons clearly demonstrate that the spectral-grid method developed in this study achieves superior accuracy while maintaining a relatively low computational cost.

Keywords: spectral-grid method, two-parameter singularly perturbed differential equation, Chebyshev polynomials, maximum absolute error, boundary layers, numerical approximation, high-accuracy algorithm.

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1 Introduction

Differential equations containing small parameters in the coefficients of the highest derivatives are referred to as singularly perturbed differential equations. The solutions of such problems possess specific features: the sought solution domain usually includes rapidly varying boundary layers and regions where the solution changes smoothly. Traditional numerical methods are not efficient for solving such problems because they require extremely fine meshes to obtain satisfactory accuracy. However, ensuring stability in the numerical solution of singularly perturbed problems often leads to large algebraic systems with high computational costs. Therefore, the development of special numerical methods adapted to boundary layers and capable of efficiently handling such problems is of significant importance. In this paper, we consider the following problem:

$$eqn1Lu(x) = -\varepsilon u''(x) - \mu a(x)u'(x) + b(x)u(x) = f(x), \quad 0 \leqslant x \leqslant 1, \tag{1}$$

subject to the boundary conditions

$$eqn2u(0) = \alpha_0, \quad u(1) = \alpha_1. \tag{2}$$

Here, α_0 and α_1 are given constants, $0 < \varepsilon, \mu \ll 1$ are two small parameters. The functions $a(x)$, $b(x)$, and $f(x)$ are assumed to be sufficiently smooth. Problems of this type are generally referred to as two-parameter singularly perturbed boundary value problems.

When $\mu = 0$, the problem (1)–(2) reduces to the well-studied reaction–diffusion boundary value problem with a single perturbation parameter. When $\mu = 1$, it transforms into a convection–diffusion problem. Numerous studies have been devoted to the numerical solutions of these particular cases. For instance, reaction–diffusion problems corresponding to $\mu = 0$ have been thoroughly analyzed in [1–4], whereas the convection–diffusion case with $\mu = 1$ has been discussed in [5–7]. However, the genuinely two-parameter case, i.e., when $0 < \varepsilon, \mu \ll 1$, has received comparatively little attention in the literature. Several studies addressing such problems can be mentioned. The authors [8] proposed a finite difference method on Shishkin-type meshes and proved its first-order convergence. Later a B-spline collocation method [9] on a similar mesh was developed and demonstrated its second-order convergence. Kadabaioo and co-authors [10] showed that the Ritz–Galerkin method is highly effective for problems involving boundary layers. In [11], the authors proposed a B-spline collocation method on graded meshes and analyzed its efficiency for small values of the parameters ε, μ, and for refined mesh densities, demonstrating the robustness of the method.

Although several methods have been developed to solve such types of problems, there is still a lack of numerical approaches that ensure high accuracy for singularly perturbed problems involving small parameters. Therefore, in this paper, a spectral-grid method is proposed for the numerical solution of two-parameter singularly perturbed differential equations. The results presented in this study demonstrate that the proposed spectral-grid method is several times more efficient than the graded B-spline collocation method.

In [12], the use of Chebyshev polynomials for solving eigenvalue problems arising from systems of higher-order nonlinear differential equations containing small parameters was discussed. In [13, 14], the authors considered parabolic-type partial differential equations with initial–boundary conditions and developed a Chebyshev perturbation approach based on continuous integration. The obtained error estimates correctly reflect the convergence order of the method.

More recently, a highly accurate and efficient numerical method based on Chebyshev spectral techniques was proposed for the analysis of the dynamics of derivatives of different orders in singularly perturbed equations [15]. In addition, a high-accuracy method for solving the biharmonic equation using spectral approaches was developed and successfully applied, demonstrating superior numerical efficiency and precision [16].

Papers [18] and [19] established theoretical foundations for hydrodynamic and spectral methods based on Chebyshev polynomials and analyzed several of their efficient implementations. In particular, [18] investigated the mathematical modeling of two-dimensional parallel flows in hydrodynamics, analyzing their dependence on boundary conditions.

In [19], the Orr–Sommerfeld equation was studied, and an efficient spectral method was proposed for its numerical solution. The results of these studies have laid the groundwork for further research on small-parameter problems using numerical techniques based on Chebyshev polynomials. Hydrodynamic stability of two-phase liquid flows has been analyzed using spectral methods in [20]. Furthermore, for the case of a bounded layer, this problem was examined through the application of the spectral collocation method in [21].

2 Statement of the problem

Consider the following two-parameter singularly perturbed differential equation with small parameters in the highest derivatives:

$$-\varepsilon u''(x) - \mu u'(x) + u(x) = e^{1-x}, \quad x \in [0,1], \tag{3}$$

subject to the boundary conditions

$$u(0) = u(1) = 0, \tag{4}$$

where ε and μ are small positive parameters.

To numerically model the differential problem (3)-(4), we employ the spectral-grid method based on the first-kind Chebyshev polynomials as basis functions. Typically, when analyzing the convergence and accuracy of numerical methods, an exact analytical or test function is used for verification [17]. According to this approach, an auxiliary analytical function $u_e(x)$ satisfying the boundary conditions (4) is chosen. Substituting it into equation (3) allows us to compute the corresponding right-hand side function. The considered problem is then solved using the spectral-grid method, and the obtained numerical solution is compared with the chosen analytical test function.

For the current study, the following analytical function satisfying the boundary conditions of equation (3) is selected as the test function:

$$u(x) = \frac{1}{\varepsilon - \mu - 1}\left[\frac{e - e^{a_1}}{1 - e^{-\frac{\sqrt{\mu^2+4\varepsilon}}{\varepsilon}}}\, e^{-a_2 x} - e^{1-x} + \frac{e^{1-a_2} - 1}{e^{-\frac{\sqrt{\mu^2+4\varepsilon}}{\varepsilon}} - 1}\, e^{a_1(1-x)}\right],$$

where

$$\alpha_1 = \frac{\mu - \sqrt{\mu^2 + 4\varepsilon}}{2\varepsilon}, \quad \alpha_2 = \frac{\mu + \sqrt{\mu^2 + 4\varepsilon}}{2\varepsilon}.$$

3 Solution method

We now present the spectral-grid algorithm for the numerical modeling of problem (3)–(4). First, the integration interval $[\eta_0, \eta_1]$ is divided into a uniform grid $\eta_0 < \eta_1 < \cdots < \eta_N = \eta_l$, resulting in N subintervals (elements):

$$[\eta_0, \eta_1], [\eta_1, \eta_2], \ldots, [\eta_j, \eta_{j+1}], \ldots, [\eta_{N-1}, \eta_N], \quad j = 1, 2, \ldots, N-1, \tag{5}$$

where $\eta_0 = 0$ and $\eta_N = 1$.

The differential equation (3) within each subinterval takes the following form:

$$-\varepsilon \frac{d^2 u_j}{d\eta^2} - \mu \frac{du_j}{d\eta} + u_j(\eta) = f_j(\eta), \quad j = 1, 2, \ldots, N. \tag{6}$$

The boundary conditions (4) at the endpoints η_0 and η_N are written as:

$$u(\eta_0) = 0, \quad u(\eta_N) = 0. \tag{7}$$

At the internal nodes, the solution of equation (3) and its first derivative are required to be continuous. These continuity conditions can be expressed as follows:

$$u_j^{(t)}(\eta_j) = u_{j+1}^{(t)}(\eta_j), \quad t = 0, 1; \quad j = 1, 2, \ldots, N-1, \tag{8}$$

where t denotes the order of the derivative.

The approximate solution $u_j(x)$ of problem (6)–(8) is represented as a truncated series in terms of the first-kind Chebyshev polynomials. Each subinterval $[\eta_j, \eta_{j+1}]$ is mapped onto the standard interval $[-1, 1]$ using the following transformation:

$$\eta = \frac{m_j}{2} + \frac{l_j}{2}x, \qquad m_j = \eta_j + \eta_{j+1}, \quad l_j = \eta_{j+1} - \eta_j, \quad j = 1, 2, \ldots, N-1, \tag{9}$$

where $\eta \in [\eta_j, \eta_{j+1}]$, $x \in [-1, 1]$, and l_j denotes the length of the $(j+1)$-th subinterval.

As a result, equation (6) can be rewritten in the transformed variable x as follows:

$$-\varepsilon \left(\frac{2}{l_j}\right)^2 \frac{d^2 u_j}{dx^2} - \mu \left(\frac{2}{l_j}\right) \frac{du_j}{dx} + u_j(x) = f_j(x), \qquad j = 1, 2, \ldots, N. \tag{10}$$

The boundary conditions (7) and the continuity conditions (8) take the form:

$$u_1(0) = 0, \qquad u_N(+1) = 0,$$

$$\left(\frac{2}{l_j}\right)^t u_j^{(t)}(+1) = \left(\frac{2}{l_{j+1}}\right)^t u_{j+1}^{(t)}(-1), \quad t = 0, 1; \quad j = 1, 2, \ldots, N-1, \tag{11}$$

where t denotes the order of the derivative.

We seek the approximate solution of problem (10)–(11) and its right-hand side $f_j(x)$ in each subinterval in the form of truncated series of the first-kind Chebyshev polynomials:

$$u_a^{(j)}(x) = \sum_{n=0}^{p_j} a_n^{(j)} T_n(x), \qquad f_j\left(\overline{x_l}^{(j)}\right) = \sum_{n=0}^{p_j} b_n^{(j)} T_n\left(x_l^{(j)}\right), \qquad \overline{x_l}^{(j)} = \frac{m_j}{2} + \frac{l_j}{2} x_l^{(j)}, \tag{12}$$

with

$$x_l^{(j)} = \cos\left(\frac{\pi l}{p_j}\right), \quad l = 0, 1, \ldots, p_j, \quad j = 1, 2, \ldots, N.$$

Here $u_a^{(j)}(x)$ denotes the approximate solution, $T_n(x)$ are the first-kind Chebyshev polynomials, and $x_l^{(j)}$ are their corresponding nodes. The parameter p_j represents the number of polynomials used to approximate the solution in the j-th subinterval, while $a_n^{(j)}$ are the unknown expansion coefficients.

For the given function $f_j(x)$, the expansion coefficients $b_n^{(j)}$ in expression (12) are determined by the inverse transformation formula [16]:

$$b_n^{(j)} = \frac{2}{p_j c_n} \sum_{l=0}^{p_j} \frac{1}{c_l} f_j\left(\overline{x_l}^{(j)}\right) T_n\left(x_l^{(j)}\right), \quad n = 0, 1, 2, \ldots, p_j, \tag{13}$$

where $c_0 = c_{p_j} = 2$, $c_m = 1$ $(m \neq 0, p_j)$, $j = 1, 2, \ldots, N$, and $M = \sum\limits_{j=1}^{N} (p_j + 1)$ is the total number of Chebyshev polynomials used in the spectral-grid approximation of the solution, while N denotes the number of grid elements. To compute derivatives of various orders of the approximate solution $u_a^{(j)}(x)$ from (12), the following relations are used, which are convenient for numerical implementation:

$$u_a^{(j)}(y) = \sum_{n=0}^{p_j} a_n^{(j)} T_n(y),$$

$$u_a^{(1)}(y) = \sum_{n=0}^{p_j} a_n^{(1)} T_n(y), \qquad a_n^{(1)} = \frac{2}{c_n} \sum_{\substack{q=n+1 \\ q+n \equiv 1 \,(\mathrm{mod}\, 2)}}^{p_j} q\, a_q, \quad n \geqslant 0, \tag{14}$$

$$u_a^{(2)}(y) = \sum_{n=0}^{p_j} a_n^{(2)} T_n(y), \qquad a_n^{(2)} = \frac{2}{c_n} \sum_{\substack{q=n+2 \\ q+n \equiv 0 \,(\mathrm{mod}\, 2)}}^{p_j} q\left(q^2 - n^2\right) a_q, \quad n \geqslant 0.$$

Substituting the approximate solution and its derivatives into the differential equation (3), together with the additional conditions (11), yields the following algebraic system for determining the unknown coefficients $a_n^{(j)}$ $(n = 0, 1, 2, \ldots, p_j;\ j = 1, 2, \ldots, N)$:

$$-\varepsilon \frac{1}{c_n} \sum_{\substack{p=n+2 \\ p+n \equiv 0 \,(\mathrm{mod}\, 2)}}^{p_j} p\left(p^2 - n^2\right) a_p - \mu \frac{2}{c_n} \sum_{\substack{p=n+1 \\ p+n \equiv 1 \,(\mathrm{mod}\, 2)}}^{p_j} p\, a_p + a_n = b_n^{(j)},$$
$$n = 0, 1, \ldots, p_j - 2,$$

$$\sum_{n=0}^{p_1} (-1)^n a_n^{(1)} = 0, \quad \sum_{n=0}^{p_N} a_n^{(N)} = 0, \quad \sum_{n=0}^{p_j} a_n^{(j)} = \sum_{n=0}^{p_{j+1}} (-1)^n a_n^{(j+1)}, \tag{15}$$
$$\frac{l_j}{2} \sum_{n=0}^{p_j} n^2 a_n^{(j)} = \frac{l_{j+1}}{2} \sum_{n=0}^{p_{j+1}} (-1)^{n-1} n^2 a_n^{(j+1)}, \quad j = 1, 2, \ldots, N.$$

The number of unknowns is given by

$$\mathbf{x}^T = \left(a_0^{(1)}, a_1^{(1)}, \ldots, a_{p_1}^{(1)}, a_0^{(2)}, a_1^{(2)}, \ldots, a_{p_2}^{(2)}, \ldots, a_0^{(N)}, a_1^{(N)}, \ldots, a_{p_N}^{(N)}\right),$$

and is computed using the relation

$$M = \sum_{j=1}^{N} (p_j + 1) = (p_1 + p_2 + \cdots + p_N + N).$$

Accordingly, the total number of equations in system (15) is equal to

$$M = (p_1 + p_2 + \cdots + p_N + N).$$

In addition, the number of basic equations in system (15) is equal to $N(p_j - 1)$, and two boundary conditions and $2(N - 1)$ continuity conditions are added to them. System (15) can be written in the following matrix form:

$$Ax = b, \tag{16}$$

where

$$b^T = \left(b_0^{(1)}, b_1^{(1)}, \ldots, b_{p_1}^{(1)}, b_0^{(2)}, b_1^{(2)}, \ldots, b_{p_2}^{(2)}, \ldots, b_0^{(N)}, b_1^{(N)}, \ldots, b_{p_N}^{(N)}\right),$$

and x^T and b^T denote the transposed vectors of unknown coefficients and right-hand side terms, respectively.

4 Discussion of results

The singularly perturbed problem (3)–(4) was solved using the spectral-grid method described in (15). The spectral-grid method can be implemented in several variants:

1) Uniform grid and uniform approximation. The computational domain is divided into subintervals of equal length, and in each subinterval, the approximate solution is represented using Chebyshev polynomials of the same degree; 2) Uniform grid and non-uniform approximation. The domain is divided into equal subintervals, but in each subinterval, the approximate solution is expressed using Chebyshev polynomials of varying degrees; 3) Non-uniform grid and uniform approximation. The interval is divided into subintervals of varying lengths, but the same number of Chebyshev polynomials is used in each subinterval; 4) Non-uniform grid and non-uniform approximation. Both the grid and the degree of approximation vary across subintervals.

Among these, the fourth variant is used in this study due to its universality and adaptability. This approach allows the localization of singular boundary layers within sufficiently small subintervals and enables the concentration of Chebyshev nodes where the solution changes rapidly. Consequently, the method provides high accuracy even for very small parameter values.

The convergence and accuracy of the proposed method were tested using the analytical test function (5). This test function was selected according to [11], and the numerical results obtained using the spectral-grid method were compared with those reported in [11]. Tables 1–3 present the computed values of the maximum absolute error at the Chebyshev collocation nodes. The maximum absolute error is evaluated using the following expression:

$$\Delta = \max_{0\leqslant k\leqslant p_j} |u_e - u_a|, \quad j = 1, 2, \ldots, N.$$

The nodes of the first-kind Chebyshev polynomials are defined as follows:

$$x_l^{(j)} = \cos\left(\frac{\pi l}{p_j}\right), \quad l = 0, 1, \ldots, p_j, \quad j = 1, 2, \ldots, N.$$

Furthermore, $0 < \varepsilon, \mu \ll 1$ are small parameters, and the total number of Chebyshev polynomials used in the approximation is given by $M = \sum_{j=1}^{N}(p_j + 1)$, where $(p_j,\ j = 1, 2, \ldots, N)$ denotes the number of Chebyshev polynomials used in the subintervals.

Table 1 presents the computed values of the maximum absolute error Δ for a uniform partition of the interval into four elements ($N = 4$) with subintervals $[0, 0.25]$, $[0.25, 0.5]$, $[0.5, 0.75]$, $[0.75, 1]$. In this case, the first and second variants of the spectral- grid method were applied.

Table 1. Maximum absolute error of the spectral-grid method ($N = 4$).

ε	μ	$M = \sum_{j=1}^{N}(p_j+1)$	B-spline method [11] Δ	Spectral-grid method				
				p_1+1	p_2+1	p_3+1	p_4+1	Δ
10^{-6}	10^{-8}	64	9.6×10^{-3}	16	16	16	16	1.0×10^{0}
				24	8	8	24	1.8×10^{-1}
				28	4	4	28	6.9×10^{-2}
		128	2.3×10^{-3}	32	32	32	32	2.2×10^{-2}
				48	16	16	48	9.2×10^{-5}
				56	8	8	56	2.9×10^{-6}
		256	5.8×10^{-4}	64	64	64	64	6.1×10^{-8}
				96	32	32	32	2.4×10^{-14}
				112	16	16	112	4.5×10^{-14}

From the results presented in table 1, it can be observed that increasing the number of grid elements, as well as refining the grid near the boundaries or boundary layers, leads to a noticeable accumulation of Chebyshev nodes. This refinement results in a significant reduction of the maximum absolute error. Table 2 presents the numerical results obtained using the spectral-grid method. Here, the parameters are chosen as $N = 4$, $\varepsilon = 10^{-8}$, $\mu = 10^{-3}, 10^{-8}$. The subinterval lengths are selected as $[0, 0.01]$, $[0.01, 0.5]$, $[0.5, 0.99]$, $[0.99, 1]$. For comparison, the results reported in [11] are also provided.

When using the spectral-grid method, it is important to select both the number and the lengths of the subintervals based on the characteristics of the problem being studied. For example, in the considered two-parameter singularly perturbed problem, when the parameters

$$\varepsilon = 10^{-10}, \quad \mu = 10^{-8},$$

take very small values, the solution exhibits sharp gradients in the boundary regions. In such cases, if the number and lengths of the subintervals are not chosen appropriately, the method may fail to provide sufficiently accurate results, even when the total number of Chebyshev polynomials M is large. Table 3 presents the comparison of the results obtained for the parameters $\varepsilon = 10^{-10}$, $\mu = 10^{-8}$, using the gradient B-spline collocation method, the spectral method with $N = 1$ subinterval, and the proposed spectral-grid method.

Table 2. Maximum absolute error of the spectral-grid method ($N = 4$).

ε	μ	$M = \sum_{j=1}^{N}(p_j+1)$	B-spline method [11] Δ	Spectral-grid method				
				p_1+1	p_2+1	p_3+1	p_4+1	Δ
10^{-8}	10^{-3}	128	5.6×10^{-3}	32	32	32	32	1.6×10^{-2}
				32	16	48	32	7.6×10^{-4}
				34	4	58	34	8.0×10^{-5}
		256	1.4×10^{-3}	64	64	64	64	1.4×10^{-5}
				72	32	80	72	1.7×10^{-7}
				80	8	88	80	6.4×10^{-9}
	10^{-8}	128	2.3×10^{-3}	32	32	32	32	3.3×10^{-5}
				48	16	16	48	1.5×10^{-10}
				56	8	8	56	5.5×10^{-13}
		256	5.8×10^{-4}	64	64	64	64	3.8×10^{-13}
				96	32	32	96	3.1×10^{-13}
				112	16	16	112	5.5×10^{-13}

Table 3. Maximum absolute error for different methods.

ε	μ	$M=\sum_{j=1}^{N}(p_j+1)$	B-spline method [11] Δ	Spectral method Δ	Spectral-grid method Δ
10^{-10}	10^{-8}	64	9.6×10^{-3}	2.7×10^{0}	3.9×10^{-4}
		128	2.3×10^{-4}	2.6×10^{0}	6.2×10^{-12}
		256	5.9×10^{-4}	1.6×10^{0}	7.6×10^{-12}

From Tables 1–3, it can be seen that for small parameter values ε, μ and different numbers of Chebyshev polynomials M, the proposed spectral-grid method provides significantly smaller maximum absolute errors compared to the gradient B-spline collocation and classical spectral methods. This demonstrates the universality and high efficiency of the proposed method.

5 Conclusion

The following conclusions can be drawn from this study: A universal, highly accurate, and efficient spectral-grid method has been developed for solving boundary value problems of two-parameter singularly perturbed differential equations. The convergence and accuracy of the proposed spectral-grid method have been justified and verified using the method of test functions. Using the spectral-grid method, the two-parameter singularly perturbed boundary value problem has been approximated and reduced to an equivalent system of algebraic equations. Based on the numerical experiments, it has been demonstrated that the proposed spectral-grid method possesses universality, high accuracy, efficiency, and adaptability in solving boundary value problems for two-parameter singularly perturbed differential equations.

References

[1] Salih M.H., Duressa G.F., Debela H.G. *Numerical solution of singularly perturbed self-adjoint boundary value problem using Galerkin method // International Journal of Engineering Science and Technology.* – 2020. – Vol. 12, Issue 3. – P. 26-37. – doi: `http://dx.doi.org/10.4314/ijest.v12i3.3`.

[2] Kaushik A., Gupta A. *An adaptive mesh generation and higher-order difference approximation for the system of singularly perturbed reaction-diffusion problem // Partial Differential Equations in Applied Mathematics.* – 2024. – Art. no. 100750. – doi: `http://dx.doi.org/10.1016/j.padiff.2024.100750`.

[3] Balasubramani N., Prasad M.G., Natesan S. *Fractal quintic spline solutions for singularly perturbed reaction-diffusion boundary-value problems // Applied Numerical Mathematics.* – 2024. – doi: `http://dx.doi.org/10.1016/j.apnum.2024.04.015`.

[4] Liu Y., Cheng Y. *Local discontinuous Galerkin method for a singularly perturbed fourth-order problem of reaction-diffusion type // Journal of Computational and Applied Mathematics.* – 2024. – Art. no. 115641. – doi: `http://dx.doi.org/10.1016/j.jcam.2023.115641`.

[5] Barzekhar N., Barati A., Jalilian R. *Sinc approximation method for solving system of singularly perturbed parabolic convection-diffusion equations // Applied Numerical Mathematics.* – 2025. – doi: `http://dx.doi.org/10.1016/j.apnum.2025.05.005`.

[6] Debela H.G., Duressa G.F. *Accelerated exponentially fitted operator method for singularly perturbed problems with integral boundary condition // International Journal of Differential Equations.* – 2020. – P. 1-8. – doi: `http://dx.doi.org/10.1155/2020/9268181`.

[7] Kusi G.R., Habte A.H., Bullo T.A. *Layer resolving numerical scheme for singularly perturbed parabolic convection-diffusion problem with an interior layer* // *MethodsX*. – 2023. – Vol. 10. – Art. no. 101953. – doi: http://dx.doi.org/10.1016/j.mex.2022.101953.

[8] Roy N., Jha A. *A parameter-uniform method for two-parameter singularly perturbed boundary value problems with discontinuous data* // *MethodsX*. – 2023. – Art. no. 102004. – doi: http://dx.doi.org/10.1016/j.mex.2023.102004.

[9] Kadalbajoo M.K., Yadaw A.S. *B-spline collocation method for a two-parameter singularly perturbed convection-diffusion boundary value problems* // *Applied Mathematics and Computation*. – 2008. – Vol. 201, Issue 1-2. – P. 504-513. – doi: http://dx.doi.org/10.1016/j.amc.2007.12.038.

[10] Kadalbajoo M.K., Yadaw A.S. *Parameter-uniform Ritz-Galerkin finite element method for two-parameter singularly perturbed boundary value problems* // *International Journal of Pure and Applied Mathematics*. – 2009. – Vol. 55, Issue 2. – P. 287-300.

[11] Andisso F.S., Duressa G.F. *Graded mesh B-spline collocation method for two-parameter singularly perturbed problems* // *MethodsX*. – 2023. – Art. no. 102336. – doi: http://dx.doi.org/10.1016/j.mex.2023.102336.

[12] Normurodov C.B., Tursunova B.A. *Numerical modeling of the boundary value problem of an ordinary differential equation with a small parameter at the highest derivative by Chebyshev polynomials of the second kind* // *Results in Applied Mathematics*. – 2023. – Vol. 19. – Art. no. 100388. – doi: http://dx.doi.org/10.1016/j.rinam.2023.100388.

[13] Normurodov Ch.B., Abduraximov B.F., Djuraeva N.T. *On estimating the rate of convergence of the initial integration method* // *AIP Conference Proceedings*. – 2024. – Vol. 3244. – doi: http://dx.doi.org/10.1063/5.0242041.

[14] Normurodov C., Toyirov A., Ziyakulova S., Viswanathan K.K. *Convergence of spectral-grid method for Burgers equation with initial-boundary conditions* // *Mathematics and Statistics*. – 2024. – Vol. 12, Issue 2. – P. 115-125. – doi: http://dx.doi.org/10.13189/ms.2024.120201.

[15] Normurodov Ch.B., Dzhuraeva N.T., Normatova M.M. *A high-accuracy and efficient method for studying the dynamics of derivatives of different orders of a singularly perturbed equation* // *Chebyshev Collection*. – 2025. – Vol. 26, Issue 4. – P. 357-369. – doi: http://dx.doi.org/10.22405/2226-8383-2025-26-4-357-369.

[16] Normurodov Ch.B., Ziyakulova Sh.A., Murodov S.K. *On one highly accurate and efficient method for solving the biharmonic equation* // *International Journal of Applied Mathematics*. – 2025. – Vol. 38, Issue 4. – P. 437-453. – doi: http://dx.doi.org/10.12732/ijam.v38i4.1.

[17] Bouakkaz M., Arar N., Meflah M. *Enhanced numerical resolution of the Duffing and Van der Pol equations via the spectral homotopy analysis method employing Chebyshev polynomials of the first kind* // *Journal of Applied Mathematics and Computation*. – 2024. – doi: http://dx.doi.org/10.1007/s12190-024-02271-5.

[18] Normurodov Ch.B. *Mathematical modeling of hydrodynamic problems for two-phase plane-parallel flows* // *Mathematical Modeling*. – 2007. – Vol. 19, Issue 6. – P. 53-60.

[19] Normurodov Ch.B. *On an efficient method for solving the Orr-Sommerfeld equation* // *Mathematical Modeling*. – 2005. – Vol. 17, Issue 9. – P. 35-42.

[20] Normurodov Ch.B., Solovyev A.S. *The influence of weighted particles on the stability of plane Poiseuille flow* // *Fluid Mechanics and Gas Dynamics*. – 1986. – Issue 1. – P. 46-50.

[21] Normurodov Ch.B., Solovyev A.S. *Stability of a two-phase flow of gas-solid particles in the boundary layer* // *Mechanics of Fluids and Gases*. – 1987. – Issue 2. – P. 60-64.

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ЧИСЛЕННОЕ МОДЕЛИРОВАНИЕ КРАЕВОЙ ЗАДАЧИ ДЛЯ ДВУХПАРАМЕТРИЧЕСКОГО СИНГУЛЯРНО ВОЗМУЩЁННОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ С ИСПОЛЬЗОВАНИЕМ СПЕКТРАЛЬНО-СЕТОЧНОГО МЕТОДА

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Известно, что решения краевых задач, связанных с двухпараметрическими сингулярно возмущенными дифференциальными уравнениями, демонстрируют образование двух различных пограничных слоев, обычно возникающих вблизи концов области. Наличие этих узких областей, характеризующихся крутыми градиентами в решении, представляет собой значительную проблему для классических численных методов. В результате стандартные методы конечных разностей, конечных элементов или дискретизации низкого порядка часто не позволяют точно воспроизвести поведение слоев, если не используются чрезмерно мелкие сетки, что приводит к увеличению вычислительных затрат и снижению эффективности. Для решения этих проблем в данной работе предлагается эффективный спектрально -сеточный метод для численного решения краевых задач, включающих двухпараметрические сингулярные возмущения. Основная идея предлагаемого подхода заключается в сочетании высокой точности спектральных методов с тщательно разработанной сеткой, которая адаптируется к структуре пограничного слоя решения. Используя эту спектрально-сеточную структуру, исходная краевая задача преобразуется в эквивалентную систему алгебраических уравнений, которая может быть эффективно решена с помощью стандартных методов линейной алгебры. Для оценки эффективности предложенного метода были проведены обширные численные эксперименты. Полученные результаты систематически сравнивались с результатами, представленными в существующей литературе. Эти сравнения наглядно демонстрируют, что разработанный в данном исследовании спектрально-сеточный метод обеспечивает более высокую точность при относительно низких вычислительных затратах.

Ключевые слова: спектрально-сеточный метод, двухпараметрическое сингулярно возмущённое дифференциальное уравнение, полиномы Чебышёва, максимальная абсолютная ошибка, пограничные слои, численная аппроксимация, высокоточный алгоритм.

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NUMERICAL MODELING OF THE BOUNDARY VALUE PROBLEM FOR A TWO-PARAMETER SINGULARLY PERTURBED DIFFERENTIAL EQUATION USING THE SPECTRAL-GRID METHOD

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The solutions of boundary value problems associated with two-parameter singularly perturbed differential equations are well known to exhibit the formation of two distinct boundary layers, typically occurring near the endpoints of the domain. The presence of these narrow regions, characterized by steep gradients in the solution, poses significant challenges for classical numerical techniques. As a result, standard finite difference, finite element, or low-order discretization methods often fail to capture the layer behavior accurately unless excessively fine meshes are employed, leading to increased computational cost and reduced efficiency. To address these difficulties, the present work proposes an efficient spectral-grid method for the numerical solution of boundary value problems involving two-parameter singular perturbations. The core idea of the proposed approach is to combine the high accuracy of spectral methods with a carefully designed grid that adapts to the boundary layer structure of the solution. By employing this spectral-grid framework, the original boundary value problem is transformed into an equivalent system of algebraic equations, which can be solved efficiently using standard linear algebra techniques. Extensive numerical experiments are carried out to assess the performance of the proposed method. The computed results are systematically compared with those available in the existing literature. These comparisons clearly demonstrate that the spectral-grid method developed in this study achieves superior accuracy while maintaining a relatively low computational cost.

Keywords: spectral-grid method, two-parameter singularly perturbed differential equation, Chebyshev polynomials, maximum absolute error, boundary layers, numerical approximation, high-accuracy algorithm.

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1 Introduction

Differential equations containing small parameters in the coefficients of the highest derivatives are referred to as singularly perturbed differential equations. The solutions of such problems possess specific features: the sought solution domain usually includes rapidly varying boundary layers and regions where the solution changes smoothly. Traditional numerical methods are not efficient for solving such problems because they require extremely fine meshes to obtain satisfactory accuracy. However, ensuring stability in the numerical solution of singularly perturbed problems often leads to large algebraic systems with high computational costs. Therefore, the development of special numerical methods adapted to boundary layers and capable of efficiently handling such problems is of significant importance. In this paper, we consider the following problem:

$$eqn1Lu(x) = -\varepsilon u''(x) - \mu a(x)u'(x) + b(x)u(x) = f(x), \quad 0 \leqslant x \leqslant 1, \tag{1}$$

subject to the boundary conditions

$$eqn2u(0) = \alpha_0, \quad u(1) = \alpha_1. \tag{2}$$

Here, α_0 and α_1 are given constants, $0 < \varepsilon, \mu \ll 1$ are two small parameters. The functions $a(x)$, $b(x)$, and $f(x)$ are assumed to be sufficiently smooth. Problems of this type are generally referred to as two-parameter singularly perturbed boundary value problems.

When $\mu = 0$, the problem (1)–(2) reduces to the well-studied reaction–diffusion boundary value problem with a single perturbation parameter. When $\mu = 1$, it transforms into a convection–diffusion problem. Numerous studies have been devoted to the numerical solutions of these particular cases. For instance, reaction–diffusion problems corresponding to $\mu = 0$ have been thoroughly analyzed in [1–4], whereas the convection–diffusion case with $\mu = 1$ has been discussed in [5–7]. However, the genuinely two-parameter case, i.e., when $0 < \varepsilon, \mu \ll 1$, has received comparatively little attention in the literature. Several studies addressing such problems can be mentioned. The authors [8] proposed a finite difference method on Shishkin-type meshes and proved its first-order convergence. Later a B-spline collocation method [9] on a similar mesh was developed and demonstrated its second-order convergence. Kadabaioo and co-authors [10] showed that the Ritz–Galerkin method is highly effective for problems involving boundary layers. In [11], the authors proposed a B-spline collocation method on graded meshes and analyzed its efficiency for small values of the parameters ε, μ, and for refined mesh densities, demonstrating the robustness of the method.

Although several methods have been developed to solve such types of problems, there is still a lack of numerical approaches that ensure high accuracy for singularly perturbed problems involving small parameters. Therefore, in this paper, a spectral-grid method is proposed for the numerical solution of two-parameter singularly perturbed differential equations. The results presented in this study demonstrate that the proposed spectral-grid method is several times more efficient than the graded B-spline collocation method.

In [12], the use of Chebyshev polynomials for solving eigenvalue problems arising from systems of higher-order nonlinear differential equations containing small parameters was discussed. In [13, 14], the authors considered parabolic-type partial differential equations with initial–boundary conditions and developed a Chebyshev perturbation approach based on continuous integration. The obtained error estimates correctly reflect the convergence order of the method.

More recently, a highly accurate and efficient numerical method based on Chebyshev spectral techniques was proposed for the analysis of the dynamics of derivatives of different orders in singularly perturbed equations [15]. In addition, a high-accuracy method for solving the biharmonic equation using spectral approaches was developed and successfully applied, demonstrating superior numerical efficiency and precision [16].

Papers [18] and [19] established theoretical foundations for hydrodynamic and spectral methods based on Chebyshev polynomials and analyzed several of their efficient implementations. In particular, [18] investigated the mathematical modeling of two-dimensional parallel flows in hydrodynamics, analyzing their dependence on boundary conditions.

In [19], the Orr–Sommerfeld equation was studied, and an efficient spectral method was proposed for its numerical solution. The results of these studies have laid the groundwork for further research on small-parameter problems using numerical techniques based on Chebyshev polynomials. Hydrodynamic stability of two-phase liquid flows has been analyzed using spectral methods in [20]. Furthermore, for the case of a bounded layer, this problem was examined through the application of the spectral collocation method in [21].

2 Statement of the problem

Consider the following two-parameter singularly perturbed differential equation with small parameters in the highest derivatives:

$$-\varepsilon u''(x) - \mu u'(x) + u(x) = e^{1-x}, \quad x \in [0, 1], \tag{3}$$

subject to the boundary conditions

$$u(0) = u(1) = 0, \tag{4}$$

where ε and μ are small positive parameters.

To numerically model the differential problem (3)-(4), we employ the spectral-grid method based on the first-kind Chebyshev polynomials as basis functions. Typically, when analyzing the convergence and accuracy of numerical methods, an exact analytical or test function is used for verification [17]. According to this approach, an auxiliary analytical function $u_e(x)$ satisfying the boundary conditions (4) is chosen. Substituting it into equation (3) allows us to compute the corresponding right-hand side function. The considered problem is then solved using the spectral-grid method, and the obtained numerical solution is compared with the chosen analytical test function.

For the current study, the following analytical function satisfying the boundary conditions of equation (3) is selected as the test function:

$$u(x) = \frac{1}{\varepsilon - \mu - 1}\left[\frac{e - e^{a_1}}{1 - e^{-\frac{\sqrt{\mu^2+4\varepsilon}}{\varepsilon}}}\, e^{-a_2 x} - e^{1-x} + \frac{e^{1-a_2} - 1}{e^{-\frac{\sqrt{\mu^2+4\varepsilon}}{\varepsilon}} - 1}\, e^{a_1(1-x)}\right],$$

where

$$\alpha_1 = \frac{\mu - \sqrt{\mu^2 + 4\varepsilon}}{2\varepsilon}, \quad \alpha_2 = \frac{\mu + \sqrt{\mu^2 + 4\varepsilon}}{2\varepsilon}.$$

3 Solution method

We now present the spectral-grid algorithm for the numerical modeling of problem (3)–(4). First, the integration interval $[\eta_0, \eta_1]$ is divided into a uniform grid $\eta_0 < \eta_1 < \cdots < \eta_N = \eta_l$, resulting in N subintervals (elements):

$$[\eta_0, \eta_1], [\eta_1, \eta_2], \ldots, [\eta_j, \eta_{j+1}], \ldots, [\eta_{N-1}, \eta_N], \quad j = 1, 2, \ldots, N-1, \tag{5}$$

where $\eta_0 = 0$ and $\eta_N = 1$.

The differential equation (3) within each subinterval takes the following form:

$$-\varepsilon \frac{d^2 u_j}{d\eta^2} - \mu \frac{du_j}{d\eta} + u_j(\eta) = f_j(\eta), \quad j = 1, 2, \ldots, N. \tag{6}$$

The boundary conditions (4) at the endpoints η_0 and η_N are written as:

$$u(\eta_0) = 0, \quad u(\eta_N) = 0. \tag{7}$$

At the internal nodes, the solution of equation (3) and its first derivative are required to be continuous. These continuity conditions can be expressed as follows:

$$u_j^{(t)}(\eta_j) = u_{j+1}^{(t)}(\eta_j), \quad t = 0, 1; \quad j = 1, 2, \ldots, N-1, \tag{8}$$

where t denotes the order of the derivative.

The approximate solution $u_j(x)$ of problem (6)–(8) is represented as a truncated series in terms of the first-kind Chebyshev polynomials. Each subinterval $[\eta_j, \eta_{j+1}]$ is mapped onto the standard interval $[-1, 1]$ using the following transformation:

$$\eta = \frac{m_j}{2} + \frac{l_j}{2}x, \qquad m_j = \eta_j + \eta_{j+1}, \quad l_j = \eta_{j+1} - \eta_j, \quad j = 1, 2, \ldots, N-1, \tag{9}$$

where $\eta \in [\eta_j, \eta_{j+1}]$, $x \in [-1, 1]$, and l_j denotes the length of the $(j+1)$-th subinterval.

As a result, equation (6) can be rewritten in the transformed variable x as follows:

$$-\varepsilon \left(\frac{2}{l_j}\right)^2 \frac{d^2 u_j}{dx^2} - \mu \left(\frac{2}{l_j}\right) \frac{du_j}{dx} + u_j(x) = f_j(x), \qquad j = 1, 2, \ldots, N. \tag{10}$$

The boundary conditions (7) and the continuity conditions (8) take the form:

$$u_1(0) = 0, \qquad u_N(+1) = 0,$$

$$\left(\frac{2}{l_j}\right)^t u_j^{(t)}(+1) = \left(\frac{2}{l_{j+1}}\right)^t u_{j+1}^{(t)}(-1), \quad t = 0, 1; \quad j = 1, 2, \ldots, N-1, \tag{11}$$

where t denotes the order of the derivative.

We seek the approximate solution of problem (10)–(11) and its right-hand side $f_j(x)$ in each subinterval in the form of truncated series of the first-kind Chebyshev polynomials:

$$u_a^{(j)}(x) = \sum_{n=0}^{p_j} a_n^{(j)} T_n(x), \qquad f_j\left(\overline{x_l}^{(j)}\right) = \sum_{n=0}^{p_j} b_n^{(j)} T_n\left(x_l^{(j)}\right), \qquad \overline{x_l}^{(j)} = \frac{m_j}{2} + \frac{l_j}{2} x_l^{(j)}, \tag{12}$$

with

$$x_l^{(j)} = \cos\left(\frac{\pi l}{p_j}\right), \quad l = 0, 1, \ldots, p_j, \quad j = 1, 2, \ldots, N.$$

Here $u_a^{(j)}(x)$ denotes the approximate solution, $T_n(x)$ are the first-kind Chebyshev polynomials, and $x_l^{(j)}$ are their corresponding nodes. The parameter p_j represents the number of polynomials used to approximate the solution in the j-th subinterval, while $a_n^{(j)}$ are the unknown expansion coefficients.

For the given function $f_j(x)$, the expansion coefficients $b_n^{(j)}$ in expression (12) are determined by the inverse transformation formula [16]:

$$b_n^{(j)} = \frac{2}{p_j c_n} \sum_{l=0}^{p_j} \frac{1}{c_l} f_j\left(\overline{x_l}^{(j)}\right) T_n\left(x_l^{(j)}\right), \quad n = 0, 1, 2, \ldots, p_j, \tag{13}$$

where $c_0 = c_{p_j} = 2$, $c_m = 1$ $(m \neq 0, p_j)$, $j = 1, 2, \ldots, N$, and $M = \sum\limits_{j=1}^{N}(p_j+1)$ is the total number of Chebyshev polynomials used in the spectral-grid approximation of the solution, while N denotes the number of grid elements. To compute derivatives of various orders of the approximate solution $u_a^{(j)}(x)$ from (12), the following relations are used, which are convenient for numerical implementation:

$$u_a^{(j)}(y) = \sum_{n=0}^{p_j} a_n^{(j)} T_n(y),$$

$$u_a^{(1)}(y) = \sum_{n=0}^{p_j} a_n^{(1)} T_n(y), \qquad a_n^{(1)} = \frac{2}{c_n} \sum_{\substack{q=n+1 \\ q+n\equiv 1 \,(\mathrm{mod}\, 2)}}^{p_j} q\, a_q, \quad n \geqslant 0, \tag{14}$$

$$u_a^{(2)}(y) = \sum_{n=0}^{p_j} a_n^{(2)} T_n(y), \qquad a_n^{(2)} = \frac{2}{c_n} \sum_{\substack{q=n+2 \\ q+n\equiv 0 \,(\mathrm{mod}\, 2)}}^{p_j} q\left(q^2-n^2\right) a_q, \quad n \geqslant 0.$$

Substituting the approximate solution and its derivatives into the differential equation (3), together with the additional conditions (11), yields the following algebraic system for determining the unknown coefficients $a_n^{(j)}$ $(n = 0, 1, 2, \ldots, p_j;\ j = 1, 2, \ldots, N)$:

$$-\varepsilon \frac{1}{c_n} \sum_{\substack{p=n+2 \\ p+n\equiv 0 \,(\mathrm{mod}\, 2)}}^{p_j} p\left(p^2-n^2\right) a_p - \mu \frac{2}{c_n} \sum_{\substack{p=n+1 \\ p+n\equiv 1 \,(\mathrm{mod}\, 2)}}^{p_j} p\, a_p + a_n = b_n^{(j)},$$
$$n = 0, 1, \ldots, p_j - 2,$$

$$\sum_{n=0}^{p_1}(-1)^n a_n^{(1)} = 0, \quad \sum_{n=0}^{p_N} a_n^{(N)} = 0, \quad \sum_{n=0}^{p_j} a_n^{(j)} = \sum_{n=0}^{p_{j+1}}(-1)^n a_n^{(j+1)}, \tag{15}$$
$$\frac{l_j}{2}\sum_{n=0}^{p_j} n^2 a_n^{(j)} = \frac{l_{j+1}}{2}\sum_{n=0}^{p_{j+1}}(-1)^{n-1} n^2 a_n^{(j+1)}, \quad j = 1, 2, \ldots, N.$$

The number of unknowns is given by

$$\mathbf{x}^T = \left(a_0^{(1)}, a_1^{(1)}, \ldots, a_{p_1}^{(1)}, a_0^{(2)}, a_1^{(2)}, \ldots, a_{p_2}^{(2)}, \ldots, a_0^{(N)}, a_1^{(N)}, \ldots, a_{p_N}^{(N)}\right),$$

and is computed using the relation

$$M = \sum_{j=1}^{N}(p_j+1) = (p_1 + p_2 + \cdots + p_N + N).$$

Accordingly, the total number of equations in system (15) is equal to

$$M = (p_1 + p_2 + \cdots + p_N + N).$$

In addition, the number of basic equations in system (15) is equal to $N(p_j-1)$, and two boundary conditions and $2(N-1)$ continuity conditions are added to them. System (15) can be written in the following matrix form:

$$Ax = b, \tag{16}$$

where

$$b^T = \left(b_0^{(1)}, b_1^{(1)}, \ldots, b_{p_1}^{(1)}, b_0^{(2)}, b_1^{(2)}, \ldots, b_{p_2}^{(2)}, \ldots, b_0^{(N)}, b_1^{(N)}, \ldots, b_{p_N}^{(N)}\right),$$

and x^T and b^T denote the transposed vectors of unknown coefficients and right-hand side terms, respectively.

4 Discussion of results

The singularly perturbed problem (3)–(4) was solved using the spectral-grid method described in (15). The spectral-grid method can be implemented in several variants:

1) Uniform grid and uniform approximation. The computational domain is divided into subintervals of equal length, and in each subinterval, the approximate solution is represented using Chebyshev polynomials of the same degree; 2) Uniform grid and non-uniform approximation. The domain is divided into equal subintervals, but in each subinterval, the approximate solution is expressed using Chebyshev polynomials of varying degrees; 3) Non-uniform grid and uniform approximation. The interval is divided into subintervals of varying lengths, but the same number of Chebyshev polynomials is used in each subinterval; 4) Non-uniform grid and non-uniform approximation. Both the grid and the degree of approximation vary across subintervals.

Among these, the fourth variant is used in this study due to its universality and adaptability. This approach allows the localization of singular boundary layers within sufficiently small subintervals and enables the concentration of Chebyshev nodes where the solution changes rapidly. Consequently, the method provides high accuracy even for very small parameter values.

The convergence and accuracy of the proposed method were tested using the analytical test function (5). This test function was selected according to [11], and the numerical results obtained using the spectral-grid method were compared with those reported in [11]. Tables 1–3 present the computed values of the maximum absolute error at the Chebyshev collocation nodes. The maximum absolute error is evaluated using the following expression:

$$\Delta = \max_{0 \leqslant k \leqslant p_j} |u_e - u_a|, \quad j = 1, 2, \ldots, N.$$

The nodes of the first-kind Chebyshev polynomials are defined as follows:

$$x_l^{(j)} = \cos\left(\frac{\pi l}{p_j}\right), \quad l = 0, 1, \ldots, p_j, \quad j = 1, 2, \ldots, N.$$

Furthermore, $0 < \varepsilon, \mu \ll 1$ are small parameters, and the total number of Chebyshev polynomials used in the approximation is given by $M = \sum_{j=1}^{N} (p_j + 1)$, where $(p_j,\ j = = 1, 2, \ldots, N)$ denotes the number of Chebyshev polynomials used in the subintervals.

Table 1 presents the computed values of the maximum absolute error Δ for a uniform partition of the interval into four elements ($N = 4$) with subintervals $[0, 0.25]$, $[0.25, 0.5]$, $[0.5, 0.75]$, $[0.75, 1]$. In this case, the first and second variants of the spectral- grid method were applied.

Table 1. Maximum absolute error of the spectral-grid method ($N = 4$).

ε	μ	$M=\sum_{j=1}^{N}(p_j+1)$	B-spline method [11] Δ	Spectral-grid method				
				p_1+1	p_2+1	p_3+1	p_4+1	Δ
10^{-6}	10^{-8}	64	9.6×10^{-3}	16	16	16	16	1.0×10^{0}
				24	8	8	24	1.8×10^{-1}
				28	4	4	28	6.9×10^{-2}
		128	2.3×10^{-3}	32	32	32	32	2.2×10^{-2}
				48	16	16	48	9.2×10^{-5}
				56	8	8	56	2.9×10^{-6}
		256	5.8×10^{-4}	64	64	64	64	6.1×10^{-8}
				96	32	32	32	2.4×10^{-14}
				112	16	16	112	4.5×10^{-14}

From the results presented in table 1, it can be observed that increasing the number of grid elements, as well as refining the grid near the boundaries or boundary layers, leads to a noticeable accumulation of Chebyshev nodes. This refinement results in a significant reduction of the maximum absolute error. Table 2 presents the numerical results obtained using the spectral-grid method. Here, the parameters are chosen as $N = 4$, $\varepsilon = 10^{-8}$, $\mu = 10^{-3}, 10^{-8}$. The subinterval lengths are selected as $[0, 0.01]$, $[0.01, 0.5]$, $[0.5, 0.99]$, $[0.99, 1]$. For comparison, the results reported in [11] are also provided.

When using the spectral-grid method, it is important to select both the number and the lengths of the subintervals based on the characteristics of the problem being studied. For example, in the considered two-parameter singularly perturbed problem, when the parameters

$$\varepsilon = 10^{-10}, \quad \mu = 10^{-8},$$

take very small values, the solution exhibits sharp gradients in the boundary regions. In such cases, if the number and lengths of the subintervals are not chosen appropriately, the method may fail to provide sufficiently accurate results, even when the total number of Chebyshev polynomials M is large. Table 3 presents the comparison of the results obtained for the parameters $\varepsilon = 10^{-10}$, $\mu = 10^{-8}$, using the gradient B-spline collocation method, the spectral method with $N = 1$ subinterval, and the proposed spectral-grid method.

Table 2. Maximum absolute error of the spectral-grid method ($N = 4$).

ε	μ	$M=\sum_{j=1}^{N}(p_j+1)$	B-spline method [11] Δ	Spectral-grid method				
				p_1+1	p_2+1	p_3+1	p_4+1	Δ
10^{-8}	10^{-3}	128	5.6×10^{-3}	32	32	32	32	1.6×10^{-2}
				32	16	48	32	7.6×10^{-4}
				34	4	58	34	8.0×10^{-5}
		256	1.4×10^{-3}	64	64	64	64	1.4×10^{-5}
				72	32	80	72	1.7×10^{-7}
				80	8	88	80	6.4×10^{-9}
	10^{-8}	128	2.3×10^{-3}	32	32	32	32	3.3×10^{-5}
				48	16	16	48	1.5×10^{-10}
				56	8	8	56	5.5×10^{-13}
		256	5.8×10^{-4}	64	64	64	64	3.8×10^{-13}
				96	32	32	96	3.1×10^{-13}
				112	16	16	112	5.5×10^{-13}

Table 3. Maximum absolute error for different methods.

ε	μ	$M = \sum_{j=1}^{N}(p_j+1)$	B-spline method [11] Δ	Spectral method Δ	Spectral-grid method Δ
10^{-10}	10^{-8}	64	9.6×10^{-3}	2.7×10^{0}	3.9×10^{-4}
		128	2.3×10^{-4}	2.6×10^{0}	6.2×10^{-12}
		256	5.9×10^{-4}	1.6×10^{0}	7.6×10^{-12}

From Tables 1–3, it can be seen that for small parameter values ε, μ and different numbers of Chebyshev polynomials M, the proposed spectral-grid method provides significantly smaller maximum absolute errors compared to the gradient B-spline collocation and classical spectral methods. This demonstrates the universality and high efficiency of the proposed method.

5 Conclusion

The following conclusions can be drawn from this study: A universal, highly accurate, and efficient spectral-grid method has been developed for solving boundary value problems of two-parameter singularly perturbed differential equations. The convergence and accuracy of the proposed spectral-grid method have been justified and verified using the method of test functions. Using the spectral-grid method, the two-parameter singularly perturbed boundary value problem has been approximated and reduced to an equivalent system of algebraic equations. Based on the numerical experiments, it has been demonstrated that the proposed spectral-grid method possesses universality, high accuracy, efficiency, and adaptability in solving boundary value problems for two-parameter singularly perturbed differential equations.

References

[1] Salih M.H., Duressa G.F., Debela H.G. *Numerical solution of singularly perturbed self-adjoint boundary value problem using Galerkin method // International Journal of Engineering Science and Technology.* – 2020. – Vol. 12, Issue 3. – P. 26-37. – doi: `http://dx.doi.org/10.4314/ijest.v12i3.3`.

[2] Kaushik A., Gupta A. *An adaptive mesh generation and higher-order difference approximation for the system of singularly perturbed reaction-diffusion problem // Partial Differential Equations in Applied Mathematics.* – 2024. – Art. no. 100750. – doi: `http://dx.doi.org/10.1016/j.padiff.2024.100750`.

[3] Balasubramani N., Prasad M.G., Natesan S. *Fractal quintic spline solutions for singularly perturbed reaction-diffusion boundary-value problems // Applied Numerical Mathematics.* – 2024. – doi: `http://dx.doi.org/10.1016/j.apnum.2024.04.015`.

[4] Liu Y., Cheng Y. *Local discontinuous Galerkin method for a singularly perturbed fourth-order problem of reaction-diffusion type // Journal of Computational and Applied Mathematics.* – 2024. – Art. no. 115641. – doi: `http://dx.doi.org/10.1016/j.jcam.2023.115641`.

[5] Barzekhar N., Barati A., Jalilian R. *Sinc approximation method for solving system of singularly perturbed parabolic convection-diffusion equations // Applied Numerical Mathematics.* – 2025. – doi: `http://dx.doi.org/10.1016/j.apnum.2025.05.005`.

[6] Debela H.G., Duressa G.F. *Accelerated exponentially fitted operator method for singularly perturbed problems with integral boundary condition // International Journal of Differential Equations.* – 2020. – P. 1-8. – doi: `http://dx.doi.org/10.1155/2020/9268181`.

[7] Kusi G.R., Habte A.H., Bullo T.A. *Layer resolving numerical scheme for singularly perturbed parabolic convection-diffusion problem with an interior layer // MethodsX.* – 2023. – Vol. 10. – Art. no. 101953. – doi: http://dx.doi.org/10.1016/j.mex.2022.101953.

[8] Roy N., Jha A. *A parameter-uniform method for two-parameter singularly perturbed boundary value problems with discontinuous data // MethodsX.* – 2023. – Art. no. 102004. – doi: http://dx.doi.org/10.1016/j.mex.2023.102004.

[9] Kadalbajoo M.K., Yadaw A.S. *B-spline collocation method for a two-parameter singularly perturbed convection-diffusion boundary value problems // Applied Mathematics and Computation.* – 2008. – Vol. 201, Issue 1-2. – P. 504-513. – doi: http://dx.doi.org/10.1016/j.amc.2007.12.038.

[10] Kadalbajoo M.K., Yadaw A.S. *Parameter-uniform Ritz-Galerkin finite element method for two-parameter singularly perturbed boundary value problems // International Journal of Pure and Applied Mathematics.* – 2009. – Vol. 55, Issue 2. – P. 287-300.

[11] Andisso F.S., Duressa G.F. *Graded mesh B-spline collocation method for two-parameter singularly perturbed problems // MethodsX.* – 2023. – Art. no. 102336. – doi: http://dx.doi.org/10.1016/j.mex.2023.102336.

[12] Normurodov C.B., Tursunova B.A. *Numerical modeling of the boundary value problem of an ordinary differential equation with a small parameter at the highest derivative by Chebyshev polynomials of the second kind // Results in Applied Mathematics.* – 2023. – Vol. 19. – Art. no. 100388. – doi: http://dx.doi.org/10.1016/j.rinam.2023.100388.

[13] Normurodov Ch.B., Abduraximov B.F., Djuraeva N.T. *On estimating the rate of convergence of the initial integration method // AIP Conference Proceedings.* – 2024. – Vol. 3244. – doi: http://dx.doi.org/10.1063/5.0242041.

[14] Normurodov C., Toyirov A., Ziyakulova S., Viswanathan K.K. *Convergence of spectral-grid method for Burgers equation with initial-boundary conditions // Mathematics and Statistics.* – 2024. – Vol. 12, Issue 2. – P. 115-125. – doi: http://dx.doi.org/10.13189/ms.2024.120201.

[15] Normurodov Ch.B., Dzhuraeva N.T., Normatova M.M. *A high-accuracy and efficient method for studying the dynamics of derivatives of different orders of a singularly perturbed equation // Chebyshev Collection.* – 2025. – Vol. 26, Issue 4. – P. 357-369. – doi: http://dx.doi.org/10.22405/2226-8383-2025-26-4-357-369.

[16] Normurodov Ch.B., Ziyakulova Sh.A., Murodov S.K. *On one highly accurate and efficient method for solving the biharmonic equation // International Journal of Applied Mathematics.* – 2025. – Vol. 38, Issue 4. – P. 437-453. – doi: http://dx.doi.org/10.12732/ijam.v38i4.1.

[17] Bouakkaz M., Arar N., Meflah M. *Enhanced numerical resolution of the Duffing and Van der Pol equations via the spectral homotopy analysis method employing Chebyshev polynomials of the first kind // Journal of Applied Mathematics and Computation.* – 2024. – doi: http://dx.doi.org/10.1007/s12190-024-02271-5.

[18] Normurodov Ch.B. *Mathematical modeling of hydrodynamic problems for two-phase plane-parallel flows // Mathematical Modeling.* – 2007. – Vol. 19, Issue 6. – P. 53-60.

[19] Normurodov Ch.B. *On an efficient method for solving the Orr-Sommerfeld equation // Mathematical Modeling.* – 2005. – Vol. 17, Issue 9. – P. 35-42.

[20] Normurodov Ch.B., Solovyev A.S. *The influence of weighted particles on the stability of plane Poiseuille flow // Fluid Mechanics and Gas Dynamics.* – 1986. – Issue 1. – P. 46-50.

[21] Normurodov Ch.B., Solovyev A.S. *Stability of a two-phase flow of gas-solid particles in the boundary layer // Mechanics of Fluids and Gases.* – 1987. – Issue 2. – P. 60-64.

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ЧИСЛЕННОЕ МОДЕЛИРОВАНИЕ КРАЕВОЙ ЗАДАЧИ ДЛЯ ДВУХПАРАМЕТРИЧЕСКОГО СИНГУЛЯРНО ВОЗМУЩЁННОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ С ИСПОЛЬЗОВАНИЕМ СПЕКТРАЛЬНО-СЕТОЧНОГО МЕТОДА

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Известно, что решения краевых задач, связанных с двухпараметрическими сингулярно возмущенными дифференциальными уравнениями, демонстрируют образование двух различных пограничных слоев, обычно возникающих вблизи концов области. Наличие этих узких областей, характеризующихся крутыми градиентами в решении, представляет собой значительную проблему для классических численных методов. В результате стандартные методы конечных разностей, конечных элементов или дискретизации низкого порядка часто не позволяют точно воспроизвести поведение слоев, если не используются чрезмерно мелкие сетки, что приводит к увеличению вычислительных затрат и снижению эффективности. Для решения этих проблем в данной работе предлагается эффективный спектрально -сеточный метод для численного решения краевых задач, включающих двухпараметрические сингулярные возмущения. Основная идея предлагаемого подхода заключается в сочетании высокой точности спектральных методов с тщательно разработанной сеткой, которая адаптируется к структуре пограничного слоя решения. Используя эту спектрально-сеточную структуру, исходная краевая задача преобразуется в эквивалентную систему алгебраических уравнений, которая может быть эффективно решена с помощью стандартных методов линейной алгебры. Для оценки эффективности предложенного метода были проведены обширные численные эксперименты. Полученные результаты систематически сравнивались с результатами, представленными в существующей литературе. Эти сравнения наглядно демонстрируют, что разработанный в данном исследовании спектрально-сеточный метод обеспечивает более высокую точность при относительно низких вычислительных затратах.

Ключевые слова: спектрально-сеточный метод, двухпараметрическое сингулярно возмущённое дифференциальное уравнение, полиномы Чебышёва, максимальная абсолютная ошибка, пограничные слои, численная аппроксимация, высокоточный алгоритм.

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