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AN OPTIMAL QUADRATURE FORMULA FOR HADAMARD-TYPE HYPERSINGULAR INTEGRALS WITH HIGH OSCILLATION IN THE SOBOLEV SPACE

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This paper addresses the problem of constructing optimal quadrature formulas for the numerical evaluation of *Hadamard-type hypersingular integrals* with *hihg oscillation.* Such integrals frequently arise in the analysis of boundary-value problems for partial differential equations, particularly in hyperbolic and wave-type models, as well as in various applied fields such as elasticity theory, fluid dynamics, and electromagnetic scattering. Our primary focus is on deriving the *analytical expressions for the coefficients* of an optimal quadrature formula tailored to the structure of the singular and oscillatory kernel. The goal is to achieve *minimal error* in a rigorous sense by constructing formulas that are *optimal in the sense of Sard*, meaning that they minimize the worst-case error over a specified function class. To this end, we develop an optimal quadrature formula of the form (2) in the *Sobolev space*, applying the *Sobolev method for constructing optimal quadrature formulas.* This method leverages variational principles and operator-theoretic tools to systematically minimize the norm of the error functional. The resulting formulas are well-adapted to handle the challenging combination of hypersingularity and oscillation, and they offer both theoretical optimality and practical computational advantages.

Keywords: Sobolev space, Sobolev method, singular integrals, hypersingular integrals, error functional, optimal quadrature formulas, error estimation.

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1 Introduction

Many problems in science and engineering naturally reduce to singular integral equations [1]. In particular, plane problems are often reduced to one-dimensional singular integral equations. The theory of one-dimensional singular integral equations is well developed. It is known that the solutions of such equations can often be expressed in terms of singular integrals. Hadamard-type hypersingular integrals with high oscillations is very difficult to approximate using any standard methods. Therefore, the accurate numerical approximation of hypersingular integrals [2–4] is a significant and relevant problem in numerical analysis. Let $f(x)$ is a smooth function on the interval $[0,1]$

Definition 1. The Hadamard-type finite part of the integral

$$I = f.p.\int_0^1 \frac{f(x)dx}{(x-t)^2},$$

is defined as

$$f.p.\int_a^b \frac{f(x)dx}{(x-t)^2} = \lim_{\varepsilon\to 0}\left[\int_a^{t-\varepsilon} \frac{f(x)dx}{(x-t)^2} + \int_{t+\varepsilon}^b \frac{f(x)dx}{(x-t)^2} + \frac{\xi(x)}{\varepsilon}\right],$$

where $\xi(x)$ is a function chosen so as to provide the existence of the limit above, $t \in (0,1)$ denotes the singularity. The notation f.p. represents the finite part of the integral.

In some cases, it is more convenient to use the following equivalent definition.

Definition 2. Let $f(x) \in L_2^{(1)}(0,1)$. First order hypersingular integral is defined by

$$\int_0^1 \frac{f(x)dx}{(x-t)^2} = \frac{d}{dt}\int_0^1 \frac{f(x)dx}{(x-t)}.$$

We define $f(x)$ as $f(x) = e^{2\pi i\omega x}\varphi(x)$. The integral exists if the function $\varphi(x)$ satisfies Hölder's condition on $[0,1]$ which means

$$\int_0^1 \frac{e^{2\pi i\omega x}\varphi(x)}{(x-t)^2}\,dx = \int_0^1 \frac{\varphi(x)-\varphi(t)}{(x-t)^2}\,e^{2\pi i\omega x}dx - \varphi(t)\int_0^1 \frac{e^{2\pi i\omega x}}{(x-t)^2}\,dx.$$

There has been increasing interest in finite part integrals with high oscillation. Several methods have been developed for the calculation of this type of integrals. They are the fast Hermite interpolation [2], Gauss-related quadrature rule [3], interpolating $f(x)$ at practical Chebyshev points and subtracting out the singularity [4]. In the following works [5-10] quadrature formulas constructe for approximating Cauchy type singular and hypersingular integrals in the Sobolev space. We construct optimal quadrature formula for Hadamard-type of singular integrals.

The paper is organized as follows:

In Section 2, We find the norm of the error functional by using extremal function introduced by S.L.Sobolev.

In Section 3, We find optimal coefficients of the quadrature formula by Sobolev method.

In Section 4, we give the conclusions of our study.

2 The statement of the problem

In the present work we construct the optimal quadrature formula for the integral

$$I = f.p.\int_0^1 \frac{e^{2\pi i\omega x}\varphi(x)dx}{(x-t)^2}. \tag{1}$$

Consider the quadrature formula for (1)

$$\int_0^1 \frac{e^{2\pi i\omega x}\varphi(x)dx}{(x-t)^2} \cong \sum_{\beta=0}^{N} A_\beta\,\varphi(x_\beta), \tag{2}$$

where $0 < t < 1$, $e^{2\pi i\omega x}$ is the weight function, $h = \frac{1}{N}$, N is a natural number, $x_\beta = h\beta$ are the nodes of quadrature formula, $\beta = \overline{0,N}$, and $\varphi(x)$ belongs to the Sobolev space $L_2^{(1)}(0,1)$, $\omega \neq 0$. A_β are the coefficients of the quadrature formula (2).

The difference between the integral and the quadrature sum

$$(R_N, \varphi) = \int\limits_0^1 \frac{e^{2\pi i\omega x}\varphi(x)dx}{(x-t)^2} - \sum_{\beta=0}^{N} A_\beta\, \varphi\left(x_\beta\right), \tag{3}$$

is called the error of the quadrature formula (2).

The space $L_2^{(1)}(0,1)$ is a Hilbert space and the inner product of two functions is defined as

$$\langle \varphi, \psi \rangle = \int\limits_0^1 \varphi'(x)\ \overline{\psi}'(x)dx,$$

where the function $\overline{\psi}$ is the complex conjugate function of the function ψ. The norm of the function is defined by the formula

$$\|\varphi\|^2_{L_2^{(1)}} = \langle \varphi, \varphi \rangle = \int\limits_0^1 \varphi'(x)\overline{\varphi'(x)}\, dx.$$

From (3) we get the following result

$$(R_N, \varphi) = \int_{-\infty}^{\infty} \frac{\varepsilon_{[0,1]}(x)\cdot e^{2\pi i\omega x}}{(x-t)^2}\varphi(x)\, dx - \sum_{\beta=0}^{N} A_\beta \delta(x-x_\beta)\varphi(x)\, dx =$$

$$= \int_{-\infty}^{\infty} \left(\frac{\varepsilon_{[0,1]}(x)\cdot e^{2\pi i\omega x}}{(x-t)^2} - \sum_{\beta=0}^{N} A_\beta \delta(x-x_\beta) \right) \varphi(x)\, dx.$$

The error functional that corresponds to the error has the form

$$R_N(x) = \frac{\varepsilon_{[0,1]}(x)\cdot e^{2\pi i\omega x}}{\left(x-t\right)^2} - \sum_{\beta=0}^{N} A_\beta\, \delta\left(x-x_\beta\right), \tag{4}$$

where $\varepsilon_{[0,1]}(x)$ is the indicator function of the interval $[0,1]$, $\delta(x)$ is Dirac's delta-function. The quadrature formula is exact on constant functions, i.e.,

$$(R_N, 1) = 0 \Rightarrow \int_0^1 R_N(x)dx = 0 \Rightarrow \sum_{k=0}^{N} A_\beta = \int_0^1 \frac{e^{2\pi i\omega x}}{(x-t)^2}dx. \tag{5}$$

From $A_\beta = A_\beta^R + iA_\beta^I$ and $e^{2\pi i\omega x} = \cos(2\pi\omega x) + i\sin(2\pi\omega x)$ equalities we get

$$\begin{aligned} \sum_{k=0}^{N} A_\beta^R &= \int_0^1 \frac{\cos(2\pi\omega x)}{(x-t)^2}\, dx, \\ \sum_{k=0}^{N} A_\beta^I &= \int_0^1 \frac{\sin(2\pi\omega x)}{(x-t)^2}\, dx. \end{aligned} \tag{6}$$

We find the form of the norm of the error functional by using Riesz's theorem on the general form of a linear continuous functional in a Hilbert space.

Theorem [11]. (Riesz representation theorem) *Let V be a real or complex Hilbert space, $\ell \in V'$. Then there is a unique $u \in V$ for which $\ell(v) = \langle v, u\rangle \quad \forall v \in V$. In addition, $\|\ell\| = \|u\|$.*

That is, for any linear continuous functional defined on a Hilbert space, there exists a unique element ψ in this space such that the following identity holds for all $\varphi \in L_2^1(0,1)$:

$$|(R_N, \psi)| = \langle \psi_{R_N}, \varphi\rangle,$$

and

$$\|R_N(\psi)\|_{L_2^{(1)*}} = \|\psi_{R_N}\|_{L_2^{(1)}}. \tag{7}$$

So, in order to find the norm of the error functional, we need to determine the function ψ_ℓ. For this, we utilize the extremal function concept introduced by S.L. Sobolev.

Definition (S.L. Sobolev) [12]. The following equality

$$(R_n, \psi_{R_N}) = \|R_N\|_{L_2^{(1)*}(0,1)} \cdot \|\psi_{R_N}\|_{L_2^{(1)}(0,1)}, \tag{8}$$

is satisfied, and the function ψ_{R_N} that achieves this is called *the extremal function.*

From expressions (7) and (8), we have the following equality:

$$(R_N, \psi_{R_N}) = \|R_N\|_{L_2^{(1)*}(0,1)} \cdot \|\psi_{R_N}\|_{L_2^{(1)}(0,1)} = \|\psi_{R_N}\|^2_{L_2^{(1)}(0,1)} = \|R_N\|^2_{L_2^{(1)*}(0,1)}.$$

Thus, the norm of the error functional is calculated as:

$$\|R_N\|^2_{L_2^{(1)*}(0,1)} = (R_N, \psi_{R_N}). \tag{9}$$

In the space $L_2^{(1)}(0,1)$, the extremal function ψ_ℓ corresponding to the error functional ℓ was found by S.L. Sobolev and is given by:

$$\psi_{R_N}(x) = -(R_N(x) * G_1(x) + P_0(x)), \tag{10}$$

where

$$G_1(x) = \frac{|x|}{2} = \frac{x \cdot \mathrm{sign}(x)}{2}.$$

$G_1(x)$ is the fundamental solution of the 2^{nd} order differential operator, satisfying:

$$\frac{d^2}{dx^2} G_1(x) = \delta(x),$$

and $\overline{R_N}(x) - R_N(x)$ is the adjoint functional, and P_0 is a zero-degree algebraic polynomial(constant).

Now, in the space $L_2^{(2)}(0,1)$, we can compute the norm of the error functional using the extremal function from (10). From equations (9) and (5), we get:

$$\|R_N\|^2_{L_2^{(2)*}(0,1)} = (R_N(x), \psi_{R_N(x)}) = \int_0^1 R_N(x) \cdot \big(\overline{R_N}(x) * G_1(x) + P_0(x)\big)\, dx =$$

$$= \int_0^1 R_N(x) \cdot \big(\overline{R_N}(x) * G_1(x)\big)\, dx + \int_0^1 R_N(x) \cdot P_0(x) dx = (R_N(x), \overline{R_N}(x) * G_1(x)).$$

2.1 Compute the Convolution $R_N(x) * G_1(x)$

We are given:

$$G_1(x) = \frac{x \cdot \operatorname{sign}(x)}{2}, \quad R_N(x) = \frac{e^{2\pi i\omega x}}{(x-t)^2} \cdot \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} A_\beta \delta(x - x_\beta).$$

Then the convolution is:

$$(\overline{R_N(x)} * G_1)(x) = \int_{-\infty}^{\infty} \overline{R_N(y)}\, G_1(x-y)\, dy =$$
$$= \int_0^1 \frac{e^{-2\pi i\omega y}(x-y)\operatorname{sign}(x-y)}{2(y-t)^2}\, dy - \sum_{k=0}^{N} \overline{A_\beta} \cdot \frac{|x-x_\beta| \cdot \operatorname{sign}(x-x_\beta)}{2}.$$

2.2 Compute the Norm $\|R_N\|^2$

Using the inner product definition:

$$\|R_N\|^2_{L_2^{(1)*}(0,1)} = \langle R_N, R_N * G_1 \rangle = \int_0^1 R_N(x)(-\overline{R_N(x)} * G_1(x))\, dx.$$

We obtain following form of the norm

$$\|R_N\|^2 = -\int_0^1 \left[\frac{e^{2\pi i\omega x}}{(x-t)^2} \cdot \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} A_\beta \delta(x-x_\beta) \right],$$
$$\left(\int_0^1 \frac{e^{-2\pi i\omega y}(x-y)\operatorname{sign}(x-y)}{2(y-t)^2} dy - \sum_{\beta=0}^{N} \overline{A_\beta} \cdot \frac{|x-x_\beta| \cdot \operatorname{sign}(x-x_\beta)}{2} \right) dx,$$

After calculations we get

$$\|R_N\|^2_{L_2^{(1)*}(0,1)} = -\Bigg[\int_0^1 \int_0^1 \frac{e^{2\pi i\omega(x-y)}|x-y|}{2(x-t)^2(y-t)^2}\, dx\, dy - \sum_{\gamma=0}^{N} \bar{A}_\gamma \int_0^1 \frac{e^{2\pi i\omega x}|x-h\gamma|}{2(x-t)^2}\, dx -$$
$$- \sum_{\beta=0}^{N} A_\beta \int_0^1 \frac{e^{-2\pi i\omega y}|y-h\beta|}{2(y-t)^2}\, dy + \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} A_\beta \overline{A}_\gamma \frac{|h\beta-h\gamma|}{2} \Bigg].$$

After we expand all complex terms and compute the integrals, the imaginary parts cancel due to symmetry and we obtain the following as one of the main results.

Theorem 1. The norm of the error functional (4) of the quadrature formula (2) in the Sobolev space

$$\|R_N\|^2_{L_{0,1}^{(1)*}} = -\Bigg[\int_0^1 \int_0^1 \frac{\cos\big(2\pi\omega(x-y)\big)|x-y|}{2(x-t)^2(y-t)^2}\, dx\, dy -$$
$$- \sum_{\beta=0}^{N} \int_0^1 \frac{A_\beta^R \cos(2\pi\omega x) + A_\beta^I \sin(2\pi\omega x)}{(y-t)^2} |y-h\beta|\, dy + \tag{11}$$
$$+ \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} \left(A_\beta^R A_\gamma^R + A_\beta^I A_\gamma^I\right) \frac{|h\beta-h\gamma|}{2} \Bigg],$$

where $A_\beta = A_\beta^R + iA_\beta^I$. So the norm is real.

3 Optimal coefficients of the quadrature formula

Eventually, using the expressions above, the squared norm of the error functional $\|R_n\|^2$ is written as a functional of the coefficients A_β^R and A_β^I. According to constraints (5) and (6), the minimum of this functional with respect to A_β^R and A_β^I can be found using the Lagrange multipliers method. To do this, we construct the Lagrange function:

$$\Psi\left(A_\beta^R, A_\beta^I, \lambda^R, \lambda^I\right) = \|R_N\|^2 - 2\lambda^R\left(\int_0^1 \frac{\cos(2\pi\omega x)}{(x-t)^2}dx - \sum_{\beta=0}^N A_\beta^R\right) -$$
$$-2\lambda^I\left(\int_0^1 \frac{\sin(2\pi\omega x)}{(x-t)^2}dx - \sum_{\beta=0}^N A_\beta^I\right).$$

Then, we derive this function with respect to the unknowns A_k^R, A_k^I, λ^R, λ^I and equate to zero to obtain a system of linear equations.

Taking derivatives and setting them to zero we get following the system of equations

$$\begin{cases} \sum_{\beta=0}^N A_\beta \frac{|h\beta-h\gamma|}{2} + \lambda = f_\omega(h\beta), \quad \beta = 0,1,\ldots,N, \\ \sum_{\beta=0}^N A_\beta = g_\omega, \end{cases} \tag{12}$$

where

$$\begin{cases} g_\omega = \int_0^1 \frac{e^{2\pi i\omega x}}{(x-t)^2}\,dx = \\ = e^{2\pi i\omega t}\left(\frac{1}{t^2-t} + 2\pi i\omega \ln\frac{1-t}{t} + \sum_{n=1}^\infty \frac{(2\pi i\omega)^{n+1}((1-t)^n-(-t)^n)}{n(n+1)!}\right), \\ f_{1,\omega}(h\beta) = \frac{1}{2}e^{2\pi i\omega t}\left[(1+2\pi i\omega)\ln\left(\frac{t(1-t)}{(h\beta-t)^2}\right) - \frac{1}{1-t} + \frac{1}{t} + \frac{2}{h\beta-t} + \right. \\ \left. + \sum_{n=1}^\infty \frac{(2\pi i\omega)^n}{n!}\left(1+\frac{2\pi i\omega}{n(n+1)}\right)((1-t)^n + (-t)^n - 2(h\beta-t)^n)\right]. \end{cases} \tag{13}$$

Solution of the system give the optimal coefficients $A_\beta, \beta = \overline{0,N}$ for the quadrature formula minimizing the error in the $L_2^{(1)}(0,1)$ Sobolev space.

The system of equations given by (12) forms a solvable linear system with $(N + + 2)$ unknowns, which gives the conditional minimum of $\|R_N\|^2_{L_2^{(1)}}$. But finding its roots from using ordinary methods are useless. So, we apply the Sobolev method to find the coefficients of the optimal quadrature formula to the system (12).

We rewrite the system (12) in convolution form as:

$$\begin{cases} A_\beta * \dfrac{|h_\beta|}{2} + \lambda = f_{1,\omega}(h_\beta), \quad \beta = 0,\ldots,N \\ A_\beta = 0, \quad \text{for } \beta < 0, \beta > N \\ \sum_{\beta=0}^N A_\gamma = g_\omega. \end{cases}$$

We proceed with further simplification

$$A_\beta = hD_1(h_\beta) * u(h\beta),$$
$$U(h\beta) = A_\beta * \frac{|h\beta|}{2} + \lambda.$$

Since the convolution with the discrete operator $D_1(h\beta)$ [13] is determined, we need to find the form of $u(h\beta)$ for $\beta \in (-\infty, \infty)$, given that

$$\beta = 0, 1, 2, \ldots, N, \quad u(h\beta) = f_{1,\omega}(h\beta),$$

Definition 3. We define the convolution of two discrete argument functions $\varphi(h\beta)$ and $\psi(h\beta)$ as the scalar product as follows:

$$\varphi(h\beta) * \psi(h\beta) = [\varphi(h\gamma), \psi(h\beta - h\gamma)] = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma)\psi(h\beta - h\gamma).$$

In addition, to carry out calculations, we need the discrete analogue of the differential operator $\frac{d^2}{dx^2}$, denoted by $D_1(h\beta)$, which is defined by the following formula [13]:

$$D_1(h\beta) = \begin{cases} 0, & |\beta| \geqslant 2, \\ h^{-2}, & |\beta| = 1, \\ -2h^{-2}, & \beta = 0. \end{cases}$$

Together with this, the discrete analogue of the differential operator $\frac{d^2}{dx^2}$, i.e., $D_1(h\beta)$, satisfies the following properties:

$$hD_1(h\beta) * \left|\frac{h\beta}{2}\right| = \delta(h\beta),$$

$$D_1(h\beta) * (h\beta)^k = 0, \quad k = 0, 1.$$

Here, $\delta(h\beta)$ is the discrete delta function and

$$\delta(h\beta) = \begin{cases} 0, & \beta \neq 0, \\ 1, & \beta = 0. \end{cases}$$

Now, let us solve system (12). Then,

$$\beta \leqslant 0, \quad u(h\beta) = A_\beta * \frac{|h\beta|}{2} + \lambda = \sum_{\gamma=1}^{\infty} A_\gamma \cdot \frac{|h\beta - h\gamma|}{2} + \lambda =$$
$$= \sum_{\gamma=0}^{N} A_\gamma \cdot \frac{-(h\beta - h\gamma)}{2} + \lambda - \frac{h\beta}{2} \underbrace{\sum_{\gamma=0}^{N} A_\gamma}_{g_\omega} + \underbrace{\frac{1}{2}\sum_{\gamma=0}^{N} A_\gamma(h\gamma)}_{\mu} + \lambda,$$

where

$$\mu = \frac{1}{2}\sum_{\gamma=0}^{N} A_\gamma(h\gamma).$$

Then we have:

$$\beta < 0 \Rightarrow u(h\beta) = -\frac{h\beta}{2} \cdot g_\omega + \mu + \lambda, \quad a_0^- = \mu + \lambda,$$

$$\beta > N \Rightarrow u(h\beta) = \frac{h\beta}{2} \cdot g_\omega - \mu - \lambda, \quad a_0^+ = -\mu + \lambda.$$

Therefore,

$$u(h\beta) = \begin{cases} -\frac{h\beta}{2} \cdot g_\omega + a_0^-, & \beta < 0, \\ f_{1,\omega}(h\beta), & 0 \leqslant \beta \leqslant N, \\ \frac{h\beta}{2} \cdot g_\omega + a_0^+, & \beta > N. \end{cases} \tag{14}$$

From equality (14) and under the conditions $\beta = 0$ and $\beta = N$ that, we obtain the following relations

$$\beta = 0 \Rightarrow a_0^- = f_{1,\omega}(0),$$

$$\beta = N \Rightarrow a_0^+ = f_{1,\omega}(hN) - \frac{g_\omega}{2}.$$

Once the full expression for the function $u(h\beta)$ is obtained, the coefficients A_β of the quadrature formula (2) can be explicitly determined.

$$A_\beta = hD_1(h\beta) * u(h\beta) = h \sum_{\gamma=-\infty}^{\infty} \lambda_1(h\beta - h\gamma) \cdot u(h\gamma) =$$

$$= h \left[\sum_{\gamma=-\infty}^{-1} D_1(h\beta - h\gamma)u(h\gamma) + \sum_{\gamma=0}^{N} D_1(h\beta - h\gamma) f_{1,\omega}(h\gamma) + \sum_{\gamma=N+1}^{\infty} D_1(h\beta - h\gamma)u(h\gamma) \right] =$$

$$= h[\sum_{\gamma=1}^{\infty} D_1(h\beta + h\gamma)u(-h\gamma) +$$

$$+ \sum_{\gamma=0}^{N} D_1(h\beta - h\gamma) f_{1,\omega}(h\gamma) + \sum_{\gamma=1}^{\infty} D_1(h\beta - h(N+\gamma))u(h(N+\gamma))].$$

At $\beta = 0$:

$$A_0 = h^{-1} \left(\frac{h}{2} g_\omega + f_{1,\omega}(h) - f_{1,\omega}(0) \right),$$

for $1 \leqslant \beta \leqslant N-1$:

$$A_\beta = h^{-1} \left(f_{1,\omega}(h(\beta-1)) - 2f_{1,\omega}(h\beta) + f_{1,\omega}(h(\beta+1)) \right),$$

for $\beta = N$:

$$A_N = h^{-1} \left(f_{1,\omega}(h(N-1)) + f_{1,\omega}(hN) - g_\omega \right).$$

It follows from the above that the following result holds.

Theorem 2. The coefficents A_k of quadrature formula (2) with the error functional (4) are determined by the following expressions:

$$A_k = \begin{cases} h^{-1} \left(\frac{h}{2} g_\omega + f_{1,\omega}(h) - f_{1,\omega}(0) \right), & \beta = 0 \\ h^{-1} \left(f_{1,\omega}(h(\beta-1)) - 2f_{1,\omega}(h\beta) + f_{1,\omega}(h(\beta+1)) \right), & 1 \leqslant \beta \leqslant N-1 \\ h^{-1} \left(f_{1,\omega}(h(N-1)) - f_{1,\omega}(hN) + \frac{h}{2} g_\omega \right), & \beta = N. \end{cases} \tag{15}$$

4 Conclusion

In the present paper, we constructed an optimal quadrature formula in the sense of Sard for the numerical evaluation of Hadamard-type hypersingular integrals with highly oscillatory kernels in the Sobolev space $L_2^{(1)}(0,1)$. The main result consists in deriving explicit analytical expressions(15) for the coefficients of the optimal quadrature formula of the form (2), ensuring minimal error in the corresponding norm.

By the Sobolev method, we derived quadrature formulas that are optimal in the sense of Sard in the Sobolev space $L_2^{(1)}(0,1)$. The main contributions of this work are twofold: first, we derived an explicit expression for the norm of the error functional associated with the quadrature formula using the concept of extremal functions and the convolution with the fundamental solution of a differential operator; second, we obtained closed-form analytical expressions for the optimal coefficients of the quadrature formula through the application of variational techniques and discrete operator theory.

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ОПТИМАЛЬНАЯ КВАДРАТУРНАЯ ФОРМУЛА ДЛЯ ГИПЕРСИНГУЛЯРНЫХ ИНТЕГРАЛОВ ТИПА АДАМАРА С ВЫСОКОЙ ОСЦИЛЛЯЦИЕЙ В ПРОСТРАНСТВЕ СОБОЛЕВА

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В данной работе рассматривается проблема построения оптимальных квадратурных формул для численного вычисления гиперсингулярных интегралов типа Адамара с высокой осцилляцией. Такие интегралы часто возникают при анализе краевых задач для дифференциальных уравнений в частных производных, особенно в гиперболических и волновых моделях, а также в различных прикладных областях, таких как теория упругости, гидродинамика и рассеяние электромагнитных волн. Нашей основной задачей является вывод аналитических выражений для коэффициентов оптимальной квадратурной формулы, адаптированной к структуре сингулярного и осциллирующего ядра. Цель состоит в достижении минимальной погрешности в строгом смысле путем построения формул, которые являются оптимальными в смысле Сарда, то есть минимизируют наихудшую погрешности в заданном классе функций. С этой целью мы разрабатываем оптимальную квадратурную формулу вида (2) в пространстве Соболева, применяя метод Соболева для построения оптимальных квадратурных формул. Этот метод использует вариационные принципы и операторные инструменты для систематического минимизирования нормы функционала погрешности. Полученные формулы хорошо приспособлены для работы со сложной комбинацией гиперсингулярности и осцилляции и обладают как теоретической оптимальностью, так и практическими вычислительными преимуществами.

Ключевые слова: пространство Соболева, метод Соболева, сингулярные интегралы, гиперсингулярные интегралы, функционал погрешности, оптимальные квадратурные формулы, оценка погрешности

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