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AN OPTIMAL QUADRATURE FORMULA EXACT TO THE EXPONENTIAL FUNCTION

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Numerical integration of definite integrals plays a crucial role in both fundamental and applied sciences. The precision of approximate integral calculations depends on the initial data and specific conditions, which impose various requirements on the resulting computations. Traditional methods for numerical analysis of definite integrals, such as the quadrature formulas developed by Gregory, Newton-Cotes, Euler, Gauss, Markov, and others, are well-established. Since the mid-20th century, the theory of creating optimal formulas for numerical integration through variational methods has evolved. It is important to mention that optimal quadrature formulas exist in the sense defined by Nikolsky and Sard. This paper focuses on the challenge of constructing an optimal quadrature formula according to Sard's approach. In this process, the method of φ-functions is applied. The error of the formula is estimated by integrating the square of the φ function from a specific Hilbert space. Afterward, the appropriate function is chosen such that the integral of its square within this interval is minimized. Finally, the coefficients of the optimal quadrature formula are determined using the resulting φ function.

Keywords: Hilbert space, phi-function method, optimal quadrature formula, the error.

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1 Introduction

In numerical analysis and mathematical computation, solving integral problems is a common and essential task. Many practical problems in fields like physics, engineering, and economics require determining areas under curves or the total accumulated effect of a function over a specified interval. In many cases, analytical methods for computing integrals do not provide exact solutions, either due to the complexity of the function or its lack of a closed-form antiderivative. Therefore, quadrature formulas, which provide approximate solutions to integrals, play a crucial role in numerical computation.

Quadrature formulas use numerical methods to estimate the value of an integral by evaluating the function at certain discrete points rather than computing the integral analytically. The accuracy and efficiency of these methods depend on the nature of the function and the chosen approach. The phi function method is one of the effective techniques used in deriving quadrature formulas. This method relies on phi functions to approximate the integral of a given function. The phi function method has gained popularity in numerical methods due to its ability to offer high precision in the integration process.

This paper discusses the principles of the phi function method, its mathematical background, and its application in deriving quadrature formulas. Currently, in the theory of constructing quadrature and cubature formulas, there are several key approaches that are employed to improve the accuracy and efficiency of numerical integration methods. These approaches aim to find the best formula for approximating integrals and multi-dimensional integrals. Below are the main approaches used today:

1. *Algebraic Approach*
   – *Goal:* To select the nodes and coefficients of quadrature and cubature formulas such that these formulas provide exact results for all functions within a specific set of functions.
   – *Key Feature:* This approach relies on the properties of the integrand (the function being integrated). Typically, the set of functions consists of algebraic polynomials, trigonometric polynomials, or rational functions with bounded degrees.
   – *Application:* For example, quadrature formulas for one-dimensional integrals or cubature formulas for multi-dimensional integrals are constructed to ensure exactness for polynomials or specific function classes.
2. *Probabilistic Approach*
   – *Goal:* This approach focuses on statistical methods to construct formulas.
   – *Key Feature:* It is based on the Monte Carlo method, which uses random sampling to estimate the values of integrals, particularly useful for high-dimensional integrals.
   – *Application:* The probabilistic approach is often employed in scenarios where the integrand is complex, irregular, or high-dimensional, and classical deterministic methods are less effective.
3. *Number-Theoretic Approach*
   – *Goal:* To utilize concepts from number theory in constructing cubature formulas.
   – *Key Feature:* This approach uses methods related to prime numbers, lattice points, and modular arithmetic to develop formulas that can efficiently approximate integrals, especially in multi-dimensional spaces.
   – *Application:* For example, using lattice rules or other number-theoretic techniques to create more efficient integration formulas with reduced errors.
4. *Functional Approach*
   – *Goal:* To view the integrals in the context of functional spaces and optimize the formulas using functional analysis techniques.
   – *Key Feature:* In this approach, the integrands are considered as elements of some Banach space (a type of vector space), and the difference between the actual integral and the approximation is treated as a continuous linear functional (called the error functional).
   – *Application:* The error of the formula is estimated by considering the norms of the error functional, and the optimal formula is obtained by minimizing these norms. This method is widely used in optimal quadrature theory and cubature theory, where the goal is to minimize the error by adjusting the parameters of the formulas.

Quadrature refers to the approximation of integrals in one-dimensional space, while cubature involves multi-dimensional integrals. The development of optimal formulas in all these approaches aims to minimize the error in numerical integration, especially when dealing with complex functions or high-dimensional integrals. Each approach offers unique advantages depending on the nature of the problem, the function to be integrated, and the dimensionality of the problem. Combining techniques from these approaches often leads to the development of highly efficient and accurate numerical integration methods.

The construction of quadrature formulas and the study of their error estimates based on the methods of functional analysis were first extensively developed in the scientific works of prominent mathematicians, particularly Nikolsky and Sard. Their contributions in the 20th century laid the theoretical foundation for modern numerical integration, particularly in how to systematically derive and analyze quadrature formulas and estimate the associated errors.

The works of S.M. Nikolsky, N.P. Korneichuk, N.E. Lushpay, T.N. Busarova, B. Boyanov, V.P. Motorny, A.A. Ligun, A.A. Zhensykbaev, K.I. Oskolkov, M.A. Chakhkiev, and T.A.Grankina focus on the issues of minimizing the norm of the error functional across coefficients and nodes in various spaces for the one-dimensional case. For instance, comprehensive results and an extensive bibliography are presented in S.M. Nikolsky's work [19].

It should be noted that there are several methods for constructing optimal formulas, such as the spline method, the φ-functions method, and the Sobolev method, which are based on minimizing the norm of the error functional over coefficients at fixed nodes. The works of A. Sard [20, 21], L.F. Meyers [17], G. Coman [6, 7], I.J. Schoenberg [22–25], S.D. Silliman [25], and P.Köhler [15], which are based on the spline method, as well as the works of A. Ghizzetti and A. Ossicini [9], F. Lanzara [16], T. Catinaş and G. Coman [5], utilizing the method of φ -functions, contributed to the development of optimal quadrature formulas in the space W_2^2.

In the context of constructing optimal cubature formulas using the Sobolev method, S.L.Sobolev's [30] results on finding the coefficients of optimal quadrature formulas helped generalize earlier studies that applied the spline method. The algorithm proposed by S.L. Sobolev in the $L_2^{(m)}$ space was later implemented in various scientific works by Z.Z. Zhamalov, F.Y.Zagirova, Kh.M. Shadimetov, A.R. Hayotov, and others. Recent results related to optimal formulas derived through the Sobolev method can be found, for example, in the works [1, 27].

It is important to note that the outcomes of this work are closely connected to the findings of the studies [2–4, 8, 10–12, 14, 26, 28], which focus on the development of optimal quadrature formulas using the Sobolev method. Specifically, in the work [13] the optimal quadrature formula with sigma parameter was studied.

2 Statement of the problem

In this work, we study the construction of an optimal quadrature formula using the method of φ -function. In this regard, we consider quadrature formulas of the form:

$$\int_a^b f(x)g(x)dx = \sum_{k=0}^{n} A_{0k} f(x_k) + \sum_{k=0}^{n} A_{1k} f'(x_k) + R_n(f), \tag{1}$$

where $f \in C^2([a, b])$, $g(x) \in L_1(a, b)$. $g(x)$ is the weight function and is supposed to be non-zero on a set of positive measure. Here A_{0k}, A_{1k}, x_k and are the coefficients and the nodes of the quadrature formula.

Let the nodes of the formula be located on the segment $[a, b]$ as follows

$$a = x_0 < x_1 < ... < x_n = b \tag{2}$$

and $R_n(f)$ is the error of formula (1). Suppose that the integrand function $f(x)$ is from the space $L_2^{(2)}(a,b)$, where $L_2^{(2)}(a,b)$ is the Sobolev space of quadratically integrable functions with the second-order derivative on the interval $[a,b]$. The inner product of any two functions f and g from this space is defined by the following formula

$$\langle f,\, g\rangle_{L_2^{(2)}(a,b)} = \int\limits_a^b f''(x)g''(x)dx. \tag{3}$$

This space is provided with the corresponding norm

$$\|f\|_{L_2^{(2)}} = \left(\int\limits_a^b \left(f''(x)\right)^2 dx\right)^{\frac{1}{2}}. \tag{4}$$

One of the important problem in the theory of quadrature formulas is the problem of the optimality of quadrature formulas relative to the error of this formula. In this paper, we consider the problem of optimality of a formula in the sense of Sard. We use the one-to-one correspondence between quadrature formulas and φ_k - functions in this. For conveniene, we introduce the multi-index notations

$$A_0 = (A_{00}, A_{01},\, ...,\, A_{0n}\,)\,, A_1 = (A_{10}, A_{11},\, ...,\, A_{1n}\,)\,, X = (x_0, x_1,\, ...,\, x_n\,)\,. \tag{5}$$

Defeniton 1. The quadrature formula (1) is called optimal in the sense of Nikolsky in a space H, if the value

$$F_n(H,\, A_0,\, A_1, X) = \sup_{f\in H} |R_n(f)|\,, \tag{6}$$

reaches its smallest value relative to A_0, A_1 and X, where A_0, A_1 and X are defined in (5).

Defeniton 2. The quadrature formula (5) is called optimal in the sense of Sard in a space H, if the quantity

$$F_n(H,\, A_0,\, A_1) = \sup_{f\in H} |R_n(f)|\,, \tag{7}$$

reaches its smallest value relative to A_0, A_1 for fixed X , where A_0, A_1 and X are defined in (5).

In this work, we solve the problem of constructing an optimal quadrature formula of the form (1) in the sense of Sard in the Sobolev space $L_2^{(2)}(a,b)$, i.e., let us find such coefficients of the formula (1) that give the smallest value to the quantity (1) for fixed X. In this case, we use the method of φ-function.

Next, the rest of this work is organized as follows. Section 3 describes the method of φ -function for constructing quadrature formulas of the form (1) in the space $L_2^{(2)}(a,b)$ and provides the relationship between the coefficients and φ-function. Section 4 is devoted to obtaining φ-function that give the smallest error value of a quadrature formula of the form (1). Using the obtained φ -function, the coefficients of the optimal quadrature formula of the form (1) are calculated.

3 The method of φ - function for constructing quadrature formulas in the space $L_2^{(2)}(a,b)$

In this section, we explain the method of φ - function for constructing optimal quadrature formulas of the form (1) in the sense of Sard in the space $L_2^{(2)}(a,b)$. For more details on the φ - function method see, for instance, [5, 9, 16].

Let functions f be from the space $L_2^{(2)}(a,b)$ and for a given positive integer n the nodes of the quadrature formula under consideration are located as in (5). Then for each subinterval $[x_k,\, x_{k-1}]$, $k = 1, 2,\, ...,\, n$, consider the functions φ_k, $k = 1,\, 2, ...,\, n$, having the following property

$$\varphi''_k(x) = g(x), \ \ k = 1, 2,\, ...,\, n. \tag{8}$$

Then the function φ is defined as follows

$$\varphi|_{[x_{k-1},\, x_k]} = \varphi_k(x), \ \ k = 1, 2, ..., n. \tag{9}$$

Let us introduce the following notation

$$I(f) := \int_a^b f(x)g(x)dx, \tag{10}$$

$$Q_n(f) := \sum_{k=0}^{n} A_{0k} f(x_k) + \sum_{k=0}^{n} A_{1k} f'(x_k) + R_n(f). \tag{11}$$

Now, using the property of additivity of a definite integral, taking into account equalities (8), from (10), we have (see, e.g., [5, 9, 16])

$$I(f) = \int_a^b f(x)g(x)dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi''_k(x) f(x)dx =$$

$$= \sum_{k=0}^{n} f(x)\varphi'_k(x)\Big|_{x_{k-1}}^{x_k} - \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi'_k(x) f'(x)dx =$$

$$= \sum_{k=0}^{n} f(x)\varphi'_k(x)|_{x_{k-1}}^{x_k} - \sum_{k=1}^{n} f'(x)\varphi(x)|_{x_{k-1}}^{x_k} + \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k(x) f''(x)dx =$$

$$= \sum_{k=0}^{n} A_{0k} f(x_k) + \sum_{k=0}^{n} A_{1k} f'(x_k) + R_n(f).$$

From here we have

$$I(f) = \sum_{k=0}^{n} A_{0k} f(x_k) + \sum_{k=0}^{n} A_{1k} f'(x_k) + R_n(f). \tag{12}$$

From (12) we get

$$\begin{gathered} A_{00} = -\varphi'_1(x_0), \\ A_{0k} = \varphi'_k(x_k) - \varphi'_{k+1}(x_k), \quad k = \overline{1, n-1}, \\ A_{0n} = \varphi'_n(x_n), \\ A_{10} = \varphi_1(x_0), \\ A_{1k} = -\left[\varphi_k(x_k) - \varphi_{k+1}(x_k)\right], \quad k = \overline{1, n-1}, \\ A_{1n} = -\varphi_n(x_n) \end{gathered} \tag{13}$$

and the error of the formula has the form

$$R_n(f) = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k(x) f''(x) dx. \tag{14}$$

Remark 8 Knowing the function φ from (13) we can find the coefficients $A_{0k},\ A_{1k},\ k = 0,\ 1, ...,\ n$. This method of constructing a quadrature formula is called the method of φ – function (see, [5, 9, 16]).

4 The optimality problem for a quadrature formula

In this section, we discuss the problem of optimality of a quadrature formula of the firm (1) in the space $L_2^{(2)}(a, b)$. Using the Cauchy-Schwarz inequality for the absolute value of the error (14) of the quadrature formula (1) we have the following

$$|R_n(f)| \leqslant \|f\|_{L_3^{(2)}} \left(\sum_{k=1}^{n} \int_{xk-1}^{x_k} [\varphi_i(x)]^2 dx \right)^{\frac{1}{2}}. \tag{15}$$

It should be noted that here the task of constructing an optimal quadrature formula of the form (1) in the sense of Sard in the space $L_2^{(2)}(a, b)$ is the task of finding the coefficients $A_0 = (A_{00}, A_{01},\ ...,\ A_{0n}\ ), A_1 = (A_{10}, A_{11},\ ...,\ A_{1n}\ )$ (for fixed nodes $X = (x_0, x_1, ..., x_n)$ satisfying the condition (2)) giving the smallest value to the quantity

$$F_n(A_0, A_1) = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} [\varphi_k(x)]^2 dx. \tag{16}$$

In turn this problem is equivalent to finding functions $\varphi_k(x), k = 1, 2, ..., n$ satisfying equation (8)and giving the smallest value to the quantity (16) on each interval $[x_{k-1},\ x_k]\,,\ k = 1,\ 2,\ ...,\ n$. Next, for the beginning we find the functions φ_k on each interval $[x_{k-1},\ x_k]$, for $k = 1, 2, ..., n$ that give the smallest value to the quantity (16) and then using the formulas (13) we calculate coefficients $A_{0k}, A_{1k},\ \ k = 0,\ 1, ..., n$ of the optimal quadrature formula (1).

4.1 Finding fuctions $\varphi_k(x)$

Now we are engaged in finding the equations on each interval $[x_{k-1},\ x_k]$ for $x_k = a + kh,\ h = \frac{b-a}{n},\ k = 1, 2, ..., n$, which are the solution to the equation:

$$y'' = \frac{1}{\sqrt{x-a}}. \tag{17}$$

Now we can solve equation(17). Integrating both sides of (17) we have

$$y = \frac{4}{3}(x-a)^{\frac{3}{2}} + Cx + D. \tag{18}$$

Next, on each interval $[x_{k-1},\ x_k]$, $k = 1, 2, ..., n$ we take fuctions $\varphi_k(x)$ in the form (18), i.e.

$$\varphi_k\,(x) = \frac{4}{3}(x-a)^{\frac{3}{2}} + C_k x + D_k. \tag{19}$$

From here we conclude to find the functions $\varphi_k(x)$ we need to find such constatnts C_k, D_k, $k=1,2,...,n$, which give the smallest value to the quantity (16) on each of intervals $[x_{k-1},\, x_k]$ for $k=1,2,...,n$. Next, we find C_k, D_k such that the integral of the square of the function $\varphi_k(x)$ defined by equality (19) on the interval $[x_{k-1},\, x_k]$ takes the smallest value. In this regard, consider the following functions

$$F_k(C_k, D_k) = \int\limits_{x_{k-1}}^{x_k} \varphi_k^2(x)dx, \ \ k = 1,\, 2,\, ...,\, n.$$

Then from here, taking into account (19), we have

$$F_k(C_k,\, D_k) = \int\limits_{x_{k-1}}^{x_k} \left(\frac{4}{3}(x-a)^{\frac{3}{2}} + C_k x + D_k\right)^2 dx =$$

$$= \int\limits_{x_{k-1}}^{x_k} \left(\frac{16}{3}(x-a)^2 + {C_k}^2x^2 + {D_k}^2 + \frac{8}{3}(x-a)^{\frac{3}{2}}C_k x + \frac{8}{3}(x-a)^{\frac{3}{2}}D_k + 2C_kD_kx\right)dx.$$

Then calculating the first order derivatives of the functions $F_k(C_k,\, D_k)$ with respect to C_k, D_k and equating them to zero, we have

$$\frac{\partial F_k}{\partial C_k} = 2C_k\int\limits_{x_{k-1}}^{x_k} x^2dx + \frac{8}{3}\int\limits_{x_{k-1}}^{x_k} (x-a)^{\frac{3}{2}}xdx + 2D_k\int\limits_{x_{k-1}}^{x_k} xdx = 0,$$

$$\frac{\partial F_k}{\partial D_k} = 2C_k\int\limits_{x_{k-1}}^{x_k} xdx + \frac{8}{3}\int\limits_{x_{k-1}}^{x_k} (x-a)^{\frac{3}{2}}dx + 2D_k\int\limits_{x_{k-1}}^{x_k} dx = 0.$$

From the last equalities we get the following

$$C_k = -\frac{48h^{\frac{1}{2}}}{105}\left[k^{\frac{5}{2}}(7-4k) - (k-1)^{\frac{5}{2}}(3+4k)\right],$$

$$D_k = -\frac{16h^{\frac{1}{2}}}{105}\Big[k^{\frac{5}{2}}(12k^2h - 25kh + 13h + 19a - 44ak) -$$
$$-(k-1)^{\frac{5}{2}}(12ak + 12k^2h + 9a + 3kh - h)\Big]. \tag{20}$$

It is easy to check that this values of C_k, D_k give the smallest value to the function $F_k(C_k,\, D_k)$ on the interval $[x_{k-1},\, x_k]$. Then, taking into account (20), from (19) we have

$$\varphi_k(x) = \frac{4}{3}(x-a)^{\frac{3}{2}} - \frac{48xh^{\frac{1}{2}}}{105}\left[k^{\frac{5}{2}}(7-4k) - (k-1)^{\frac{5}{2}}(3+4k)\right] -$$
$$-\frac{16h^{\frac{1}{2}}}{105}\Big[k^{\frac{5}{2}}(12k^2h - 25kh + 13h + 19a - 44ak) -$$
$$-(k-1)^{\frac{5}{2}}(12ak + 12k^2h + 9a + 3kh - h)\Big], \quad k = 1,\, 2,\, \ldots,\, n. \tag{21}$$

4.2 Calculation of coefficients of the optimal quadrature formula

Now, using (21), from the formulas (13) we calculate the coefficients A_{0k}, A_{1k}, $k = = 0, 1, ..., n$ of the optimal quadrature formula of the form (1). First, let's calculate A_{00}. From (13), taking into account φ_1, we have

$$A_{00} = \frac{48}{35} h^{\frac{1}{2}}. \tag{22}$$

Now let's calculate the coefficients A_{1k}. From (13), using $\varphi_k(x)$ for $k = 1, 2, ..., n.$, we have

$$A_{0k} = \frac{48h^{\frac{1}{2}}}{105} \left[k^{\frac{5}{2}}(7-4k) + (k-1)^{\frac{5}{2}}(3+4k) \right] +$$
$$+\frac{48h^{\frac{1}{2}}}{105} \left[(k+1)^{\frac{5}{2}}(3-4k) + k^{\frac{5}{2}}(7+4k) \right], \; k = 1,\, 2,\, ...,\, n-1.$$

Now we calculate A_{0n} from (13)

$$A_{0n} = 2(nh)^{\frac{1}{2}} - \frac{48h^{\frac{1}{2}}}{105} \left[(n-1)^{\frac{5}{2}}(3+4n) - n^{\frac{5}{2}}(4n-7) \right], \, k = 1,\, 2,\, ...n-1.$$

The next step is to calculate A_{1k}, $k = 0, 1, ..., n$. First A_{10} from (13)

$$A_{10} = \frac{16}{105} h^{\frac{3}{2}}.$$

We calculate the values of A_{1k}, $k = 0, 1, ..., n-1$ from (13)

$$A_{1k} = -\frac{48(a+kh)h^{\frac{1}{2}}}{105} \left[(k+1)^{\frac{5}{2}}(3-4k) + k^{\frac{5}{2}}(7+4k) \right] -$$
$$-\frac{16h^{\frac{1}{2}}}{105} \left[(k+1)^{\frac{5}{2}}(12ka + 12k^2h - 9a - 3kh - h) - \right.$$
$$\left. -k^{\frac{5}{2}}(12k^2h + 14h + 12ak + 21a + 27kh) \right] +$$
$$+\frac{48h^{\frac{1}{2}}}{105} \left[k^{\frac{5}{2}}(7-4k) + (k-1)^{\frac{5}{2}}(3+4k) \right] +$$
$$+\frac{16h^{\frac{1}{2}}}{105} \left[k^{\frac{5}{2}}(12k^2h - 27kh + 14h - 21a - 44ak) - \right.$$
$$\left. -(k-1)^{\frac{5}{2}}(12ak + 12k^2h + 9a + 9kh - h) \right]. \tag{23}$$

Finally, let's calculate the last coefficient A_{1n}. Then, from (13), taking into account (21), we obtain

$$A_{1n} = -\frac{4}{3}(nh)^{3/2} + \frac{48(a+nh)h^{1/2}}{105} \left[n^{5/2}(7-4n) - (n-1)^{5/2}(3-4n) \right] +$$
$$+\frac{16h^{1/2}}{105} \left[n^{5/2}(12n^2h + 14h - 44an - 21a - 27nh) - \right.$$
$$\left. -(n-1)^{5/2}(12an + 9a + 12n^2h + 3nh - h) \right]. \tag{24}$$

Thus, summing up the results of (22), (23) and (24), we obtain the following main theorem of this work.

Theorem 4.1. In the space $L_2^{(2)}(a,b)$ for each fixed positive integer n, there is a unique quadrature formula that is optimal in the sense of Sard of the form

$$\int_a^b f(x)g(x)dx = \sum_{k=0}^{n} A_{0k}f(x_k) + \sum_{k=0}^{n} A_{1k}f'(x_k) + R_n(f),$$

with coefficients

$$A_{00} = \frac{48}{35}h^{\frac{1}{2}},$$

$$A_{0k} = \frac{48h^{\frac{1}{2}}}{105}\left[k^{\frac{5}{2}}(7-4k) \ +(k-1)^{\frac{5}{2}}(3+4k)\right] +$$

$$+\frac{48h^{\frac{1}{2}}}{105}\left[(k+1)^{\frac{5}{2}}(3-4k) \ +k^{\frac{5}{2}}(7+4k)\right], \ k=1,\ 2,\ ...,\ n-1,$$

$$A_{0n} = 2(nh)^{\frac{1}{2}} - \frac{48h^{\frac{1}{2}}}{105}\left[(n-1)^{\frac{5}{2}}(3+4n) - n^{\frac{5}{2}}(4n-7)\right],$$

$$A_{10} = \frac{16}{105}h^{\frac{3}{2}},$$

$$A_{1k} = -\frac{48(a+kh)h^{\frac{1}{2}}}{105}\left[(k+1)^{\frac{5}{2}}(3-4k)+ \ k^{\frac{5}{2}}(7+4k)\right] -$$

$$-\frac{16h^{\frac{1}{2}}}{105}\Big[(k+1)^{\frac{5}{2}}(12ka+12k^2h-9a-3kh-h) -$$

$$- k^{\frac{5}{2}}\left(12k^2h+14h+12ak+21a+27kh\right)\Big] +$$

$$+\frac{48h^{\frac{1}{2}}}{105}\left[k^{\frac{5}{2}}(7-4k)+(k-1)^{\frac{5}{2}}(3+4k)\right] +$$

$$+\frac{16h^{\frac{1}{2}}}{105}\Big[k^{\frac{5}{2}}(12k^2h-27kh+14h-21a-44ak) -$$

$$- (k-1)^{\frac{5}{2}}(12ak+12k^2h+9a+9kh-h)\Big], \ k=1,2,...,n-1,$$

$$A_{1n} = -\frac{4}{3}(nh)^{\frac{3}{2}} + \frac{48(a+nh)h^{\frac{1}{2}}}{105}\left[n^{\frac{5}{2}}(7-4n) \ - \ (n-1)^{\frac{5}{2}}(3-4n)\right] +$$

$$+\frac{16h^{\frac{1}{2}}}{105}\Big[n^{\frac{5}{2}}(12n^2h+14h-44an-21a-27nh) -$$

$$-(n-1)^{\frac{5}{2}}(12an+9a+12n^2h+3nh-h)\Big],$$

for fixed $x_k = a+kh$, $k=0,\ 1,\ ...,n$.

5 Conclusion

In this work, we constructed an optimal quadrature formula in the Sobolev space. Here the quadrature sum consist of a linear combination of the values $f(x_k)$ of the function $f(x)$ at the nodes $x_k \in [a,\ b]$, $x_k = a+kh$ in $[a,\ b]$ and its first derivative, where $a = x_0 < x_1 < ... < x_n = b$. The error of the quadrature formula under consideration is estimated from above using the product of the norm of the integrand and the L_2 norm of the particular $\varphi(x)$ function from the Sobolev space. Moreover, the function φ is determined by an unknown factor on each subinterval. The optimal quadrature formula is obtained by choosing these factors, which provide the smallest value of the L_2-norm of the function. In this work, we found such a φ function. In our upcoming work, we will focus on determining the error and calculating the norm of the function. The φ function played a crucial role in determining the optimal parameters, leading to a formula that minimizes the error and maximizes efficiency. This approach not only enhances the accuracy of numerical integration for exponential functions but also lays the groundwork for further optimization techniques in numerical analysis. The methodology presented can be extended to other classes of functions, providing a valuable tool for high-precision computations in both theoretical and applied mathematics. Future work will involve applying this optimal quadrature formula to more complex functions and exploring its potential in broader numerical analysis contexts.

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ОПТИМАЛЬНАЯ КВАДРАТУРНАЯ ФОРМУЛА, ТОЧНАЯ ДЛЯ ЭКСПОНЕНЦИАЛЬНОЙ ФУНКЦИИ

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Численное интегрирование определённых интегралов играет важную роль как в фундаментальных, так и в прикладных науках. Точность приближённого вычисле-

ния интегралов зависит от исходных данных и конкретных условий, которые накладывают различные требования на получаемые результаты. Традиционные методы численного анализа определённых интегралов, такие как квадратурные формулы, разработанные Грегори, Ньютоном-Котесом, Эйлером, Гауссом, Марковым и другими, являются хорошо изученными. С середины XX века развивается теория построения оптимальных формул численного интегрирования с использованием вариационных методов. Следует отметить, что существуют оптимальные квадратурные формулы в смысле, определённом Никольским и Сардом. Настоящая работа посвящена задаче построения оптимальной квадратурной формулы в подходе Сарда. В этом процессе применяется метод φ-функций. Погрешность формулы оценивается путём интегрирования квадрата φ-функции из определённого гильбертова пространства. Затем выбирается такая φ-функция, чтобы интеграл от её квадрата на данном интервале принимал минимальное значение. Наконец, на основе полученной φ-функции определяются коэффициенты оптимальной квадратурной формулы.

Ключевые слова: Гильбертово пространство, метод φ-функции, оптимальная квадратурная формула, погрешность.

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