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REGULARIZATION OF THE INITIAL-BOUNDARY VALUE PROBLEM FOR A INHOMOGENEOUS PARABOLIC EQUATION WITH CHANGING TIME DIRECTION

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In this paper, the Cauchy problem with boundary conditions for an inhomogeneous parabolic equation with changing time direction is investigated. To construct the solution, the method of separation of variables (Fourier method) is employed, leading to a spectral problem where the eigenvalues and eigenfunctions are determined. An a priori estimate for the solution is obtained, theorems of uniqueness and conditional stability in the correctness set are proved. A regularized solution corresponding to the approximate initial data is constructed. The regularization parameter is chosen based on the efficiency estimate derived from the norm of the difference between the exact and the regularized solutions. Numerical experiments were performed for several cases of given data, and the results are presented in tables and graphs. The results show that the regularized solution corresponding to the approximate data closely regularized solution for the exact data, confirming the robustness of the method to errors in the input data.

Keywords: parabolic equation with changing time direction, ill-posed problem, priori estimate, uniqueness, conditionally stability, Lagrange multiplier method, regularization, approximate solution.

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1 Introduction

Initial-boundary value problems for parabolic partial differential equations play a key role in the mathematical modeling of diffusion processes, heat transfer, and many other time-dependent phenomena. In typical parabolic problems, the direction of time is consistent and forward, ensuring that the problem is well-posed in the Hadamard sense. However, initial-boundary value problems for equations that change direction with respect to time become significantly more difficult. Such problems are usually ill-posed, that is, the solution may not exist, may not be unique, or may not depend continuously on the initial data.

This study is aimed at regularizing the initial-boundary value problem for an inhomogeneous parabolic equation that changes direction with respect to time.

This work considers the theoretical formulation of the regularization of the problem. That is, the conditional stability estimate, uniqueness, and conditional stability of the solution are studied. Based on these, a regularized solution is constructed. The research is very important not only for the theoretical development of partial differential equations but also for practical problems in inverse heat transfer, diffusion processes, and other similar time-reversible or mixed-direction processes.

Let the function $u(x,t)$ satisfy the equation

$$u_t(x,t) + k(x)\,u_{xx}(x,t) = f(x,t), \tag{1}$$

in the region $\Omega = \{(x,t) : -l < x < l,\ x \neq 0,\ 0 < t < T\}$, where $k(x) = \begin{cases} a, & \text{if } x > 0, \\ -b, & \text{if } x < 0, \end{cases}$ a, b are given constants and $f(x,t)$ is a sufficiently smooth given function.

Problem statement. Find a function $u(x,t)$ satisfying equation (1) in the region Ω and the initial

$$u(x,0) = \varphi(x), \quad -l \leqslant x \leqslant l, \tag{2}$$

boundary

$$u(-l,t) = u(l,t) = 0, \quad 0 \leqslant t \leqslant T, \tag{3}$$

and gluing

$$u(-0,t) = u(+0,t), \quad bu_x(-0,t) = -au_x(+0,t), \quad 0 \leqslant t \leqslant T, \tag{4}$$

conditions, where $\varphi(x)$ is a sufficiently smooth given function.

Problems involving parabolic equations with changing type were first investigated in the works of M. Gevrey [1]. Later, equations of this type were also examined in the scientific works of M.S. Baouendi and P. Grisvard [2], C.D. Pagani and G. Talenti [3], and L. Cattabriga [4]. The research of S. A. Tersenov [5] considered model parabolic equations with changing direction, along with several other types of model equations. For inhomogeneous parabolic type equations with changing type, V.K. Romanko [6], V.N. Vragov and A.G. Podgaev [7], S.G. Pyatkov [8] and A.A. Kerefov [9] established fundamental theorems on existence, uniqueness, and stability for a wide class of such equations using operator theory and functional analysis. The conditional well-posedness of initial-boundary value problems and the development of regularized approximate solutions for such types of equations have been addressed in the works of K.S. Fayazov, I.O. Khajiev and Ya.K. Khudaybergenov [10]- [14].

2 Eigenvalue problem

We search the solution of the problem (1) -(4) with the Fourier method, also known as separation of variables. As a result, we obtain the following spectral problem. Determine the value of the parameter λ such that problem

$$\begin{cases} k(x)X'' - \lambda X = 0, \\ X(-l) = X(l) = 0, \\ X(-0) = X(+0), \\ bX'(-0) = -aX'(+0). \end{cases} \tag{5}$$

admits a nontrivial solution.

The positive eigenvalues λ_k^+ and the negative eigenvalues λ_k^- of spectral problem (5) are the solutions of the transcendental equations $\sqrt{a}tg\sqrt{\frac{\lambda_k^+}{b}}l - \sqrt{b}th\sqrt{\frac{\lambda_k^+}{a}}l = 0$ and $\sqrt{b}tg\sqrt{\frac{-\lambda_k^-}{a}}l - \sqrt{a}th\sqrt{\frac{-\lambda_k^-}{b}}l = 0$. The problem (5) has eigenfunctions for every eigenvalue

λ_k^+, λ_k^- and they are

$$X_k^+(x) = \begin{cases} \dfrac{1}{\sqrt{P(\lambda_k^+)}} \cdot \dfrac{\sin\left(\sqrt{\dfrac{\lambda_k^+}{b}}(l+x)\right)}{\cos\left(\sqrt{\dfrac{\lambda_k^+}{b}}\, l\right)}, & -l \leqslant x < 0, \\ \dfrac{\sqrt{b}}{\sqrt{aP(\lambda_k^+)}} \cdot \dfrac{\sinh\left(\sqrt{\dfrac{\lambda_k^+}{a}}(l-x)\right)}{\cosh\left(\sqrt{\dfrac{\lambda_k^+}{a}}\, l\right)}, & 0 < x \leqslant l, \end{cases}$$

$$X_k^-(x) = \begin{cases} -\dfrac{\sqrt{a}}{\sqrt{bP(\lambda_k^-)}} \cdot \dfrac{\sinh\left(\sqrt{\dfrac{-\lambda_k^-}{b}}(l+x)\right)}{\cosh\left(\sqrt{\dfrac{-\lambda_k^-}{b}}\, l\right)}, & -\ell \leqslant x < 0, \\ -\dfrac{1}{\sqrt{P(\lambda_k^-)}} \cdot \dfrac{\sin\left(\sqrt{\dfrac{-\lambda_k^-}{a}}(l-x)\right)}{\cos\left(\sqrt{\dfrac{-\lambda_k^-}{a}}\, l\right)}, & 0 < x \leqslant \ell, \end{cases}$$

respectively, where $P\left(\lambda_k^+\right) = \frac{\ell}{2}\left(1 - \frac{b}{a} + \tan^2\left(\sqrt{\frac{\lambda_k^+}{b}}\,\ell\right) + \frac{b}{a}\tanh^2\left(\sqrt{\frac{\lambda_k^+}{a}}\,\ell\right)\right)$, $P\left(\lambda_k^-\right) =$ $= \frac{\ell}{2}\left(1 - \frac{a}{b} + \tan^2\left(\sqrt{\frac{-\lambda_k^-}{a}}\,\ell\right) + \frac{a}{b}\tanh^2\left(\sqrt{\frac{-\lambda_k^-}{b}}\,\ell\right)\right)$. The eigenvalues λ_k^+, $-\lambda_k^-$ constitute increasing sequences, which can be seen from the following graphs and tables.

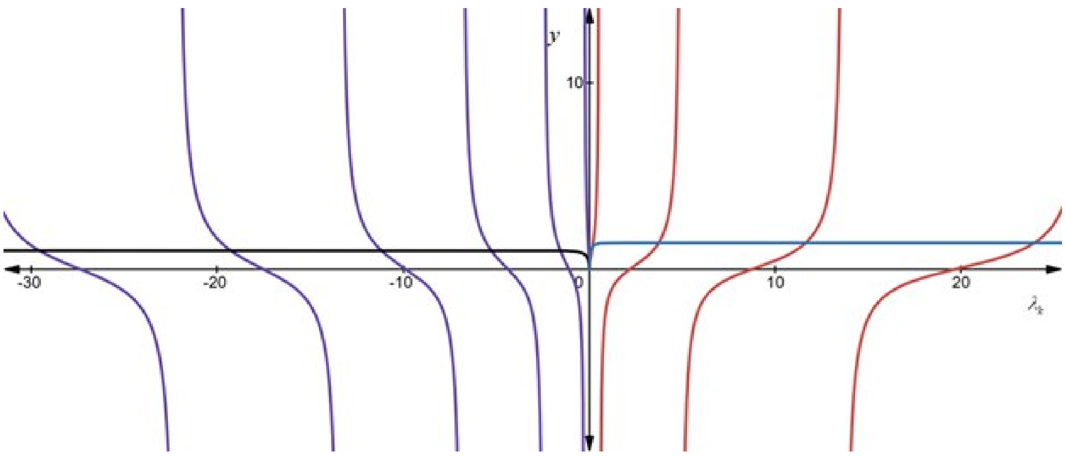

Figure 1 $a = 1$, $b = 2$, $l = 3$

Table 1 $a = 1,\ b = 2,\ l = 3$

λ_k^-	-1.5645	-5.2878	-11.2	-19.3068	-29.606	-420.686
y	0.9901	0.9998	1	1	1	1

Table 2 $a = 1,\ b = 2,\ l = 3$

λ_k^+	0.1695	3.7299	11.6435	23.9436	40.6302	61.7033
y	1.1937	1.4141	1.4142	1.4142	1.4142	1.4142

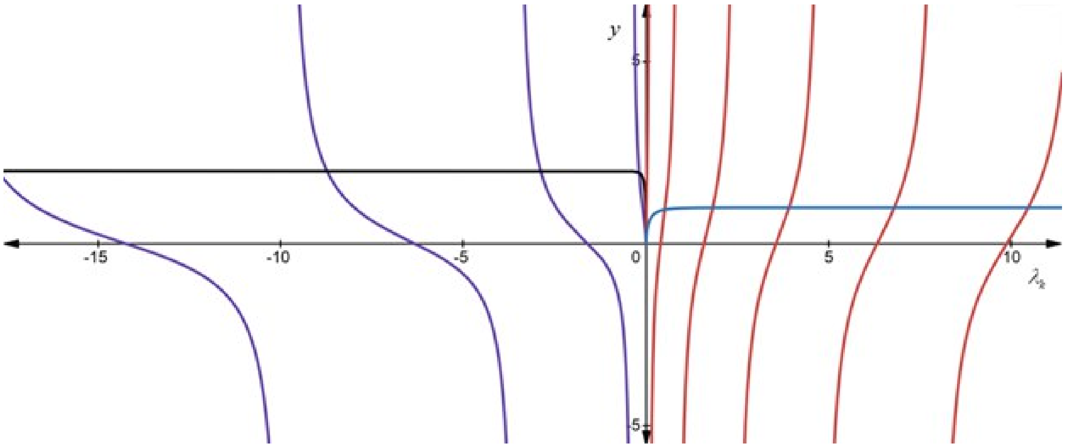

Figure 2 $a = 4,\ b = 1,\ l = 5$

Table 3 $a = 4,\ b = 1,\ l = 5$

λ_k^-	-0.1925	-2.8882	-8.7387	-17.7474	-29.9144	-45.2396
y	1.9509	2	2	2	2	2

Table 4 $a = 4,\ b = 1,\ l = 5$

λ_k^+	0.5136	1.8202	6.7912	10.4608	14.9199	20.1687
y	0.9459	0.9976	1	1	1	1

In Tables 1–4 present the values of the eigenvalues λ_k^- and λ_k^+ obtained by approximately solving the transcendental equations using Newton's method. Graphical solution of a transcendental equation involves plotting both sides of the equation as separate functions and finding the points where their graphs intersect. The intersection points in Figures 1 and Figure 2 represent the solutions to the equation.

Let the dot product in space $L_2(-l,l)$ be $(u,v) = \int\limits_{-l}^{l} uvdx$, and the norm be $\|u\|^2 = (u,u)$. According to the Hilbert–Schmidt theorem, the system of eigenfunctions in space $L_2(-l,l)$ is complete and orthogonal. That is, eigenfunctions have normalization properties.

$$\left(X_k^+, X_j^-\right) = 0, \quad \left(X_k^\pm, \quad X_j^\pm\right) = \delta_{kj}, \quad \forall k, j, n \in N,$$

where δ_{kj} is the Kronecker delta.

Lemma 2.1. Suppose that the function $u(x,t)$ satisfies the equation

$$u_t\left(x,t\right)+k\left(x\right)u_{xx}\left(x,t\right)=0$$

and the conditions $u\left(-l,t\right)=u\left(l,t\right)=0$, $u\left(-0,t\right)=u\left(+0,t\right)$, $bu_x\left(-0,t\right)=-au_x\left(+0,t\right)$, then the estimate

$$\|u\left(x,t\right)\|\leqslant\sqrt{2}\|u\left(x,0\right)\|^{1-\frac{t}{T}}\|u\left(x,T\right)\|^{\frac{t}{T}}.$$

is valid.

Proof. According to the properties of the eigenfunctions $X_k^+\left(x\right)$, $X_k^-\left(x\right)$, $(i,j=1,2,...)$ of the spectral problem (5), the function $u\left(x,t\right)$ satisfying the conditions of Lemma 1 and the solution of equation $u_t\left(x,t\right)+k\left(x\right)u_{xx}\left(x,t\right)=0$ can be written in the form

$$u\left(x,t\right)=\sum_{i=1}^{\infty}u_k^+\left(t\right)X_k^+\left(x\right)+\sum_{i=1}^{\infty}u_k^-\left(t\right)X_k^-\left(x\right),$$

where the functions $u_k^+\left(t\right)$, $u_k^-\left(t\right)$, $\quad(i,j=1,2,...)$ are the solutions of problems

$$\begin{cases}\left\{u_k^+\left(t\right)\right\}_t+\lambda_k^+u_k^+\left(t\right)=0,\\ u_k^+(0)=\varphi_k^+,\end{cases}\qquad\begin{cases}\left\{u_k^-\left(t\right)\right\}_t+\lambda_k^-u_k^-\left(t\right)=0,\\ u_k^-(0)=\varphi_k^-.\end{cases}$$

We know that for each function $u_k^+\left(t\right)$, $u_k^-\left(t\right)$, $(i,j=1,2,...)$, the inequalities

$$\left\{u_k^+\left(t\right)\right\}^2\leqslant\left(\left\{u_k^+\left(0\right)\right\}^2\right)^{1-\frac{t}{T}}\left(\left\{u_k^+\left(T\right)\right\}^2\right)^{\frac{t}{T}},$$
$$\left\{u_k^-\left(t\right)\right\}^2\leqslant\left(\left\{u_k^-\left(0\right)\right\}^2\right)^{1-\frac{t}{T}}\left(\left\{u_k^-\left(T\right)\right\}^2\right)^{\frac{t}{T}}.$$

hold, respectively [15]. According to the sum of these inequalities and Holder's inequality, we get

$$\sum_{i=1}^{\infty}\left\{u_i^+\right\}^2+\sum_{i=1}^{\infty}\left\{u_i^-\right\}^2\leqslant2\left(\sum_{i=1}^{\infty}\left\{u_i^+\left(0\right)\right\}^2+\sum_{i=1}^{\infty}\left\{u_i^-\left(0\right)\right\}^2\right)^{1-\frac{t}{T}}\times$$
$$\times\left(\sum_{i=1}^{\infty}\left\{u_i^+\left(T\right)\right\}^2+\sum_{i=1}^{\infty}\left\{u_i^-\left(T\right)\right\}^2\right)^{\frac{t}{T}},$$

or

$$\|u\left(x,t\right)\|\leqslant\sqrt{2}(\|u\left(x,0\right)\|)^{1-\frac{t}{T}}(\|u\left(x,T\right)\|)^{\frac{t}{T}}.$$

Lemma 2.2. For the solution $u(x,t)$ of problem (1)–(4), the following estimate

$$\|u\left(x,t\right)\|\leqslant\sqrt{2}(\|\varphi\left(x\right)\|+\alpha)^{\left(1-\frac{t}{T}\right)}(\|u\left(x,T\right)\|+\alpha)^{\frac{t}{T}}+\alpha.$$

holds, where $\alpha=\left(T\int\limits_0^T\|f\left(x,t\right)\|^2dt\right)^{1/2}$.

Proof. The solution of equation (1) can be represented in the form $u\left(x,t\right)=\bar{u}\left(x,t\right)+\tilde{u}\left(x,t\right)$, where $\bar{u}\left(x,t\right)$ is solution of the equation

$$\bar{u}_t\left(x,t\right)-sgnx\bar{u}_{xx}\left(x,t\right)=0$$

and $\tilde{u}(x,t)$ is solution of the equation

$$\tilde{u}_t(x,t) - sgnx\tilde{u}_{xx}(x,t) = f(x,t),$$

where the functions $\bar{u}(x,t)$, $\tilde{u}(x,t)$ satisfy the boundary conditions

$$\bar{u}(-l,t) = \bar{u}(l,t) = 0, \tilde{u}(-l,t) = \tilde{u}(l,t) = 0$$

and the gluing conditions

$$\bar{u}(-0,t) = \bar{u}(+0,t), \quad b\bar{u}(-0,t) = -a\bar{u}(+0,t),$$

$$\tilde{u}(-0,t) = \tilde{u}(+0,t), \quad b\tilde{u}(-0,t) = -a\tilde{u}(+0,t).$$

Let the solutions $\bar{u}(x,t)$, $\tilde{u}(x,t)$ of the equation (1) exist. Then they can be described in the forms

$$\bar{u}(x,t) = \sum_{k=1}^{\infty} \tilde{u}_k^+(t) X_k^+(x) + \sum_{k=1}^{\infty} \tilde{u}_k^-(t) X_k^-(x),$$

$$\tilde{u}(x,t) = \sum_{k=1}^{\infty} \tilde{u}_k^+(t) X_k^+(x) + \sum_{k=1}^{\infty} \tilde{u}_k^-(t) X_k^-(x).$$

Here, for each $k = 1, 2, ...$, $\tilde{u}_k^{\pm}(t)$ and $\bar{u}_k^{\pm}(t)$ satisfy the following problems, respectively:

$$\begin{cases} \left\{\tilde{u}_k^+(t)\right\}_t + \lambda_k^+ \tilde{u}_k^+(t) = f_k^+, \\ \tilde{u}_k^+(0) = 0, \end{cases} \qquad \begin{cases} \left\{\tilde{u}_k^-(t)\right\}_t + \lambda_k^- \tilde{u}_k^-(t) = f_k^-, \\ \tilde{u}_k^-(0) = -\int\limits_0^T e^{\lambda_k^- \tau} f_k^-(\tau)\, d\tau, \end{cases}$$

$$\begin{cases} \left\{\bar{u}_k^+(t)\right\}_t + \lambda_k^+ \bar{u}_k^+(t) = 0, \\ \bar{u}_k^+(0) = \varphi_k^-, \end{cases} \qquad \begin{cases} \left\{\bar{u}_k^-(t)\right\}_t + \lambda_k^- \bar{u}_k^-(t) = 0, \\ \tilde{u}_k^-(0) = \varphi_k^- + \int\limits_0^T e^{\lambda_k^- \tau} f_k^-(\tau)\, d\tau, \end{cases}$$

where $\varphi_k^{\pm} = \int\limits_{-l}^{l} \varphi(x) X_k^{\pm}(x)\, dx$, $\quad f_k^{\pm} = \int\limits_{-l}^{l} f(x,t) X_k^{\pm}(x) dx$. As shown in [10],

$$\tilde{u}_k^+(t) = \int\limits_0^t e^{\lambda_k^+(\tau - t)} f_k^+(\tau)\, d\tau, \quad \tilde{u}_k^-(t) = -\int\limits_t^T e^{\lambda_k^-(\tau - t)} f_k^-(\tau)\, d\tau.$$

Consequently,

$$\|\tilde{u}(x,t)\|^2 = \sum_{k=1}^{\infty} \left(\int\limits_0^t e^{\lambda_k^+(\tau - t)} f_k^+(\tau)\, d\tau \right)^2 + \sum_{k=1}^{\infty} \left(\int\limits_t^T e^{\lambda_k^-(\tau - t)} f_k^-(\tau)\, d\tau \right)^2 \leqslant$$

$$\leqslant \sum_{k=1}^{\infty} t \int\limits_0^t \left\{f_k^+(\tau)\right\}^2 d\tau + \sum_{k=1}^{\infty} (T - t) \int\limits_t^T \left\{f_k^-(\tau)\right\}^2 d\tau \leqslant T \int\limits_0^T \|f(x,t)\|^2 dt. \tag{6}$$

According to the result of Lemma 1, the estimate

$$\|\bar{u}(x,t)\| \leqslant \sqrt{2} \|\bar{u}(x,0)\|^{1-\frac{t}{T}} \|\bar{u}(x,T)\|^{\frac{t}{T}},$$

holds for the function $\bar{u}(x,t)$, and taking into account the equality $\bar{u}(x,t) = u(x,t) - \tilde{u}(x,t)$, the required inequality

$$\|u(x,t)\| \leqslant \sqrt{2}(\|u(x,0)\| + \alpha)^{\left(1-\frac{t}{T}\right)}(\|u(x,T)\| + \alpha)^{\frac{t}{T}} + \alpha, \tag{7}$$

follows, where $\alpha = \left(T \int\limits_0^T \|f(x,t)\|^2 dt\right)^{1/2}$.

Thus, we define the set of correctness M as follows:

$$M = \{u(x,t) : \|u(x,T)\| \leqslant m,\, m < \infty\}.$$

3 Uniqueness and conditional stability theorems

Theorem 3.1. Let the solution of the problem (1)–(4) exists in the set M, then it is unique.

Proof. Assume that the problem (1)-(4) has two solutions $u_1(x,t)$ and $u_2(x,t)$. Let $u(x,t) = u_1(x,t) - u_2(x,t)$ denote their difference. Then the function $u(x,t)$ satisfies equation

$$u_t(x,t) + k(x)\, u_{xx}(x,t) = 0,$$

along with the conditions $u(x,0) = 0$, and boundary (3) and gluing conditions (4). It follows that $\alpha = 0$, and by inequality (7), we get $\|u(x,t)\| \leqslant 0$ or $u(x,t) \equiv 0$. This implies $u_1(x,t) \equiv u_2(x,t)$. The uniqueness of the solution has been proven.

Theorem 3.2. Suppose that the solution of the problem (1)–(4) exists, and let $u(x,t) \in M$, $\|\varphi(x) - \varphi_\varepsilon(x)\| \leqslant \varepsilon$, $\|f(x,t) - f_\varepsilon(x,t)\| \leqslant \varepsilon$. Then the estimate

$$\|u(x,t) - u_\varepsilon(x,t)\| \leqslant \sqrt{2}(\varepsilon\,(T+1))^{\left(1-\frac{t}{T}\right)}(2m + T\varepsilon)^{\frac{t}{T}} + T\varepsilon,$$

holds.

Proof. Let $U(x,t) = u(x,t) - u_\varepsilon(x,t)$ be a function that satisfies equation

$$U_t(x,t) + k(x)\, U_{xx}(x,t) = f(x,t) - f_\varepsilon(x,t),$$

in region $\Omega = \{(x,t) : -l < x < l,\, x \neq 0,\; 0 < t < T\}$ and fulfills the following conditions: the initial condition

$$U(x,0) = \varphi(x) - \varphi_\varepsilon(x), \quad -l \leqslant x \leqslant l,$$

the boundary conditions

$$U(-l,t) = U(l,t) = 0, \quad 0 \leqslant t \leqslant T,$$

and the gluing conditions

$$U(-0,t) = U(+0,t), \quad bU_x(-0,t) = -aU_x(+0,t), \quad 0 \leqslant t \leqslant T.$$

According to the conditions of the theorem we have

$$\|U(x,0)\| = \|\varphi(x) - \varphi_\varepsilon(x)\| \leqslant \varepsilon, \quad \|U(x,T)\| \leqslant \|u(x,T)\| + \|u_\varepsilon(x,T)\| \leqslant 2m,$$

$$\alpha = \left(T \int\limits_0^T \|f(x,\tau) - f_\varepsilon(x,\tau)\|^2 d\tau\right)^{1/2} \leqslant \varepsilon T.$$

From estimate (7), we obtain

$$\|U(x,t)\| \leqslant \sqrt{2}(\varepsilon(T+1))^{\left(1-\frac{t}{T}\right)}(2m+\varepsilon T)^{\frac{t}{T}}+\varepsilon T.$$

The theorem is proved.

4 Approximate solution

Let us consider constructing the regularized approximate solution of the given problem (1)–(4). Suppose that the solution of the problem (1)–(4) exists and $u(x,t) \in M$. Then the solution can be represented in the form

$$u\left(x,t\right)=\sum_{k=1}^{\infty} u_k^+\left(t\right)X_k^+\left(x\right)+\sum_{k=1}^{\infty} u_k^-\left(t\right)X_k^-\left(x\right), \tag{8}$$

where $u_k^+\left(t\right)=e^{-\lambda_k^+ t}\varphi_k^+ +\int\limits_0^t f_k^+\left(\tau\right)e^{\lambda_k^+(\tau-t)}d\tau,\ u_k^-\left(t\right)=e^{-\lambda_k^- t}\varphi_k^- +\int\limits_0^t f_k^-\left(\tau\right)e^{\lambda_k^-(\tau-t)}d\tau.$

We denote by

$$u^N(x,t)=\sum_{k=1}^{\infty} u_k^+\left(t\right)X_k^+\left(x\right)+\sum_{k=1}^{N} u_k^-\left(t\right)X_k^-\left(x\right), \tag{9}$$

the regularized solution corresponding to the exact data, and represent the regularized solution corresponding to the approximate data in the form

$$u_\varepsilon^N(x,t)=\sum_{k=1}^{\infty} u_{\varepsilon k}^+\left(t\right)X_k^+\left(x\right)+\sum_{k=1}^{N} u_{\varepsilon k}^-\left(t\right)X_k^-\left(x\right), \tag{10}$$

where N is the regularization parameter.

The inequality

$$\left\|u-u_\varepsilon^N\right\| \leqslant \left\|u-u^N\right\|+\left\|u^N-u_\varepsilon^N\right\|, \tag{11}$$

holds for the norm of the difference between the exact solution (8) and the regularized approximate solution (10) corresponding to the approximate data.

Now, we analyze the regularized solution of the problem (1)–(4) in two cases: 1) $f\left(x,t\right)=0$, 2) $\varphi\left(x\right)=0$.

Case 1: In this case, the second norm on the right-hand side of inequality (11) is estimated as follows:

$$\left\|u^N-u_\varepsilon^N\right\|^2=\sum_{k=1}^{\infty} e^{-2\lambda_k^+ t}\left(\varphi_k^+-\varphi_{k\varepsilon}^+\right)^2+\sum_{k=1}^{N} e^{-2\lambda_k^- t}\left(\varphi_k^--\varphi_{k\varepsilon}^-\right)^2 \leqslant$$

$$\leqslant e^{-2\lambda_1^+ t}\sum_{k=1}^{\infty}\left(\varphi_k^+-\varphi_{k\varepsilon}^+\right)^2+e^{-2\lambda_N^- t}\sum_{k=1}^{\infty}\left(\varphi_k^--\varphi_{k\varepsilon}^-\right)^2 \leqslant \left(e^{-2\lambda_1^+ t}+e^{-2\lambda_N^- t}\right)\varepsilon^2. \tag{12}$$

The norm $\left\|u-u^N\right\|$ is expressed as follows:

$$\left\|u-u^N\right\|^2=\sum_{k=N+1}^{\infty} e^{-2\lambda_k^- t}\left\{\varphi_k^-\right\}^2. \tag{13}$$

Now, under condition $\|u(x,T)\| \leqslant m$, we determine the greatest value of the expression on the right side of equality (13) using the method of Lagrange multipliers.

It is known that, from condition $\|u(x,T)\| \leqslant m$:

$$\sum_{k=1}^{\infty} e^{-2\lambda_k^- T}\left\{\varphi_k^-\right\}^2 \leqslant m^2. \tag{14}$$

We construct the Lagrange function $F\left(\varphi_k^-, \xi\right)$ under condition (14) as follows:

$$F\left(\varphi_k^-, \xi\right) = \sum_{k=N+1}^{\infty} e^{-2\lambda_k^- t}\left\{\varphi_k^-\right\}^2 + \xi\left(\sum_{k=1}^{\infty} e^{-2\lambda_k^- T}\left\{\varphi_k^-\right\}^2 - m^2\right),$$

where ξ is the Lagrange parameter. Now, using the following partial derivatives of the function:

$$\frac{\partial F}{\partial \varphi_k^-} = 0, \ \frac{\partial F}{\partial \xi} = 0, \ k = 1, 2, ...$$

and the given constraints, the values of φ_k^- are determined in the following form:

$$\varphi_k^- = \begin{cases} m e^{\lambda_{N+1}^- T}, & k = N+1, \\ 0, & k \neq N+1. \end{cases}$$

By placing this result into (13), we obtain

$$\left\|u - u^N\right\|^2 \leqslant m^2 e^{2\lambda_{N+1}^-(T-t)}. \tag{15}$$

Using the estimates from (12) and (15) in inequality (11), we obtain the overall estimate

$$\left\|u - u_\varepsilon^N\right\| \leqslant m e^{\lambda_{N+1}^-(T-t)} + \varepsilon\left(e^{-2\lambda_1^+ t} + e^{-2\lambda_N^- t}\right)^{1/2}. \tag{16}$$

Case 2: In this case, the second norm on the right-hand side of inequality (11) is estimated as follows:

$$\left\|u^N - u_\varepsilon^N\right\|^2 = \sum_{k=1}^{\infty}\left(\int_0^t \left(f_k^+(\tau) - f_{k\varepsilon}^+(\tau)\right) e^{\lambda_k^+(\tau-t)} d\tau\right)^2 +$$

$$+\sum_{k=1}^{N}\left(\int_0^t \left(f_k^-(\tau) - f_{k\varepsilon}^-(\tau)\right) e^{\lambda_k^-(\tau-t)} d\tau\right)^2 \leqslant$$

$$\leqslant t\sum_{k=1}^{\infty}\int_0^t \left(f_k^+(\tau) - f_{k\varepsilon}^+(\tau)\right)^2 d\tau + \frac{1}{2\lambda_1^-}\left(1 - e^{-2\lambda_N^- t}\right)\sum_{k=1}^{N}\int_0^t \left(f_k^-(\tau) - f_{k\varepsilon}^-(\tau)\right)^2 d\tau \leqslant$$

$$\leqslant \varepsilon^2\left(t + \frac{1}{2\lambda_1^-}\left(1 - e^{-2\lambda_N^- t}\right)\right). \tag{17}$$

Now, under condition $\|u(x, y, T)\| \leqslant m$, and using the method of Lagrange multipliers for constrained extremum, the norm $\|u - u^N\|$ is estimated as follows:

$$\|u - u^N\|^2 = \sum_{k=N+1}^{\infty} \left(\int_0^T f_k^-(\tau) e^{\lambda_k^-(\tau - t)} d\tau - \int_t^T f_k^-(\tau) e^{\lambda_k^-(\tau - t)} d\tau \right)^2 \leqslant$$

$$\leqslant 2 \sum_{k=N+1}^{\infty} \left(e^{-\lambda_k^- t} \int_0^T f_k^-(\tau) e^{\lambda_k^- \tau} d\tau \right)^2 + 2 \sum_{k=N+1}^{\infty} \left(\int_t^T f_k^-(\tau) e^{\lambda_k^-(\tau - t)} d\tau \right)^2 \leqslant$$

$$\leqslant 2 \sum_{k=N+1}^{\infty} e^{-2\lambda_k^- t} \left(\int_0^T f_k^-(\tau) e^{\lambda_k^- \tau} d\tau \right)^2 + 2(T - t) \sum_{k=N+1}^{\infty} \int_t^T \left\{ f_k^-(\tau) \right\}^2 d\tau \leqslant$$

$$\leqslant 2m^2 e^{2\lambda_{N+1}^-(T-t)} + 2(T - t) \frac{1}{N^2}. \tag{18}$$

By inserting the obtained estimates (17) and (18) into inequality (11), we obtain the overall estimate

$$\|u - u_\varepsilon^N\| \leqslant \left(2m^2 e^{2\lambda_{N+1}^-(T-t)} + 2(T - t) \frac{1}{N^2} \right)^{1/2} + \varepsilon \left(t + \frac{1}{2\lambda_1^-} \left(1 - e^{-2\lambda_N^- t} \right) \right)^{1/2}. \tag{19}$$

By minimizing the right-hand side of inequalities (16) and (19), the regularization parameter N is determined corresponding to each of m, T, a, b and l.

5 Numerical results

Thus, we consider several cases as follows:

1. We present the values of the right-hand side of inequality (15) for $a = 1$, $b = 3$, $l = 3$, $T = 0,5$, $m = 10000$, $\varepsilon = 0.001$, $\Delta t = 0.05$ in the form of a table.

Table 5

t	0,05	0,1	0,15	0,2	0,25	0,3	0,35	0,4	0,45
N	3	3	3	3	2	2	2	2	1

In Table 5, by minimizing the right-hand side of estimate (15), the regularization parameter N is determined for each t_k. By selecting the step size $\Delta x = 0.6$ and the initial data $\varphi(x) = (x + l)(l - x)$, we observe the numerical results of the approximate solution relative to the initial data in Table 6 and the graphical results in Figure 3.

Table 7 and Figure 4 present the numerical and graphical representation of the approximate solution corresponding to the approximate input data.

Table 6 *Approximate solution $u^N(x,t)$ with respect to the exact initial data*

$x\backslash t$	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
-3	0	0	0	0	0	0	0	0	0	0	0
-2.4	-35.4624	-1.6087	-1.7678	-1.1034	0.0768	1.3019	2.3571	3.1922	3.8239	4.2888	4.6243
-1.8	-53.3956	-4.0518	-2.5280	0.4616	3.1719	5.2916	6.8758	8.0347	8.8696	9.4633	9.8808
-1.2	-46.7913	-5.1050	3.7880	8.9416	11.7766	13.4298	14.4428	15.0816	15.4881	15.7465	15.9123
-0.6	-22.9065	18.0774	24.7462	25.3678	24.9479	24.3392	23.7496	23.2354	22.8105	22.4782	22.2413
0	812.1108	65.7200	48.5696	41.0673	36.6865	33.7930	31.7636	30.3071	29.2715	28.5742	28.1733
0.6	28.7966	24.5608	22.3084	20.7339	19.4932	18.4334	17.4660	16.5286	15.5696	14.5395	13.3860
1.2	13.9823	13.1664	12.4993	11.9015	11.3298	10.7532	10.1440	9.4745	8.7138	7.8253	6.7641
1.8	7.9198	7.7036	7.5266	7.3835	7.2731	7.1972	7.1605	7.1706	7.2383	7.3785	7.6108
2.4	3.8272	3.8224	3.8507	3.9189	4.0364	4.2162	4.4754	4.8366	5.3292	5.9915	6.8731
3	0	0	0	0	0	0	0	0	0	0	0

Table 7 *Approximate solution of $u_\varepsilon^N(x,t)$ with respect to the approximate initial data*

$x\backslash t$	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
-3	0	0	0	0	0	0	0	0	0	0	0
-2.4	-35.4663	-1.6089	-1.7681	-1.1035	0.0768	1.3020	2.3573	3.1925	3.8243	4.2892	4.6248
-1.8	-53.4016	-4.0523	-2.5283	0.4616	3.1722	5.2921	6.8766	8.0355	8.8705	9.4643	9.8818
-1.2	-46.7966	-5.1056	3.7884	8.9426	11.7779	13.4312	14.4444	15.0832	15.4897	15.7482	15.9139
-0.6	-22.9091	18.0794	24.7489	25.3705	24.9506	24.3418	23.7521	23.2378	22.8129	22.4805	22.2435
0	812.2010	65.7272	48.5749	41.0717	36.6904	33.7966	31.7669	30.3102	29.2744	28.5769	28.1759
0.6	28.7999	24.5637	22.3110	20.7363	19.4956	18.4357	17.4682	16.5308	15.5718	14.5417	13.3883
1.2	13.9840	13.1680	12.5008	11.9030	11.3313	10.7547	10.1455	9.4760	8.7153	7.8269	6.7658
1.8	7.9207	7.7044	7.5275	7.3844	7.2739	7.1980	7.1613	7.1714	7.2390	7.3791	7.6114
2.4	3.8276	3.8227	3.8511	3.9192	4.0367	4.2164	4.4756	4.8368	5.3293	5.9915	6.8730
3	0	0	0	0	0	0	0	0	0	0	0

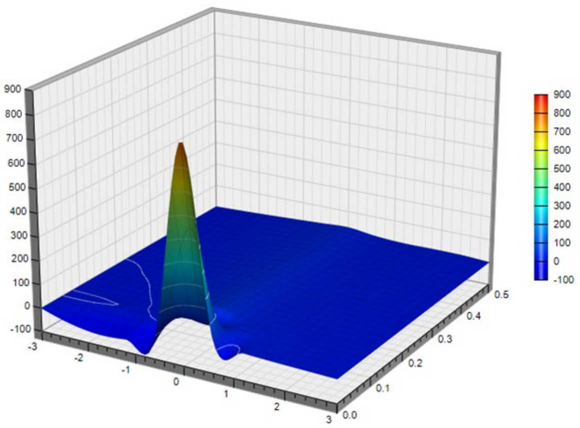

Figure 3 Graph of the approximate solution $u^N(x,t)$ with respect to the exact initial data

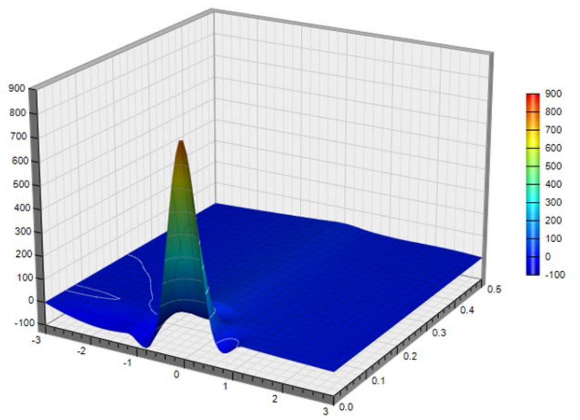

Figure 4 Graph of the approximate solution of $u_\varepsilon^N(x,t)$ with respect to the approximate initial data

2. The values of the right-hand side of inequality (15) corresponding to $a = 4$, $b = 1$, $l = 5$, $T = 1$, $m = 1000$, $\varepsilon = 0.00001$, $\Delta t = 0.2$ are presented in a table.

Table 8

t	0,2	0,4	0,6	0,8
N	3	3	2	2

Table 8 presents the regularization parameter corresponding to each t_k, as in the previous case. By choosing the step size $\Delta x = 1$ and the initial data $\varphi(x) =$

$= \sin(x+l)(l-x)$ in an appropriate form, we obtain the numerical results of the approximate solution relative to the initial data in Table 9 and the graphical representation in Figure 5.
Table 10 and Figure 6 present the numerical and graphical results of the approximate solution corresponding to the approximate input data.

Table 9 *Approximate solution $u^N(x,t)$ with respect to the exact initial data*

$x\backslash t$	0	0.2	0.4	0.6	0.8	1.0
-5	0	0	0	0	0	0
-4	-1.9558	-1.9962	-2.1009	-2.2302	-2.4025	-2.6498
-3	-4.3615	-4.5055	-4.8306	-5.2904	-5.9867	-7.0932
-2	-8.0139	-8.5835	-9.7598	-11.6609	-14.8029	-20.1374
-1	-15.4133	-18.2388	-23.6796	-32.8269	-48.6636	-76.4462
0	-43.0649	-53.5479	-80.5379	-128.1879	-212.5492	-362.2790
1	-21.9716	-26.6764	-34.7971	-48.8165	-73.3047	-116.4183
2	-1.2968	7.4464	23.3581	52.0864	103.6752	196.0174
3	11.8766	27.9099	56.7272	108.3414	200.5901	365.2564
4	11.2220	23.4472	45.3605	84.5442	154.5078	279.3231
5	0	0	0	0	0	0

Table 10 *Approximate solution of $u_\varepsilon^N(x,t)$ with respect to the approximate initial data*

$x\backslash t$	0	0.2	0.4	0.6	0.8	1.0
-5	0	0	0	0	0	0
-4	-1.9560	-1.9964	-2.1010	-2.2304	-2.4027	-2.6500
-3	-4.3618	-4.5058	-4.8309	-5.2908	-5.9872	-7.0938
-2	-8.0145	-8.5842	-9.7606	-11.6618	-14.8041	-20.1390
-1	-15.4145	-18.2402	-23.6814	-32.8294	-48.6673	-76.4520
0	-43.0674	-53.5519	-80.5439	-128.1975	-212.5653	-362.3063
1	-21.9733	-26.6784	-34.7997	-48.8201	-73.3102	-116.4271
2	-1.2969	7.4470	23.3599	52.0904	103.6830	196.0322
3	11.8775	27.9120	56.7314	108.3496	200.6052	365.2840
4	11.2228	23.4489	45.3639	84.5506	154.5195	279.3442
5	0	0	0	0	0	0

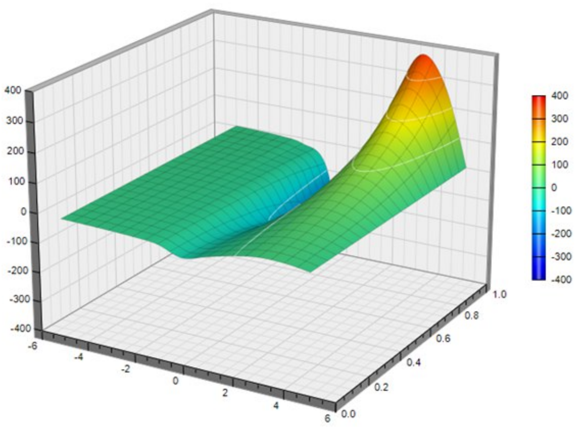

Figure 5 Graph of the approximate solution $u^N(x,t)$ with respect to the exact initial data

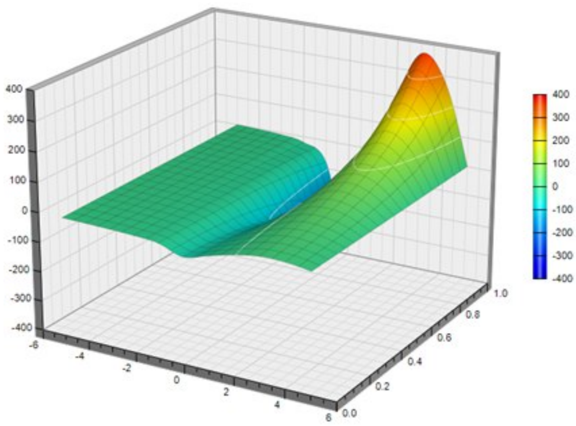

Figure 6 Graph of the approximate solution of $u_\varepsilon^N(x,t)$ with respect to the approximate initial data

6 Conclusion

In this work, the ill-posed problem in the Hadamard sense for an inhomogeneous parabolic equation with changing direction was studied to be conditionally correct. A conditional stability estimate was obtained for the solution of this problem, and a set of correctness was determined. In this set, theorems on the uniqueness and conditional stability of the solution of the problem were proved. As a result, a regularized solution to the problem under consideration was constructed. In this case, the regularization parameter was determined by reaching the minimum of the norm of the difference between the exact solution and the regularized solution for the approximate data.

The results of the numerical experiment showed that if the regularization parameter is chosen in such a way, the regularized solution will be continuously dependent on the change of the initial data.

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РЕГУЛЯРИЗАЦИЯ НАЧАЛЬНО-КРАЕВОЙ ЗАДАЧИ ДЛЯ НЕОДНОРОДНОГО ПАРАБОЛИЧЕСКОГО УРАВНЕНИЯ С МЕНЯЮЩИМСЯ НАПРАВЛЕНИЕМ ВРЕМЕНИ

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В данной работе исследована задача Коши с граничными условиями для неоднородного параболического уравнения с меняющимся направлением времени. Для построения решения применён метод разделения переменных (метод Фурье), в результате чего изучена соответствующая спектральная задача, определены собственные значения и собственные функции. Получена априорная оценка решения, доказаны теоремы единственности и условной устойчивости в множестве корректности. Построено регуляризованное решение задачи, соответствующее приближённым начальными данным. Для выбора параметра регуляризации оценена эффективность по норме разности между точным и регуляризованным решениями. Проведены численные эксперименты для различных наборов исходных данных, результаты которых представлены в виде таблиц и графиков. Из результатов видно, что регуляризованное решение, полученное для приближённых данных, близко к решению, полученному для точных данных, что подтверждает устойчивость метода к погрешностям в исходной информации.

Ключевые слова: параболическое уравнение с меняющимся направлением времени, некорректная задача, априорная оценка, единственность, условная устойчивость, метод множителей Лагранжа, регуляризация, приближенное решение.

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