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AN OPTIMAL METHOD FOR THE APPROXIMATE SOLUTION OF THE HYPERSINGULAR INTEGRAL EQUATIONS

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Hypersingular integral equations are encountered in a number of fields, such as the dynamics of air and fluid flow, elasticity, and wave propagation theory. Analytical solutions for these integral equations exist, but the solutions themselves are also expressed through singular integrals. Therefore, it becomes necessary to develop formulas for the approximate calculation of these solutions, and such formulas have been developed. Among these formulas, very few are highly accurate. However, they are not optimal. Our work is devoted to constructing an optimal quadrature formula for the approximate calculation of the analytical solutions of a hypersingular integral equation.

Keywords: optimal quadrature formulas, extremal function, Sobolev space, optimal coefficients, Hadamard type singular integral.

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1 Introduction

Hypersingular integral equations are a type of integral equation where the kernel—the function inside the integral–is highly "singular,"meaning it has an infinitely large value at a certain point. This singularity is of a higher order than what's found in more common singular integral equations. These equations are a powerful tool for analyzing complex, three-dimensional problems in various fields, including:

Aerodynamics and Fluid Dynamics: Studying how air and fluids move around objects; Elasticity: Analyzing how materials deform under stress; Wave Theory: Understanding the diffraction of electromagnetic and acoustic waves; Ecology: Modeling certain environmental systems;

Often, hypersingular integral equations are derived from Neumann boundary value problems. These are mathematical problems for equations like the Laplace or Helmholtz equations, where you're given the values of the function's derivative on the boundary of a region, not the function's value itself. The process of converting these boundary value problems into integral equations involves using a concept called the double-layer potential.

We consider the following hypersingular integral equation first kind

$$\int_{-1}^{1} \frac{g(x)}{(x-t)^2} dx = \varphi(t), \tag{1}$$

here the functions $g(x)$ and $\varphi(x)$ satisfy the Hölder condition (or belongs to the class H), $-1 < t < 1$. We integrate the left side of the integral equation (1) by parts and obtain the following

$$\frac{g(-1)}{-1-t} + \frac{g(1)}{1-t} + \int_{-1}^{1} \frac{g'(x)}{(x-t)} dx = \varphi(t), \tag{2}$$

For integral equation (1) or (2) to have a unique solution in the given class, the value of the function g(x) at the boundaries of the segment must be equal to zero, i.e.:

$$g(1) = g(-1) = 0$$

or

$$\int_{-1}^{1} g'(x) dx = 0. \tag{3}$$

We introduce denotation $g'(x) = \rho(x)$, and get

$$\int_{-1}^{1} \frac{\rho(x)}{x-t} dx = \varphi(t), \quad -1 < t < 1. \tag{4}$$

The unique exact analytical solution of the integral equation (4) that satisfies condition (3) is equal to the following [13]

$$\rho(t) = -\frac{1}{\pi^2 \sqrt{1-t^2}} \int_{-1}^{1} \frac{\sqrt{1-x^2}\varphi(x)}{x-t} dx. \tag{5}$$

The function under this integral (5) is very complex in practical problems. As a result, it is impossible to find the antiderivative of this function. Therefore, to approximate such singular integral, scientists have used the following methods: Discrete vortex method [7, 12], Interpolation methods [2, 8, 9, 11], Piecewise interpolation method [3, 4, 6] and other numerical quadrature methods [5, 10]. The Discrete Vortex Method employs a singular point in the middle of successive nodes. Numerous scientific studies have focused on constructing quadrature formulas using Interpolation methods. These formulas use the roots of Chebyshev polynomials of the first kind as nodes. However, in several physical applications, singular integral equations need to be solved even when the function g is not very smooth or unknown. In such cases, Gaussian and transformation methods work efficiently only if the function g is smooth enough. The accuracy of numerical integration is significantly influenced by mesh selection. The Gaussian method's utility is constrained since it relies on approximating integrals at Gaussian points. Conversely, the Newton-Cotes method becomes impractical when the singular point is in close proximity to the quadrature formula nodes and even more so when it coincides with them. Scholars have underscored the paramount importance of mesh selection in guaranteeing the accuracy of numerical integration. During the initial phases, considerable emphasis was placed on mesh selection to position the singular point at the centre of a subinterval, highlighting its significant impact on the overall accuracy of the integration. But the methods mentioned above are not an optimal approximation technique. In the present paper, we construct an optimal quadrature formula in the space $L_2^{(1)}(-1, 1)$.

2 Statement of the problem

We consider the following quadrature formula

$$\int_{-1}^{1} \frac{\sqrt{1-x^2}\varphi(x)}{x-t} \mathtt{d}x \cong \sum_{\beta=0}^{N} C[\beta]\varphi(x_\beta), -1 < t < 1, \tag{6}$$

in the Sobolev space $L_2^{(1)}(-1,1)$. This space is a Hilbert space of classes of all real valued functions φ defined in the interval $[-1,1]$ that differ by a polynomial of degree first and square integrable with derivative of order two. Here $C[\beta]$ are the coefficients, x_β are the nodes of the quadrature formula, N is a natural number.

The following difference is called *the error* of quadrature formula (6):

$$(\ell,\varphi)=\int\limits_{-1}^{1}\frac{\sqrt{1-x^2}\varphi(x)}{x-t}\mathtt{d}x-\sum_{\beta=0}^{N}C[\beta]\varphi(x_\beta)=\int\limits_{-\infty}^{\infty}\ell(x)\varphi(x)\mathtt{d}x,$$

where

$$\ell(x)=\frac{\sqrt{1-x^2}\varepsilon_{[-1,1]}(x)}{x-t}-\sum_{\beta=0}^{N}C[\beta]\delta(x-x_\beta), \tag{7}$$

$\varepsilon_{[-1,1]}(x)$ is the indicator of the interval $[-1,1]$, δ is the Dirac delta-function, $\ell(x)$ is the error functional of quadrature formula (7).

Since the functional (7) defined in the space $L_2^{(1)}(-1,1)$ [14], then we have

$$(\ell,1)=0. \tag{8}$$

The main aim of the present paper is to construct optimal quadrature formulas in the sense of Sard of the form (1) in the space $L_2^{(1)}(-1,1)$ by the Sobolev method for approximate integration of the Cauchy type singular integral. This means to find the coefficients $C[\beta]$ which satisfy the following equality

$$\|\mathring{\ell}|L_2^{(1)*}\|:=\inf_{C[\beta]}\|\ell|L_2^{(1)*}\|. \tag{9}$$

Thus, in order to construct optimal quadrature formulas in the form (6) in the sense of Sard we have to consequently solve the following problems.

Problem 1. Find the norm of the error functional (7) of the quadrature formula (6) in the space $L_2^{(1)*}(-1,1)$.

Problem 2 Find the coefficients $C[\beta]$ which satisfy the equality (9).

In the works [1, 15, 16] for the norm of the error functional the following form was obtained

$$\begin{gathered}\|\ell|L_2^{(1)*}(-1,1)\|^2=\Bigg[\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N}C[\beta]C[\gamma]\frac{|x_\beta-x_\gamma|}{2}-\\ -2\sum_{\beta=0}^{N}C[\beta]\int\limits_{-1}^{1}\frac{\sqrt{1-x^2}|x-x_\beta|}{2(x-t)}dx+\int\limits_{-1}^{1}\int\limits_{-1}^{1}\frac{\sqrt{1-x^2}\sqrt{1-y^2}|x-y|}{2(x-t)(y-t)}dxdy\Bigg].\end{gathered} \tag{10}$$

Thus Problem 1 is solved for quadrature formulas of the form (6) in the space $L_2^{(1)}(-$ $-1,1)$.

3 The main results

Assume that the nodes x_β of the quadrature formula (6) are fixed. The error functional (7) satisfies conditions (8). The norm of the error functional $\ell(x)$ is a multidimensional function with respect to the coefficients $C[\beta]$ $(\beta=\overline{0,N})$. For finding the point of the conditional minimum of the expression (10), under the conditions (8), we apply the Lagrange method.

We denote $\mathbf{C}=(C[0],C[1],...,C[N])$ and λ. Consider the function

$$\Psi(\mathbf{C},\lambda)=\|\ell\|^2-2\lambda(\ell(x),1).$$

Equating to zero the partial derivatives of $\Psi(\mathbf{C},\lambda)$ by $C[\beta]$ $(\beta=\overline{0,N})$ and λ, we get the following system of linear equations

$$\sum_{\gamma=0}^{N}C[\gamma]\frac{|x_\beta-x_\gamma|}{2}+\lambda=f_1(x_\beta),\quad \beta=0,1,2,...,N, \tag{11}$$

$$\sum_{\gamma=0}^{N}C[\gamma]=g_0, \tag{12}$$

where

$$f_2(x_\beta)=\int_{-1}^{1}\frac{\sqrt{1-x^2}|x-x_\beta|}{2(x-t)}dx, \tag{13}$$

$$g_0=\int_{-1}^{1}\frac{\sqrt{1-x^2}}{x-t}dx, \tag{14}$$

and $C[\gamma]$, $\gamma=0,1,...,N$ and λ, are unknowns.

The system (11)–(12) has a unique solution and this solution gives the minimum to $\|\ell\|^2$ under the conditions (8). The uniqueness of the solution of such type of systems was discussed in [14, 17].

We give the algorithm for solution of system (11)-(12) when the nodes x_β are equally spaced, i.e., $x_\beta=h\beta-1$, $h=\frac{2}{N}$, $N\geqslant 0$. Here we use similar method suggested by S.L. Sobolev [14] for finding the coefficients of optimal quadrature formulas in the Sobolev space $L_2^{(1)}(-1,1)$.

Suppose that $C[\beta]=0$ when $\beta<0$ and $\beta>N$. Using the definition of convolution, we rewrite system (11)-(12) in the following form:

$$G_1(h\beta)*C[\beta]+\lambda=f_1(h\beta),\quad \beta=0,1,...,N, \tag{15}$$

$$\sum_{\beta=0}^{N}C[\beta]=g_0, \tag{16}$$

where

$$G_1(h\beta)=\frac{(h\beta)\mathtt{sgn}(h\beta)}{2},$$

λ is an arbitrary constant, $\mathtt{sgn}(h\beta)$ is the signum function.

Thus we have the following problem.

Problem 3. Find the discrete function $C[\beta]$ and polynomial λ of degree zero which satisfy the system (15)-(16).

Further we investigate Problem 3. Instead of $C[\beta]$ we introduce the following functions

$$v(h\beta)=G_1(h\beta)*C[\beta], \tag{17}$$

$$u(h\beta)=v\,(h\beta)+\lambda. \tag{18}$$

In such statement the coefficients $C[\beta]$ are expressed by the function $u(h\beta)$, i.e. taking into account

$$hD_1(h\beta) * u(h\beta) = \delta(h\beta),$$

where

$$D_1(h\beta) = \begin{cases} 0, & |\beta| \geqslant 2, \\ h^{-2}, & |\beta| = 1, \\ -2h^{-2}, & \beta = 0. \end{cases} \tag{19}$$

There are (19) and (18) , for the coefficients we have

$$C[\beta] = hD_1(h\beta) * u(h\beta). \tag{20}$$

Thus, if we find the function $u(h\beta)$, then the coefficients $C[\beta]$ will be found from equality (20). To calculate the convolution (20) it is required to find the representation of the function $u(h\beta)$ for all integer values of β. From equality (15) we get that $u(h\beta) = f_1(h\beta)$ when $h\beta - 1 \in [-1, 1]$, i.e. $\beta = 0, 1, ..., N$. Now we need to find the representation of the function $u(h\beta)$ when $\beta < 0$ and $\beta > N$.

Since $C[\beta] = 0$ when $h\beta \notin [-1, 1]$ then $C[\beta] = hD_1(h\beta) * u(h\beta) = 0, \;\; h\beta \notin [-1, 1]$.

Now we calculate the convolution $v(h\beta) = G_1(h\beta) * C[\beta]$ when $\beta \leqslant 0$ and $\beta \geqslant N$. Suppose $\beta \leqslant 0$, then taking into account that $G_1(h\beta) = \frac{|h\beta|}{2}$ and equality (16), we have

$$v(h\beta) = -\frac{1}{2}(h\beta)g_0 - p_0 + \lambda, \tag{21}$$

Similarly, in the case $\beta \geqslant N$ for the convolution $v(h\beta) = G_1(h\beta) * C[\beta]$ we obtain

$$v(h\beta) = \frac{1}{2}(h\beta)g_0 + p_0 + \lambda. \tag{22}$$

We denote

$$a_0^- = -p_0 + \lambda, \tag{23}$$

$$a_0^+ = p_0 + \lambda. \tag{24}$$

Taking into account (18), (21) and (22) we get the following problem

Problem 4. Find the solution of the equation

$$hD_1(h\beta) * u(h\beta) = 0, \qquad h\beta \notin [-1, 1], \tag{25}$$

having the form:

$$u(h\beta) = \begin{cases} -\frac{1}{2}(h\beta)g_0 + a_0^-, & \beta < 0, \\ f_1(h\beta), & 0 \leqslant \beta \leqslant N, \\ \frac{1}{2}(h\beta)g_0 + a_0^+, & \beta > N. \end{cases} \tag{26}$$

Here a_0^- and a_0^+ are unknown constant (or polynomials of degree zero) with respect to $h\beta$.

If we find a_0^- and a_0^+ then from (23), (24) we have

$$\lambda = \frac{1}{2}(a_0^- + a_0^+), \tag{27}$$

$$p_0 = \frac{1}{2}(a_0^+ - a_0^-). \tag{28}$$

Unknowns a_0^- and a_0^+ can be found from equation (25), using the function $D_1(h\beta)$ defined by (19). Then we obtain explicit form of the function $u(h\beta)$ and from (20) we find the coefficients $C[\beta]$. Furthermore from (27) we get λ.

Thus Problem 4 and respectively Problems 3 will be solved.

Then, using the above algorithm, we obtain explicit formulas for coefficients of the optimal quadrature formula (6). It should be noted that the quadrature formula (6) is exact for linear function.

The following holds

Theorem. Coefficients of the optimal quadrature formulas (6), with equally spaced nodes in the space $L_2^{(1)}(-1,1)$, have the following form

$$C[0] = h^{-1}\left[f_1(h) + \frac{\pi}{4}\left(2t^2+2t-1\right) - \frac{h}{2}\pi t\right],$$

$$C[\beta] = h^{-1}\left[f_1(h\beta-h) - 2f_1(h\beta) + f_1(h\beta+h)\right], \qquad \beta=\overline{1,N-1},$$

$$C[N] = h^{-1}\left[f_1(1-h) - \frac{\pi}{4}\left(2t^2-2t-1\right) - \frac{h}{2}\pi t\right],$$

where

$$f_1(h\beta) = \left(\frac{h\beta-1}{2} - t\right)\sqrt{1-(h\beta-1)^2} + \left(t^2 - t(h\beta-1) - \frac{1}{2}\right)\texttt{arcsin}(h\beta-1) +$$
$$+(t-(h\beta-1))\sqrt{1-t^2}ln\left|\frac{1-t(h\beta-1)+\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t}\right|,$$

Proof. From (26) with $\beta=0$ and $\beta=N$ we immediately obtain (26), i.e.

$$a_0^- = f_1(0),$$

$$a_0^+ = f(1) - g_0.$$

This means that we have obtained an explicit form of the function $u(h\beta)$.

Further, using (19) and (26), from (20) calculating the convolution $hD_1(h\beta) * u(h\beta)$ for $\beta=\overline{0,N}$, respectively, we obtain results of the theorem. Theorem 1 is proved. □

Remark 1. So, the approximate calculation of equality (5) is as follows

$$\rho(t) \cong -\frac{1}{\pi^2\sqrt{1-t^2}}\sum_{\beta=0}^{N} C[\beta]\varphi(x_\beta).$$

4 Conclusion

In the introduction of the article, several problems in various fields were listed. It was stated that the solutions to these problems lead to first-kind hypersingular integral equations. Since there are no exact analytical methods to solve these types of integral equations, approximate solution methods have been proposed. In this work, we have provided a new method to partially overcome the shortcomings of the proposed methods. That is, an optimal quadrature formula was constructed to approximate singular integrals with high accuracy, and its analytical form was found.

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ОПТИМАЛЬНЫЙ МЕТОД ПРИБЛИЖЁННОГО РЕШЕНИЯ ГИПЕРСИНГУЛЯРНЫХ ИНТЕГРАЛЬНЫХ УРАВНЕНИЙ

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Гиперсингулярные интегральные уравнения встречаются в ряде областей, таких как динамика воздушных и жидкостных потоков, теория упругости и распространения волн. Для этих интегральных уравнений существуют аналитические решения, но сами решения также выражаются через сингулярные интегралы. Поэтому возникает необходимость в разработке формул для приближённого вычисления этих решений, и такие формулы уже разработаны. Среди этих формул очень немногие обладают высокой точностью. Однако они не являются оптимальными. Наша работа посвящена построению оптимальной квадратурной формулы для приближённого вычисления аналитических решений гиперсингулярного интегрального уравнения.

Ключевые слова: оптимальные квадратурные формулы, экстремальная функция, пространство Соболева, оптимальные коэффициенты, сингулярный интеграл типа Адамара.

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