UDC 519.644

ALGEBRAIC-TRIGONOMETRIC OPTIMAL FORMULAS FOR NUMERICAL INTEGRATION

[1,2*]***Hayotov A.R.***, [1,3]***Khaitov T.O.***

[*]hayotov@mail.ru

[1]V.I.Romanovskiy Institute of Mathematics,
9, University str., Tashkent, 100174 Uzbekistan;
[2]Central Asian University,
264, Milliy bog str., Barkamol MFY, Tashkent, 111221 Uzbekistan;
[3]Tashkent State Technical University,
2A, University Str., Tashkent, 100095 Uzbekistan.

In this article, we present a detailed study on the determination of the coefficients of a quadrature formula that involves derivatives, making use of the so-called the φ – function method. This method provides a systematic framework for constructing optimal quadrature formulas in the context of approximate integration. In particular, the φ – function approach not only simplifies the process of finding the required coefficients, but also ensures that the resulting formulas achieve a high degree of accuracy. Furthermore, the functional error associated with the constructed quadrature formula is carefully analyzed. The error analysis is supported by precise mathematical expressions, which confirm the reliability and effectiveness of the derived formula. By considering quadrature formulas with arbitrarily fixed nodes, the optimality conditions are rigorously examined, and the procedure for determining the corresponding components and coefficients is described in detail. As a key outcome of this investigation, explicit analytical expressions for the coefficients of the optimal quadrature formula are successfully obtained. Of particular interest is the case of equally spaced nodes, for which the derived quadrature formula naturally reduces to a classical Euler-Maclaurin type formula, demonstrating both the generality and the practical significance of the φ – function method.

Keywords: the φ – function method, quadrature formula, definite integral, nodal points, optimality.

Citation: Hayotov A.R., Khaitov T.O. 2025. Algebraic-trigonometric optimal formulas for numerical integration. *Problems of Computational and Applied Mathematics.* 5(69): 102-113.

DOI: https://doi.org/10.71310/pcam.5_69.2025.08

1 Introduction

The definite integral is one of the fundamental concepts of mathematical analysis and, in particular, it can be used to get the area or volume of a solid under the graph of a function. However, in many cases, finding an antiderivative function of a integrand is difficult or even impossible. Therefore, several approximate calculation methods for numerical integration have been developed. It is known from the course of mathematical analysis that if the antiderivative function of the function under the integral is known, the integral can be calculated using the Newton - Leibnitz formula. For cases where it is difficult to find the antiderivative of the functions under the integral, the problem of approximating the value of the definite integral arises. To solve this problem, various formulas have been found in mathematics, which are generally called quadrature and cubature formulas [1, 3, 12–16].

In this work, we construct optimal quadrature formulas in the space $K_2^{(3,1)}$ using the φ – function method.

The space $K_2^{(3,1)}$ that we are looking at is a Hilbert space. This space is a special case of the spaces considered in the works [6, 8, 12]. If the nodal points of the quadrature formula are arbitrarily fixed, then the Spline method, the φ function method, and the Sobolev method are available to construct quadrature formulas obtained by minimizing the norm of the error functional depending on the coefficients.

Using the φ function method in several spaces, A.Chizzetti and A.Ossicini [17], F.Lanzara [13], T.Catinaş and G.Coman [2] constructed optimal quadrature formulas. In [6], the authors studied the problem of constructing optimal quadrature formulas in the sense of Sard. The coefficients of the optimal quadrature formula were calculated using the resulting φ-function. The optimal quadrature formula in that work is exact for the functions $e^{\sigma x}$ and $e^{-\sigma x}$, where σ is a nonzero real parameter.

In present work, we consider the following type of quadrature formula

$$\int_a^b f(x)\,dx = \sum_{k=0}^{n} A_{0k} f(x_k) + \sum_{k=0}^{n} A_{1k} f'(x_k) + \sum_{k=0}^{n} A_{2k} f''(x_k) + R_n(f), \tag{1}$$

where A_{0k}, A_{1k}, A_{2k}, and x_k are the coefficients and nodes of quadrature formula (1), respectively. Let the nodal points be partitioned on the interval $[a,\, b]$ as follows,

$$a = x_0 < x_1 < \ldots < x_n = b, \tag{2}$$

where $R_n(f)$ is the error of quadrature formula (1).

Let us assume that the function f under the integral we are looking at is taken from the space $K_2^{(3,1)}$, where

$$K_2^{(3,1)} := \{f : [a,\, b] \to \mathbb{R} | f'' \text{ – absolute continuous and } f''' \in L_2(a,\, b)\}.$$

In this space, the inner product for arbitrary functions $f(x)$ and $g(x)$ is defined as

$$\langle f, g\rangle_{K_2^{(3,1)}} = \int_a^b (f'''(x) + f'(x))(g'''(x) + g'(x))\,dx.$$

The corresponding norm in this space is determined as follows

$$\|f\|_{K_2^{(3,1)}} = \left(\int_a^b (f'''(x) + f'(x))^2 dx \right)^{1/2}.$$

We consider the construction of the quadrature formula of the form (1) with the smallest error for functions of the space $K_2^{(3,1)}$ considered in a certain form in the space of quadrature formulas. For convenience, we introduce the following designations

$$A_0 = (A_{00},\, A_{01},\, \ldots,\, A_{0n}),\;\; A_1 = (A_{10},\, A_{11},\, \ldots,\, A_{1n}),\;\; A_2 = (A_{20},\, A_{21},\, \ldots,\, A_{2n})$$

and

$$X = (x_0,\, x_1,\, \ldots,\, x_n). \tag{3}$$

Below we give the definitions of optimality, of the quadrature formulas [3, 6, 17].

Definition 1. A quadrature formula of the form (1) is called optimal in the sense of Nikolysky in the space $K_2^{(3,1)}$ if the quantity

$$F_n\left(K_2^{(3,1)},\, A_0,\, A_1,\, A_2,\, X\right) = \sup_{f\in K_2^{(3,1)}} \left|R_n\left(f\right)\right|,$$

reaches its smallest value with respect to A_0, A_1, A_2 and X, and A_0, A_1, A_2 and X are determined by equality (3).

Definition 2. A quadrature formula of the form (1) is called optimal in the sense of Sard in the space $K_2^{(3,1)}$ if the quantity

$$F_n\left(K_2^{(3,1)},\, A_0,\, A_1,\, A_2\right) = \sup_{f\in K_2^{(3,1)}} \left|R_n\left(f\right)\right|,$$

reaches its smallest value relative to A_0, A_1, A_2 for fixed X, where A_0, A_1, A_2 and X are defined in (3).

In this paper we construct an optimal quadrature formula of the form (1) in the sense of Sard in the Hilbert space $K_2^{(3,1)}$ which is exact for trigonometric functions $\sin(x)$, $\cos(x)$ and constant term.

Next, we discuss the φ - function method for construction of quadrature formulas of the form (1) in the space $K_2^{(3,1)}$. In Section 3, we consider the optimization of the quadrature formulas in the form (1). We get the explicite expressions of coefficients for the optimal formula. In particular, we get the Euler-Maclaurin type quadrature formula in the space $K_2^{(3,1)}$.

2 The method for construction of quadrature formulas

Let a function f be taken from the space $K_2^{(3,1)}$ and the nodes for the given natural numbers n be distributed as in the form (2). Then, for each interval $[x_{k-1},\, x_k]$ $(k = 1,\, 2,\, \ldots, n)$. We consider the functions φ_k, $k = 1,\, 2,\, \ldots,\, n$ with the following property

$$\varphi_k'''(x) + \varphi_k'(x) = -1, \qquad k = 1,\, 2,\, \ldots,\, n. \tag{4}$$

Then the function φ is defined as follows,

$$\varphi|_{[x_{k-1},\, x_k]} = \varphi_k, \qquad k = 1,\, 2,\, \ldots,\, n.$$

Therefore, the reduction of the function φ on the interval $[x_{k-1},\, x_k]$ is equal to φ_k.

We introduce the following notations:

$$I\left(f\right) := \int\limits_a^b f(x)dx,$$

$$Q_n\left(f\right) = \sum_{k=0}^{n} A_{0k} f(x_k) + \sum_{k=0}^{n} A_{1k} f'(x_k) + \sum_{k=0}^{n} A_{2k} f''(x_k).$$

Now, using the property of addition of definite integrals and taking into account equality (4), we get

$$I\left(f\right) = \int\limits_a^b f(x)\,dx = -\sum_{k=1}^{n} \int\limits_{x_{k-1}}^{x_k} (-1)\cdot\ f(x)\,dx = -\sum_{k=1}^{n} \int\limits_{x_{k-1}}^{x_k} (\varphi_k'''\left(x\right) + \varphi_k'\left(x\right))f(x)\,dx.$$

Then integrating by parts we have

$$
\begin{gathered}
I\left(f\right)=\varphi_1''\left(x_0\right)f\left(x_0\right)+\sum_{k=1}^{n-1}\left(\varphi_{k+1}''\left(x_k\right)-\varphi_k''\left(x_k\right)\right)f\left(x_k\right)-\varphi_n''\left(x_n\right)f\left(x_n\right)-\\
-\varphi_1'(x_0)f'(x_0)-\sum_{k=1}^{n-1}(\varphi_{k+1}'(x_k)-\varphi_k'(x_k))f'(x_k)+\varphi_n'(x_n)f'(x_n)+\\
+\varphi_1(x_0)f''(x_0)+\sum_{k=1}^{n-1}(\varphi_{k+1}(x_k)-\varphi_k(x_k))f''(x_k)-\varphi_n(x_n)f''(x_n)+\\
+\varphi_1(x_0)f(x_0)+\sum_{k=1}^{n-1}(\varphi_{k+1}(x_k)-\varphi_k(x_k))f(x_k)-\varphi_n(x_n)f(x_n)+\\
+\sum_{k=1}^{n}\int\limits_{x_{k-1}}^{x_k}\left(f'''\left(x\right)+f'\left(x\right)\right)\varphi_k(x)dx.
\end{gathered}
\tag{5}
$$

From equality (5) we get:

$$
\begin{gathered}
A_{00}=\varphi_1''(x_0)+\varphi_1(x_0),\\
A_{0k}=\varphi_{k+1}''(x_k)-\varphi_k''(x_k)+\varphi_{k+1}(x_k)-\varphi_k(x_k),\qquad k=1,\,2,\,\ldots,\,n-1,\\
A_{0n}=-\varphi_n''(x_n)-\varphi_n(x_n),\\
A_{10}=-\varphi_1'(x_0),\\
A_{1k}=-\varphi_{k+1}'(x_k)+\varphi'{}_k(x_k),\quad k=1,\,2,\,\ldots,\,n-1,\\
A_{1n}=\varphi_n'(x_n),\\
A_{20}=\varphi_1\left(x_0\right),\\
A_{2k}=\varphi_{k+1}(x_k)-\varphi_k(x_k),\quad k=1,\,2,\,\ldots,\,n-1,\\
A_{2n}=-\varphi_n(x_n).
\end{gathered}
\tag{6}
$$

and the error formula is as follows

$$
R_n\left(f\right)=\sum_{k=1}^{n}\int\limits_{x_{k-1}}^{x_k}\left(f'''(x)+f'\left(x\right)\right)\varphi_k\left(x\right)dx=\int\limits_a^b\left(f'''(x)+f'(x)\right)\varphi(x)dx. \tag{7}
$$

Now we solve this nonhomogeneous second-order differential equation:

$$
y'''+y'=-1 \tag{8}
$$

and obtain a general solution in the following form

$$
y=C_1+C_2\cos\left(x\right)+C_3\sin\left(x\right)-x. \tag{9}
$$

Remark 1. Knowing the function φ, one can find the coefficients A_{0k}, A_{1k} and A_{2k}, $k = 0, 1, \ldots, n$ from equality (6). This method is called the φ-function method of constructing a quadrature formula.

Remark 2. As can be seen from expression (7) above, the quadrature formula (1) is exact to the functions that are solutions of the equation

$$f'''(x) + f'(x) = 0.$$

In the following sections, we deal with finding the coefficients of the optimal quadrature formula (1) in the space $K_2^{(3,1)}$.

3 An optimal quadrature formula

In this section, we consider the optimality of the quadrature formula (1) in the space $K_2^{(3,1)}$. Using the Cauchy-Schwarz inequality, the absolute value of the error of expression (7) of the quadrature formula (1) is determined as follows

$$|R_n(f)| \leqslant \|f''' + f'\|_{L_2(a,b)} \cdot \left(\int_a^b \varphi^2(x)dx \right)^{1/2} = \|f(x)\|_{K_2^{(3,1)}} \cdot \|\varphi(x)\|_{L_2(a,\,b)}.$$

Now we consider the function $\varphi_k(x)$ in the interval $[x_{k-1},\, x_k]$ $(k = 1, 2, \ldots, n)$ as a solution of equation (8). Then based on (9) for φ_k, we have

$$\varphi_k(x) = C_1^{(k)} + C_2^{(k)} \cdot \cos x + C_3^{(k)} \cdot \sin x - x, \tag{10}$$

where $C_1^{(k)}$, $C_2^{(k)}$, $C_3^{(k)}$, $(k = 1, 2, \ldots, n)$ are arbitrary constants.

In conclusion, to find the functions $\varphi_k(x)$, it is necessary to find all unknowns $C_1^{(k)}, C_2^{(k)}$ and $C_3^{(k)}$, $k = 1, 2, \ldots, n$. We find $C_1^{(k)}, C_2^{(k)}$ and $C_3^{(k)}$ such that when the integral of the square of the function $\varphi_k\,(x)$ takes the smallest value. Let's look at this function that is related to these,

$$\mathcal{F}_k(C_1^{(k)},\, C_2^{(k)},\, C_3^{(k)}) = \int_{x_{k-1}}^{x_k} (\varphi_k(x))^2 dx, \qquad k = 1, 2, \ldots, n.$$

Then, taking into account (10), we have the following.

$$\mathcal{F}_k(C_1^{(k)},\, C_2^{(k)},\, C_3^{(k)}) = \int_{x_{k-1}}^{x_k} \Big(C_1^{(k)} + C_2^{(k)} \cos{(x)} + C_3^{(k)} \sin{(x)} - x \Big)^2 dx.$$

We calculate the first particular derivatives of this function with respect to $C_1^{(k)}, C_2^{(k)}$ and $C_3^{(k)}$ and equating them to zero we have the following system of linear equations

$$\mathcal{F}'\Big(C_1^{(k)}, C_2^{(k)}, C_3^{(k)}\Big)_{C_1^{(k)}} = 2 \int_{x_{k-1}}^{x_k} \Big(C_1^{(k)} + C_2^{(k)} \cos{(x)} + C_3^{(k)} \sin{(x)} - x \Big)\, dx,$$

$$\mathcal{F}'\Big(C_1^{(k)}, C_2^{(k)}, C_3^{(k)}\Big)_{C_2^{(k)}} = 2 \int_{x_{k-1}}^{x_k} \Big(C_1^{(k)} + C_2^{(k)} \cos{(x)} + C_3^{(k)} \sin{(x)} - x \Big) \cos{(x)}\, dx,$$

$$\mathcal{F}'\left(C_1^{(k)}, C_2^{(k)}, C_3^{(k)}\right)_{C_3^{(k)}} = 2\int\limits_{x_{k-1}}^{x_k} \left(C_1^{(k)} + C_2^{(k)}\cos(x) + C_3^{(k)}\sin(x) - x\right)\sin(x)\,dx.$$

We can write the system in the following form by introducing notations, and we present these notations below:

$$\begin{cases} a_{11}\cdot C_1^{(k)} + a_{12}\cdot C_2^{(k)} + a_{13}\cdot C_3^{(k)} = b_1, \\ a_{21}\cdot C_1^{(k)} + a_{22}\cdot C_2^{(k)} + a_{23}\cdot C_3^{(k)} = b_2, \\ a_{31}\cdot C_1^{(k)} + a_{32}\cdot C_2^{(k)} + a_{33}\cdot C_3^{(k)} = b_3, \end{cases}$$

where

$$a_{11} = 2\int\limits_{x_{k-1}}^{x_k} dx,$$

$$a_{12} = a_{21} = 2\int\limits_{x_{k-1}}^{x_k} \cos(x)\,dx,$$

$$a_{13} = a_{31} = 2\int\limits_{x_{k-1}}^{x_k} \sin(x)\,dx,$$

$$a_{22} = 2\int\limits_{x_{k-1}}^{x_k} \cos^2(x)\,dx,$$

$$a_{23} = a_{32} = 2\int\limits_{x_{k-1}}^{x_k} \sin(x)\cos(x)\,dx,$$

$$a_{33} = 2\int\limits_{x_{k-1}}^{x_k} \sin^2(x)\,dx,$$

$$b_1 = 2\int\limits_{x_{k-1}}^{x_k} x\,dx,$$

$$b_2 = 2\int\limits_{x_{k-1}}^{x_k} x\cos(x)\,dx,$$

$$b_3 = 2\int\limits_{x_{k-1}}^{x_k} x\sin(x)\,dx.$$

We present the calculations of these designations below

$$a_{11} = 2\int\limits_{x_{k-1}}^{x_k} dx = 2\left(x_k - x_{k-1}\right),$$

$$a_{12} = a_{21} = 2\int\limits_{x_{k-1}}^{x_k} \cos(x)\,dx = 2\left(\sin(x_k) - \sin(x_{k-1})\right) =$$
$$= 4\sin\left(\frac{x_k - x_{k-1}}{2}\right)\cos\left(\frac{x_k + x_{k-1}}{2}\right),$$

$$a_{13} = a_{31} = 2\int\limits_{x_{k-1}}^{x_k} \sin(x)\,dx = -2\left(\cos(x_k) - \cos(x_{k-1})\right) =$$
$$= 4\sin\left(\frac{x_k - x_{k-1}}{2}\right)\sin\left(\frac{x_k + x_{k-1}}{2}\right),$$

$$a_{22}=2\int\limits_{x_{k-1}}^{x_k}\cos^2(x)\,dx=x_k-x_{k-1}+\sin(x_k-x_{k-1})\cos(x_k+x_{k-1}),$$

$$a_{23}=a_{32}=2\int\limits_{x_{k-1}}^{x_k}\sin(x)\cos(x)\,dx=\sin(x_k-x_{k-1})\sin(x_k+x_{k-1}),$$

$$a_{33}=2\int\limits_{x_{k-1}}^{x_k}\sin^2(x)\,dx=x_k-x_{k-1}-\sin(x_k-x_{k-1})\cos(x_k+x_{k-1}),$$

$$b_1=2\int\limits_{x_{k-1}}^{x_k}x dx=x_k^2-x_{k-1}^2,$$

$$b_2=2\int\limits_{x_{k-1}}^{x_k}x\cos(x)\,dx=2\left(x_k\sin(x_k)-x_{k-1}\sin(x_{k-1})+\cos(x_k)-\cos(x_{k-1})\right),$$

$$b_3=2\int\limits_{x_{k-1}}^{x_k}x\sin(x)\,dx=2\left(-x_k\cos(x_k)+x_{k-1}\cos(x_{k-1})+\sin(x_k)-\sin(x_{k-1})\right).$$

According to the result of the solved system, we have the following

$$C_1^{(k)}=\frac{x_k+x_{k-1}}{2},$$

$$C_2^{(k)}=\frac{2\sin(\frac{x_k+x_{k-1}}{2})((x_k-x_{k-1})\cos(\frac{x_k-x_{k-1}}{2})-2\sin(\frac{x_k-x_{k-1}}{2}))}{x_k-x_{k-1}-\sin(x_k-x_{k-1})},$$

$$C_3^{(k)}=-\frac{2\cos(\frac{x_k+x_{k-1}}{2})((x_k-x_{k-1})\cos(\frac{x_k-x_{k-1}}{2})-2\sin(\frac{x_k-x_{k-1}}{2}))}{x_k-x_{k-1}-\sin(x_k-x_{k-1})}. \tag{11}$$

Then using (11) from (10) we come

$$\varphi_k(x)=\frac{x_k+x_{k-1}}{2}-x-$$

$$-\frac{2\sin\left(x-\frac{x_k+x_{k-1}}{2}\right)\left((x_k-x_{k-1})\cos\left(\frac{x_k-x_{k-1}}{2}\right)-2\sin\left(\frac{x_k-x_{k-1}}{2}\right)\right)}{x_k-x_{k-1}-\sin(x_k-x_{k-1})}. \tag{12}$$

For the first and second derivatives of φ_k we have

$$\varphi_k'(x)=-1-$$

$$-\frac{2\cos\left(x-\frac{x_k+x_{k-1}}{2}\right)\left((x_k-x_{k-1})\cos\left(\frac{x_k-x_{k-1}}{2}\right)-2\sin\left(\frac{x_k-x_{k-1}}{2}\right)\right)}{x_k-x_{k-1}-\sin(x_k-x_{k-1})}, \tag{13}$$

$$\varphi_k''(x)=$$

$$=\frac{2\sin\left(x-\frac{x_k+x_{k-1}}{2}\right)\left((x_k-x_{k-1})\cos\left(\frac{x_k-x_{k-1}}{2}\right)-2\sin\left(\frac{x_k-x_{k-1}}{2}\right)\right)}{x_k-x_{k-1}-\sin(x_k-x_{k-1})}. \tag{14}$$

Now we can calculate the coefficients $A_{0k},\ A_{1k}, A_{2k},\ k=1,\,2,\,\ldots,\,n$ based on (6) for the optimal quadrature formula of the form (1) in the space $K_2^{(3,1)}$.

Thus we have obtained the following main result of thus work.

Theorem 3.1. *For fixed nodes $a = x_0 < x_1 < \ldots < x_n = b$ there exists a unique optimal quadrature formula of the form*

$$\int_a^b f(x)\,dx \cong \sum_{k=0}^{n} A_{0k} f(x_k) + \sum_{k=0}^{n} A_{1k} f'(x_k) + \sum_{k=0}^{n} A_{2k} f''(x_k), \tag{15}$$

in the sense of Sard in the space $K_2^{(3,1)}$ with coefficients A_{0k}, A_{1k}, A_{2k} where A_{0k}, A_{1k}, and A_{2k} are the coefficients and nodes of the quadrature formula (1)*, respectively*

$$A_{00} = \frac{x_1 - x_0}{2},$$

$$A_{0k} = \frac{x_{k+1} - x_{k-1}}{2}, \quad k = 1, 2, \ldots, n-1,$$

$$A_{0n} = \frac{x_n - x_{n-1}}{2},$$

$$A_{10} = \frac{(x_1 - x_0)(2 + \cos(x_1 - x_0)) - 3\sin(x_1 - x_0)}{x_1 - x_0 - \sin(x_1 - x_0)},$$

$$A_{1k} = \frac{(x_{k+1} - x_k)(1 + \cos(x_{k+1} - x_k)) - 2\sin(x_{k+1} - x_k)}{x_{k+1} - x_k - \sin(x_{k+1} - x_k)} -$$

$$-\frac{(x_k - x_{k-1})(1 + \cos(x_k - x_{k-1})) - 2\sin(x_k - x_{k-1})}{x_k - x_{k-1} - \sin(x_k - x_{k-1})}, \quad k = 1, 2, \ldots, n-1,$$

$$A_{1n} = -\frac{(x_n - x_{n-1})(2 + \cos(x_n - x_{n-1})) - 3\sin(x_n - x_{n-1})}{x_n - x_{n-1} - \sin(x_n - x_{n-1})},$$

$$A_{20} = \frac{x_1 - x_0}{2} + \frac{(x_1 - x_0)\sin(x_1 - x_0) - 2 + 2\cos(x_1 - x_0)}{x_1 - x_0 - \sin(x_1 - x_0)},$$

$$A_{2k} = \frac{x_{k+1} - x_{k-1}}{2} + \frac{(x_{k+1} - x_k)\sin(x_{k+1} - x_k) - 2 + 2\cos(x_{k+1} - x_k)}{x_{k+1} - x_k - \sin(x_{k+1} - x_k)} -$$

$$-\frac{(x_k - x_{k-1})\sin(x_k - x_{k-1}) - 2 + 2\cos(x_k - x_{k-1})}{x_k - x_{k-1} - \sin(x_k - x_{k-1})}, \quad k = 1, 2, \ldots, n-1,$$

$$A_{2n} = \frac{x_n - x_{n-1}}{2} + \frac{(x_n - x_{n-1})\sin(x_n - x_{n-1}) - 2 + 2\cos(x_n - x_{n-1})}{x_n - x_{n-1} - \sin(x_n - x_{n-1})}.$$

The formula is exact for trigonometric functions $\sin(x)$, $\cos(x)$ *and constant term.*

The proof of Theorem 3.1 is obtained putting (11), (12), (13), (14) to (6) and doing some calculations.

From Theorem 3.1 when the nodes are equally spaced we get the optimal quadrature formula of the Euler - Maclaurin type in the space $K_2^{(3,1)}$. That is the following holds.

Corollary 3.1 *For equally spaced nodes $x_k = a + kh$, $k = 0, 1, \ldots, n$, $h = \frac{b-a}{n}$, there exist a unique optimal quadrature formula*

$$\int_a^b f(x)\,dx \cong \sum_{k=0}^{n} A_{0k} f(a + kh) + A_{10} f'(a) + A_{1n} f'(b) + \sum_{k=0}^{n} A_{2k} f''(a + kh),$$

of the Euler - Maclaurin type with coefficients

$$A_{00}=\frac{h}{2},$$

$$A_{0k}=h,\ \ k=1,\,2,\,\dots,\,n-1,$$

$$A_{0n}=\frac{h}{2},$$

$$A_{10}=\frac{h(2+\cos(h))-3\sin(h)}{h-\sin(h)},$$

$$A_{1n}=-\frac{h\,(2+\cos\,(h))-3\sin\,(h)}{h-\sin\,(h)},$$

$$A_{20}=\frac{h}{2}+\frac{(h)\sin\,(h)-2+2\cos\,(h)}{h-\sin\,(h)},$$

$$A_{2k}=h+\frac{h\sin\,(h)-2+2\cos\,(h)}{h-\sin\,(h)}-\frac{h\sin\,(h)-2+2\cos\,(h)}{h-\sin\,(h)},\ k=1,\,2,\,\dots,\,n-1,$$

$$A_{2n}=\frac{h}{2}+\frac{h\sin\,(h)-2+2\cos\,(h)}{h-\sin\,(h)}.$$

4 Estimation of the error of the optimal quadrature formula

Next, we evaluate the square of the function $\varphi_k(x)$ given by (12) and perform integration over the interval $[x_{k-1},x_k]$

$$\int\limits_{x_{k-1}}^{x_k}\varphi_k^2\,(x)\,dx=\int\limits_{x_{k-1}}^{x_k}\Bigg(\left(x-\frac{x_k+x_{k-1}}{2}\right)^2-$$

$$-\frac{2\,(x_k+x_{k-1})\sin\left(x-\frac{x_k+x_{k-1}}{2}\right)\left((x_k-x_{k-1})\cos\left(\frac{x_k-x_{k-1}}{2}\right)-2\sin\left(\frac{x_k-x_{k-1}}{2}\right)\right)}{x_k-x_{k-1}-\sin\,(x_k-x_{k-1})}+$$

$$+\frac{4x\sin\left(x-\frac{x_k+x_{k-1}}{2}\right)\left((x_k-x_{k-1})\cos\left(\frac{x_k-x_{k-1}}{2}\right)-2\sin\left(\frac{x_k-x_{k-1}}{2}\right)\right)}{x_k-x_{k-1}-\sin\,(x_k-x_{k-1})}+$$

$$+\frac{2\Big((x_k-x_{k-1})\cos\left(\frac{x_k-x_{k-1}}{2}\right)-2\sin\left(\frac{x_k-x_{k-1}}{2}\right)\Big)^2}{(x_k-x_{k-1}-\sin\,(x_k-x_{k-1}))^2}+$$

$$-\frac{2\cos\,(2x-x_k-x_{k-1})\left((x_k-x_{k-1})\cos\left(\frac{x_k-x_{k-1}}{2}\right)-2\sin\left(\frac{x_k-x_{k-1}}{2}\right)\right)^2}{(x_k-x_{k-1}-\sin\,(x_k-x_{k-1}))^2}\Bigg)\,dx=$$

$$=\int\limits_{x_{k-1}}^{x_k}\left(x-\frac{x_k+x_{k-1}}{2}\right)^2dx-$$

$$-\frac{2\,(x_k+x_{k-1})\left((x_k-x_{k-1})\cos\left(\frac{x_k-x_{k-1}}{2}\right)-2\sin\left(\frac{x_k-x_{k-1}}{2}\right)\right)}{x_k-x_{k-1}-\sin\,(x_k-x_{k-1})}\int\limits_{x_{k-1}}^{x_k}\sin\left(x-\frac{x_k+x_{k-1}}{2}\right)dx+$$

$$+\frac{4\left(\left(x_k-x_{k-1}\right)\cos\left(\frac{x_k-x_{k-1}}{2}\right)-2\sin\left(\frac{x_k-x_{k-1}}{2}\right)\right)}{x_k-x_{k-1}-\sin\left(x_k-x_{k-1}\right)}\int\limits_{x_{k-1}}^{x_k} x\sin\left(x-\frac{x_k+x_{k-1}}{2}\right)dx+$$

$$+\int\limits_{x_{k-1}}^{x_k}\left(\frac{2\left(\left(x_k-x_{k-1}\right)\cos\left(\frac{x_k-x_{k-1}}{2}\right)-2\sin\left(\frac{x_k-x_{k-1}}{2}\right)\right)^2}{\left(x_k-x_{k-1}-\sin\left(x_k-x_{k-1}\right)\right)^2}\right)dx-$$

$$-\int\limits_{x_{k-1}}^{x_k}\frac{2\cos\left(2x-x_k-x_{k-1}\right)\left(\left(x_k-x_{k-1}\right)\cos\left(\frac{x_k-x_{k-1}}{2}\right)-2\sin\left(\frac{x_k-x_{k-1}}{2}\right)\right)^2}{\left(x_k-x_{k-1}-\sin\left(x_k-x_{k-1}\right)\right)^2}dx=$$

$$=\frac{\left(x_k-x_{k-1}\right)^3}{12}-\frac{2\left(\left(x_k-x_{k-1}\right)\cos\left(\frac{x_k-x_{k-1}}{2}\right)-2\sin\left(\frac{x_k-x_{k-1}}{2}\right)\right)^2}{x_k-x_{k-1}-\sin\left(x_k-x_{k-1}\right)}.$$

Hence we have

$$\int\limits_{x_{k-1}}^{x_k}\varphi_k^2\left(x\right)dx=\frac{\left(x_k-x_{k-1}\right)^3}{12}-\frac{2\left(\left(x_k-x_{k-1}\right)\cos\left(\frac{x_k-x_{k-1}}{2}\right)-2\sin\left(\frac{x_k-x_{k-1}}{2}\right)\right)^2}{x_k-x_{k-1}-\sin\left(x_k-x_{k-1}\right)}. \tag{16}$$

Now, we proceed to find the norm of the function $\varphi(x)$:

$$\|\varphi\|^2_{L_2(a,b)}=\sum_{k=1}^{n}\left[\frac{\left(x_k-x_{k-1}\right)^3}{12}-\frac{2\left(\left(x_k-x_{k-1}\right)\cos\left(\frac{x_k-x_{k-1}}{2}\right)-2\sin\left(\frac{x_k-x_{k-1}}{2}\right)\right)^2}{x_k-x_{k-1}-\sin\left(x_k-x_{k-1}\right)}\right]. \tag{17}$$

Then using (17), based on the Cauchy-Schwarz inequality, for the error of the optimal quadrature formula we have the following estimation

$$|R_n\left(f\right)|\leqslant\|f''+f\|_{L_2(a,b)}\cdot\left(\int\limits_a^b\varphi^2(x)dx\right)^{1/2}=||f||_{K_2^{(3,1)}}\cdot||\varphi||_{L_2(a,\,b)}=$$

$$=||f||_{K_2^{(3,1)}}\cdot\sqrt{\sum_{k=1}^{n}\left[\frac{\left(x_k-x_{k-1}\right)^3}{12}-\frac{2\left(\left(x_k-x_{k-1}\right)\cos\left(\frac{x_k-x_{k-1}}{2}\right)-2\sin\left(\frac{x_k-x_{k-1}}{2}\right)\right)^2}{x_k-x_{k-1}-\sin\left(x_k-x_{k-1}\right)}\right]}.$$

From (17), for the error of the Euler-Maclaurin type quadrature formula, when nodes are equally spaced, we get the estimation

$$|R_n\left(f\right)|\leqslant\|f\|_{K_2^{(3,1)}}\sum_{k=1}^{n}\left[\frac{h^3}{12}-\frac{2\left(h\cos\left(\frac{h}{2}\right)-2\sin\left(\frac{h}{2}\right)\right)^2}{h-\sin\left(h\right)}\right]. \tag{18}$$

5 Conclusion

In this paper, we developed an optimal quadrature formula in the space $K_2^{(3,1)}$, which is a Hilbert space consisting of absolutely continuous functions whose first derivatives are square-integrable over the interval $[a,b]$. This space provides a suitable framework for constructing accurate numerical integration methods.

A central aspect of the study is the application of the φ-function method, which played an important role in optimizing the quadrature formulas. This method helped improve the precision and structure of the formula, making it more effective for practical use.

The resulting quadrature formula can be applied in both theoretical investigations and real-world problems related to mathematical analysis. Moreover, the approach used in this work may be extended to develop more general and complex quadrature formulas in future research.

Overall, the findings of this study contribute to the development of efficient integration techniques and have potential for further application in scientific and engineering computations.

References

[1] Boltaev N.D., Hayotov A.R., Shadimetov Kh.M. Construction of optimal quadrature formula for numerical calculation of Fourier coefficients in Sobolev space $L_2^{(1)}$ // *American Journal of Numerical Analysis*. – 2016. – Vol. 4, No. 1. – P. 1-7. – doi: http://dx.doi.org/10.12691/ajna-4-1-1.

[2] Catinaş T., Coman Gh. Optimal quadrature formulas based on the φ-function method // *Stud. Univ. Babeş-Bolyai Math.*. – 2006. – Vol. 51, No. 1. – P. 49-64.

[3] Coman Gh. Formule de cuadratura de tip Sard // *Stud. Univ. Babeş-Bolyai Math. Mech.*. – 1972. – Vol. 17, No. 2. – P. 73-77.

[4] Ahlberg J.H., Nilson E.N., Walsh J.L. *The theory of splines and their applications*. – New York: Academic Press, 1967.

[5] Hayotov A.R. Construction of Interpolation Splines Minimizing the Semi-norm in the Space $K_2(P_m)$ // *Journal of Siberian Federal University. Mathematics & Physics*. – 2018. – Vol. 11, No. 3. – P. 383-396.

[6] Hayotov A.R., Babayev S.S., Abduakhadov A.A., Davronov J.R. An optimal quadrature formula exact to the exponential function by the φ-function method // *Stud. Univ. Babeş-Bolyai Math.*. – 2024. – Vol. 69, No. 3. – P. 651-663. – doi: http://dx.doi.org/10.24193/subbmath.2024.3.11.

[7] Hayotov A.R., Kuldoshev H.M. An optimal quadrature formula with sigma parameter // *Problems of Computational and Applied Mathematics*. – 2023. – Vol. 48, No. 2/1. – P. 7-19.

[8] Hayotov A.R., Babaev S.S. Optimal quadrature formula for numerical integration of fractional integrals in a Hilbert space // *Journal of Mathematical Sciences (Springer)*. – 2023. – Vol. 277, No. 3. – P. 403-419. – doi: http://dx.doi.org/10.1007/s10958-023-06844-w.

[9] Hayotov A.R., Babaev S.S. Optimal quadrature formulas for computing of Fourier integrals in $W_2^{(m,m-2)}$ space // *AIP Conference Proceedings*. – 2021. – Vol. 2365. – 020021.

[10] Hayotov A.R., Jeon S., Lee C.-O. On an optimal quadrature formula for approximation of Fourier integrals in the space $L_2^{(1)}$ // *J. Comput. Appl. Math.*. – 2020. – Vol. 372. – 112713.

[11] Hayotov A.R., Jeon S., Shadimetov Kh.M. Application of optimal quadrature formulas for reconstruction of CT images // *J. Comput. Appl. Math.*. – 2021. – Vol. 388. – 113313.

[12] Hayotov A.R., Rasulov R.G. The order of convergence of an optimal quadrature formula with derivative in the space $W_2^{(1,0)}$ // *Filomat*. – 2020. – Vol. 34, No. 11. – P. 3835-3844.

[13] Lanzara F. On optimal quadrature formulae // *J. Inequal. Appl.*. – 2000. – No. 5. – P. 201-225.

[14] Nikolsky S.M. On the issue of estimates of approximations by quadrature formulas // *Advances in Math. Sciences*. – 1950. – Vol. 5, No. 3. – P. 165-177.

[15] Nikolsky S.M. *Quadrature Formulas*. – 4th ed. – Moscow: Nauka, 1988.

[16] Shadimetov Kh.M., Hayotov A.R. *Optimal approximation of the error functionals of quadrature and interpolation formulas in spaces of differentiable functions.* – Tashkent: Muhr Press, 2022.

[17] Chizzetti A., Ossicini A. *Quadrature Formulae.* – Berlin: Academie Verlag, 1970.

УДК 519.644

АЛГЕБРО-ТРИГОНОМЕТРИЧЕСКИЕ ОПТИМАЛЬНЫЕ ФОРМУЛЫ ЧИСЛЕННОГО ИНТЕГРИРОВАНИЯ

[1,2]***Хаётов А.Р.***, [1,3]***Хаитов Т.О.***

*hayotov@mail.ru

[1]Институт математики им. В.И. Романовского АН РУз,
100174, Узбекистан, Ташкент ул. Университетская, 9;
[2]Центрально-Азиатский университет,
111221, Узбекистан, Ташкент, ул. Миллий бог, 264;
[3]Ташкентский государственный технический университет,
100095, Узбекистан, Ташкент, ул. Университетская, 2А.

В данной статье предаставлен детальный анализ определения коэффициентов квадратурной формулы с производными на основе так называемого метода φ – функции. Данный метод обеспечивает систематический подход к построению оптимальных квадратурных формул в задаче приближённого интегрирования. В частности, применение φ - функции не только упрощает процесс нахождения требуемых коэффициентов, но и гарантирует высокую точность получаемых формул. Кроме того, тщательно исследуется функциональная погрешность построенной квадратурной формулы. Анализ погрешности подтверждается строгими математическими выражениями, которые демонстрируют надёжность и эффективность полученной формулы. При рассмотрении квадратурных формул с произвольно фиксированными узлами оптимальные условия анализируются строго, а процедура определения соответствующих компонент и коэффициентов подробно описана. Ключевым результатом данного исследования является получение явных аналитических выражений для коэффициентов оптимальной квадратурной формулы. Особый интерес представляет случай равномерно расположенных узлов, для которого полученная квадратурная формула естественным образом сводится к классической формуле типа Эйлера–Маклорена, что подчёркивает как общность, так и практическую значимость метода φ – функции.

Ключевые слова: метод φ функции, квадратурная формула, определённый интеграл, узловые точки, оптимальность.

Цитирование: *Хаётов А.Р., Хаитов Т.О.* Алгебро-тригонометрические оптимальные формулы численного интегрирования // Проблемы вычислительной и прикладной математики. – 2025. – № 5(69). – С. 102-113.

DOI: https://doi.org/10.71310/pcam.5_69.2025.08

ПРОБЛЕМЫ ВЫЧИСЛИТЕЛЬНОЙ И ПРИКЛАДНОЙ МАТЕМАТИКИ

№ 5(69) 2025

Журнал основан в 2015 году.
Издается 6 раз в год.

Учредитель:
Научно-исследовательский институт развития цифровых технологий и искусственного интеллекта.

Журнал зарегистрирован в Агентстве информации и массовых коммуникаций при Администрации Президента Республики Узбекистан.
Регистрационное свидетельство №0856 от 5 августа 2015 года.

ISSN 2181-8460, eISSN 2181-046X

Адрес редакции:
100125, г. Ташкент, м-в. Буз-2, 17А.
Тел.: +(998) 712-319-253, 712-319-249.
Э-почта: journals@airi.uz.
Веб-сайт: https://journals.airi.uz.

Дизайн и вёрстка:
Шарипов Х.Д.

Отпечатано в типографии НИИ РЦТИИ.
Подписано в печать 29.10.2025 г.
Формат 60x84 1/8. Заказ №7. Тираж 100 экз.

PROBLEMS OF COMPUTATIONAL AND APPLIED MATHEMATICS

No. 5(69) 2025

The journal was established in 2015.
6 issues are published per year.

Founder:
Digital Technologies and Artificial Intelligence Development Research Institute.

The journal is registered by Agency of Information and Mass Communications under the Administration of the President of the Republic of Uzbekistan.
The registration certificate No. 0856 of 5 August 2015.

ISSN 2181-8460, eISSN 2181-046X

Address:
100125, Tashkent, Buz-2, 17A.
Tel.: +(998) 712-319-253, 712-319-249.
E-mail: `journals@airi.uz`.
Web-site: `https://journals.airi.uz`.

Layout design:
Sharipov Kh.D.

DTAIDRI printing office.
Signed for print 29.10.2025
Format 60x84 1/8. Order No. 7. Print run of 100 copies.

Содержание

Contents

№ 5(69) 2025 ISSN 2181-8460

HISOBLASH VA AMALIY MATEMATIKA MUAMMOLARI

ПРОБЛЕМЫ ВЫЧИСЛИТЕЛЬНОЙ И ПРИКЛАДНОЙ МАТЕМАТИКИ

PROBLEMS OF COMPUTATIONAL AND APPLIED MATHEMATICS

RAQAMLI TEXNOLOGIYALAR VA
SUN'IY INTELLEKTNI RIVOJLANTIRISH
ILMIY-TADQIQOT INSTITUTI