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NUMERICAL SOLUTION OF PLANE PROBLEMS OF THE THEORY OF ELASTICITY DIRECTLY IN STRESSES

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Usually, the solution of a plane problem of the theory of elasticity in stresses is reduced to solving a biharmonic equation for the Airy stress function. In this paper, two (A and B) variants of plane boundary value problems of the theory of elasticity are formulated directly in terms of stresses. In the first case (A), the boundary value problem consists of two equilibrium equations and one Beltrami-Michell equation with the corresponding boundary and additional boundary conditions. In the formulation of the second plane boundary value problem (B), in contrast to the first, the equations of equilibrium differentiated with respect to x and y , respectively, are used. Symmetric finite-difference equations are constructed and the known Timoshenko-Goodier problem of stretching a rectangular plate with a parabolic load is solved for comparison. The discrete analogs of boundary value problems A and B are composed by the finite-difference method and the iterative method and the marching method are used to solve them. By comparing the numerical results of boundary value problems, A and B obtained by two methods, the validity of the formulated boundary value problems and the reliability of the obtained results are ensured.

Keywords: stress, Beltrami-Michell equations, equilibrium equations, difference equations, iterative method, marching method.

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1 Introduction

The formulation of the boundary value problem of the theory of elasticity with respect to stresses is an actual problem of applied mathematics and mathematical modeling. It is known that model equations with respect to stresses are usually formulated within the framework of the Saint-Venant compatibility equations. The Saint-Venant conditions, using Hooke's law and the equation of equilibrium, can be written in the form of Beltrami-Michell equations. It is known that the boundary value problem in stresses consists of six Beltrami-Michell differential equations, three equilibrium equations, and three boundary conditions. In this case, problems of overdetermination of the number of equations and insufficiency of boundary conditions arise. These problems underlie the research in the field of formulation and numerical solution of boundary value problems with respect to stresses.

The boundary value problem in stresses, in the three-dimensional case, was solved in the work of Filonchenko-Borodich [9] by the variational method based on the expansion

of the stress tensor in a series of trigonometric functions, etc. [11, 12, 20, 21]. In [16], the Beltrami-Michell equations were solved using Fourier integral transformation. In [17, 19], for the solution of three-dimensional problems in stresses, functions of stress of the Maxwell and Morera type were used, similar to the two-dimensional case.

In the works of Pobedrya [2, 3, 5], a new formulation of the boundary value problem in stresses is proposed, where the equilibrium equations are considered as boundary conditions. In a special case, the new formulation follows from the classical Beltrami-Michell equations. The works [10, 13] are also devoted to the study of the new boundary value problem of Pobedri in stresses. The issues of the existence and uniqueness of the solution of boundary value problems in stresses, and the equivalence of new and classical formulations are considered in the following works [8, 15]. Dynamic boundary value problems with respect to stresses are considered in the works of Konovalov [14]. Two-dimensional problems in stresses are usually solved by introducing the potential of the stress function of Eri satisfying the equilibrium equations and is reduced to solving the biharmonic equation [4, 17]. In [22], a method for solving the three-dimensional problem of elastoplastic deformation of a transversely isotropic body using the finite element method (FEM) is presented. As an example, a solution to the problem of deforming a fiber composite in the shape of a rectangle with a system of round holes is given.

This work is devoted to the formulation and numerical solution of the plane problem of the theory of elasticity directly with respect to stresses. The plane problem based on the Beltrami-Michell equations is formulated in two variants. At the same time, it consists of two equilibrium equations, and one Beltrami-Michell equation, and two boundaries and one additional conditions on each side of the rectangle. Discrete equations are constructed by the finite-difference method, and are solved by the iterative method and the marching method.

2 Boundary value problem in elasticity in stresses

It is known that the boundary value problem of the theory of elasticity in stresses consists of the equilibrium equation, the Beltrami-Michell equation, and the corresponding boundary conditions [1, 2]. In the absence of body forces, the boundary value problem takes the form

$$\sigma_{ij,j} = 0, \quad (1)$$

$$\nabla^2 \sigma_{ij} + \frac{1}{1+\nu} S_{,ij} = 0, \quad S = \sigma_{kk}, \quad (2)$$

$$\sigma_{ij} n_j|_{\Sigma_1} = S_i. \quad (3)$$

Where S_i — is the surface load, Σ — is the surface of the volume V consisting of two parts Σ_1 —, Σ_2 —, n_i — is the components of the external normal to the surface Σ , ν — is the Poisson's ratio, ∇^2 — is the Laplace operator.

The system of equations (1-3) consists of six Beltrami-Michell equations and three equilibrium equations with the corresponding boundary conditions. Clearly, the number of differential equations is equal to nine, and the boundary conditions are three, and the boundary value problem is open.

According to the studies of Pobedrya [2, 5, 8], considering the equilibrium equation on the boundary of a given domain as a boundary condition, i.e.

$$\sigma_{ij,j}|_{\Sigma} = 0. \quad (4)$$

This, in combination with the equations (1-3), represents the boundary value problem of the theory of elasticity in stresses. Note that considering the equilibrium equation on the boundary of the given domain V allows us to formulate the boundary value problem of the theory in a closed form with respect to stresses.

3 Plane problem of the theory of elasticity in stresses

In the case of plane stress ($\sigma_{zz} = \sigma_{zy} = \sigma_{zx}$), equations (1-3) take the following form:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \quad (5)$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0, \quad (6)$$

$$\frac{\partial^2 \sigma_{xy}}{\partial x^2} + \frac{\partial^2 \sigma_{xy}}{\partial y^2} + \frac{1}{1+v} \left(\frac{\partial^2 \sigma_x}{\partial x \partial y} + \frac{\partial^2 \sigma_y}{\partial x \partial y} \right) = 0, \quad (7)$$

with boundary conditions

$$\begin{aligned} (\sigma_x n_1 + \sigma_{xy} n_2)|_{\Gamma} &= S_1, \\ (\sigma_{xy} n_1 + \sigma_y n_2)|_{\Gamma} &= S_2, \end{aligned} \quad (8)$$

and, with additional boundary conditions

$$\begin{aligned} \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} \right) \Big|_{\Gamma} &= 0, \\ \left(\frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} \right) \Big|_{\Gamma} &= 0. \end{aligned} \quad (9)$$

Thus, equations (5-9) represent a plane boundary value problem of the theory of elasticity in stresses (Problem A), which can be solved numerically directly with respect to stresses. By differentiating equations (5) and (6), respectively, with respect to x and the second with respect to y , the following boundary value problem in stresses (Problem B) can be formulated.

$$\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = 0, \quad (10)$$

$$\frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = 0, \quad (11)$$

$$\frac{\partial^2 \sigma_{xy}}{\partial x^2} + \frac{\partial^2 \sigma_{xy}}{\partial y^2} + \frac{1}{1+v} \left(\frac{\partial^2 \sigma_x}{\partial x \partial y} + \frac{\partial^2 \sigma_y}{\partial x \partial y} \right) = 0. \quad (12)$$

But the boundary conditions are the same as in problem A (8-9). Note that the order of approximation of the finite-difference equations (10-11) is higher than that of (5-6). Let problems A and B be considered in a rectangular region and the plate is under tensile forces on two sides along the OX axis, the other sides are free of loads, i.e.

$$\text{for } x = \pm a : \sigma_x|_{x=\pm a} = S, \quad \sigma_{xy}|_{x=\pm a} = 0, \quad (13)$$

$$\text{for } y = \pm b : \sigma_y|_{y=\pm b} = 0, \quad \sigma_{xy}|_{y=\pm b} = 0. \quad (14)$$

In this case, the additional boundary conditions (9) for a rectangular domain are as follows

$$\text{for } x = \pm a : \left[\frac{\partial \sigma_y}{\partial y} \right]_{x=\pm a} = - \left[\frac{\partial \sigma_{xy}}{\partial x} \right]_{x=\pm a}, \quad (15)$$

$$\text{for } y = \pm b : \left[\frac{\partial \sigma_x}{\partial x} \right]_{y=\pm b} = - \left[\frac{\partial \sigma_{xy}}{\partial y} \right]_{y=\pm b}. \quad (16)$$

Thus, it is not difficult to see from the relations (13-16) that there are three boundary conditions on each side of the rectangle.

4 Finite-difference equations for the plane problem of the theory of elasticity in stresses

Let boundary value problem A (5-9) be considered in a rectangle with side lengths $l_1 = 2a$ and $l_2 = 2b$. By dividing the lengths of the sides of the rectangle into N_k , we can find the grid step $h_k = l_k/N_k$, $k = 1, 2$ along the coordinate axes. Then the coordinates of the nodal points have the form $x_i = -a + ih_1$ ($i = \overline{0, n_1}$), $y_j = -b + jh_2$ ($j = \overline{0, n_2}$) [3, 6]. Replacing the derivatives in equations (5-7) with the corresponding finite difference relations, we can find that.

$$\frac{\sigma_{i+1,j}^x - \sigma_{i,j}^x}{h_1} + \frac{\sigma_{i,j+1}^{xy} - \sigma_{i,j-1}^{xy}}{2h_2} = 0, \quad (17)$$

$$\frac{\sigma_{i,j+1}^y - \sigma_{i,j}^y}{h_2} + \frac{\sigma_{i+1,j}^{xy} - \sigma_{i-1,j}^{xy}}{2h_1} = 0, \quad (18)$$

$$\begin{aligned} & \frac{\sigma_{i+1,j}^{xy} - 2\sigma_{ij}^{xy} + \sigma_{i-1,j}^{xy}}{h_1^2} + \frac{\sigma_{i,j+1}^{xy} - 2\sigma_{ij}^{xy} + \sigma_{i,j-1}^{xy}}{h_2^2} + \\ & + \frac{1}{1+v} \left\{ \frac{\sigma_{i+1,j+1}^x - \sigma_{i-1,j+1}^x - \sigma_{i+1,j-1}^x + \sigma_{i-1,j-1}^x}{4h_1h_2} + \right. \\ & \left. + \frac{\sigma_{i+1,j+1}^y - \sigma_{i-1,j+1}^y - \sigma_{i+1,j-1}^y + \sigma_{i-1,j-1}^y}{4h_1h_2} \right\} = 0. \end{aligned} \quad (19)$$

Replacing the first term in equation (17) with the left boundary condition, we can find that

$$\frac{\sigma_{i,j}^x - \sigma_{i-1,j}^x}{h_1} + \frac{\sigma_{i,j+1}^{xy} - \sigma_{i,j-1}^{xy}}{2h_2} = 0. \quad (20)$$

From the equations (17) and (20), we can find that

$$2\sigma_{ij}^x = \sigma_{i+1,j}^x + \sigma_{i-1,j}^x. \quad (21)$$

Similarly, from equation (18), we find that

$$2\sigma_{ij}^y = \sigma_{i,j+1}^y + \sigma_{i,j-1}^y. \quad (22)$$

Solving the finite-difference equations (19-21) and (22) with respect to σ_{ij}^x , σ_{ij}^y and σ_{ij}^{xy} and taking the following renaming's of $\sigma_{ij}^x = \sigma_{xij}^{(k)}$, $\sigma_{ij}^y = \sigma_{yij}^{(k)}$ and $\sigma_{ij}^{xy} = \sigma_{xyij}^{(k)}$ the following relationships can be found:

$$\sigma_{xij}^{(k+1)} = \frac{\sigma_{xi+1,j}^{(k)} + \sigma_{xi-1,j}^{(k)}}{2}, \quad (23)$$

$$\sigma_{yij}^{(k+1)} = \frac{\sigma_{yi,j+1}^{(k)} + \sigma_{yi,j-1}^{(k)}}{2}, \quad (24)$$

$$\begin{aligned}
\sigma_{xyij}^{(k+1)} = & \left\{ \frac{\sigma_{xyi+1,j}^{(k)} + \sigma_{xyi-1,j}^{(k)}}{h_1^2} + \frac{\sigma_{xyi,j+1}^{(k)} + \sigma_{xyi,j-1}^{(k)}}{h_2^2} + \right. \\
& + \frac{1}{1+v} \left\{ \frac{\sigma_{xi+1,j+1}^{(k)} - \sigma_{xi-1,j+1}^{(k)} - \sigma_{xi+1,j-1}^{(k)} + \sigma_{xi-1,j-1}^{(k)}}{4h_1h_2} + \right. \\
& \left. \left. + \frac{\sigma_{yi+1,j+1}^{(k)} - \sigma_{yi-1,j+1}^{(k)} - \sigma_{yi+1,j-1}^{(k)} + \sigma_{yi-1,j-1}^{(k)}}{4h_1h_2} \right\} \right\} / \left\{ \frac{2}{h_1^2} + \frac{2}{h_2^2} \right\}.
\end{aligned} \tag{25}$$

where the indices are changed in the interior points, i.e., $1 \leq i \leq n_1 - 1$, $1 \leq j \leq n_2 - 1$, and k is iteration number.

According to (13) and (14), the boundary conditions are as follows:

$$\begin{aligned}
for : x = \mp a : \\
\sigma_{x0j}^{(0)} = S, \sigma_{xy0j}^{(0)} = 0, \\
\sigma_{xN_1j}^{(0)} = S, \sigma_{xyN_1j}^{(0)} = 0,
\end{aligned} \tag{26}$$

$$\begin{aligned}
for : y = \mp b : \\
\sigma_{yi0}^{(0)} = 0, \sigma_{xyi0}^{(0)} = 0, \\
\sigma_{yin_2}^{(0)} = 0, \sigma_{xyin_2}^{(0)} = 0.
\end{aligned} \tag{27}$$

Additional boundary conditions (13-14) with respect to nodal points have the form: for $x = \mp a$:

$$\begin{aligned}
\sigma_{y0j}^{(0)} = \sigma_{y0,j+1}^{(0)} + h_2 \frac{\sigma_{xy1j}^{(0)} - \sigma_{xy0j}^{(0)}}{h_1}, \\
\sigma_{y0j}^{(0)} = \sigma_{yn_1,j+1}^{(0)} + h_2 \frac{\sigma_{xyn_1,j}^{(0)} - \sigma_{xyn_1-1,j}^{(0)}}{h_1},
\end{aligned} \tag{28}$$

for $y = \mp b$:

$$\begin{aligned}
\sigma_{xi0}^{(0)} = \sigma_{xi+1,0}^{(0)} + h_1 \frac{\sigma_{xyi1}^{(0)} - \sigma_{xyi0}^{(0)}}{h_2}, \\
\sigma_{xin_2}^{(0)} = \sigma_{xi+1,n_2}^{(0)} + h_1 \frac{\sigma_{xyin_2}^{(0)} - \sigma_{xyi,n_2-1}^{(0)}}{h_2}.
\end{aligned} \tag{29}$$

The finite-difference equations (23-29) can be solved using an iterative method.

The finite-difference equations of problem B (10-12), similarly to problem A, can be written in the following form:

$$\frac{\sigma_{i+1,j}^x - 2\sigma_{i,j}^x + \sigma_{i-1,j}^x}{h_1^2} + \frac{\sigma_{i+1,j+1}^{xy} - \sigma_{i+1,j-1}^{xy} - \sigma_{i-1,j+1}^{xy} + \sigma_{i-1,j-1}^{xy}}{4h_1h_2} = 0, \tag{30}$$

$$\frac{\sigma_{i,j+1}^y - 2\sigma_{i,j}^y + \sigma_{i,j-1}^y}{h_2^2} + \frac{\sigma_{i+1,j+1}^{xy} - \sigma_{i-1,j+1}^{xy} - \sigma_{i+1,j-1}^{xy} + \sigma_{i-1,j-1}^{xy}}{4h_1h_2} = 0, \tag{31}$$

$$\begin{aligned}
& \frac{\sigma_{i+1,j}^{xy} - 2\sigma_{ij}^{xy} + \sigma_{i-1,j}^{xy}}{h_1^2} + \frac{\sigma_{i,j+1}^{xy} - 2\sigma_{ij}^{xy} + \sigma_{i,j-1}^{xy}}{h_2^2} + \\
& + \frac{1}{1+v} \left\{ \frac{\sigma_{i+1,j+1}^x - \sigma_{i-1,j+1}^x - \sigma_{i+1,j-1}^x + \sigma_{i-1,j-1}^x}{4h_1h_2} + \right. \\
& \left. + \frac{\sigma_{i+1,j+1}^y - \sigma_{i-1,j+1}^y - \sigma_{i+1,j-1}^y + \sigma_{i-1,j-1}^y}{4h_1h_2} \right\} = 0.
\end{aligned} \tag{32}$$

Solving the finite difference equations (30, 31) and (32) for σ_{ij}^x , σ_{ij}^y and σ_{ij}^{xy} , and taking the following notations $\sigma_{ij}^x = \sigma_{ij}^{(k)}$, $\sigma_{ij}^y = \sigma_{ij}^{(k)}$, and $\sigma_{ij}^{xy} = \sigma_{ij}^{(k)}$, we can find, respectively

$$\sigma_{xj}^{(k+1)} = \left(\frac{\sigma_{xi+1,j}^{(k)} + \sigma_{xi-1,j}^{(k)}}{h_1^2} + \frac{\sigma_{xyi+1,j+1}^{(k)} - \sigma_{xyi+1,j-1}^{(k)} - \sigma_{xyi-1,j+1}^{(k)} + \sigma_{xyi-1,j-1}^{(k)}}{4h_1h_2} \right) / \left(\frac{2}{h_1^2} \right), \quad (33)$$

$$\sigma_{yi,j}^{(k+1)} = \left(\frac{\sigma_{yi+1,j}^{(k)} + \sigma_{yi-1,j}^{(k)}}{h_2^2} + \frac{\sigma_{xyi+1,j+1}^{(k)} - \sigma_{xyi+1,j-1}^{(k)} - \sigma_{xyi-1,j+1}^{(k)} + \sigma_{xyi-1,j-1}^{(k)}}{4h_1h_2} \right) / \left(\frac{2}{h_2^2} \right), \quad (34)$$

$$\begin{aligned} \sigma_{xyij}^{(k+1)} = & \left\{ \frac{\sigma_{xyi+1,j}^{(k)} + \sigma_{xyi-1,j}^{(k)}}{h_1^2} + \frac{\sigma_{xyi,j+1}^{(k)} + \sigma_{xyi,j-1}^{(k)}}{h_2^2} + \right. \\ & + \frac{1}{1+v} \left\{ \frac{\sigma_{xi+1,j+1}^{(k)} - \sigma_{xi-1,j+1}^{(k)} - \sigma_{xi+1,j-1}^{(k)} + \sigma_{xi-1,j-1}^{(k)}}{4h_1h_2} + \right. \\ & \left. \left. + \frac{\sigma_{yi+1,j+1}^{(k)} - \sigma_{yi-1,j+1}^{(k)} - \sigma_{yi+1,j-1}^{(k)} + \sigma_{yi-1,j-1}^{(k)}}{4h_1h_2} \right\} \right\} / \left\{ \frac{2}{h_1^2} + \frac{2}{h_2^2} \right\}. \end{aligned} \quad (35)$$

Boundary conditions have the same form (26-28). Finite-difference equations (33-35) can be solved by the iterative method.

To solve problem B, the method of alternating directions can be applied within the framework of the marching method. For this purpose, the finite-difference equations (33-35) can be written in the following form:

$$\begin{aligned} a_i \sigma_{i+1,j}^x + b_i \sigma_{i,j}^x + c_i \sigma_{i-1,j}^x &= f_{ij}^x, \\ \dot{a}_i \sigma_{i,j+1}^y + \dot{b}_i \sigma_{i,j}^y + \dot{c}_i \sigma_{i,j-1}^y &= \dot{f}_{ij}^y, \\ \ddot{a}_i \sigma_{i+1,j}^{xy} + \ddot{b}_i \sigma_{i,j}^{xy} + \ddot{c}_i \sigma_{i-1,j}^{xy} &= \ddot{f}_{ij}^{xy}, \\ \ddot{a}_i \sigma_{i+1,j}^{xy} + \ddot{b}_i \sigma_{i,j}^{xy} + \ddot{c}_i \sigma_{i-1,j}^{xy} &= \ddot{f}_{ij}^{xy}. \end{aligned} \quad (36)$$

Thus, the solution of problem B is reduced to the sequential application of the marching method for the system of four equations (36). In this case, the initial parameters of the marching method are determined according to the boundary conditions (26-28).

5 Numerical examples

This section is devoted to the numerical solution and justification of the validity of the formulated boundary value problems A and B with respect to stresses. The finite difference equations of boundary value problems A and B were solved using the iterative and the marching methods. The problem of stretching a rectangular plate with a parabolic load was solved and the numerical results obtained by different methods were compared.

Let a rectangular plate of size $l_1 = 2a$, $l_2 = 2b$ be under the action of a parabolic load applied to the opposite sides perpendicular to the axis OX . The other sides are free of loads. This problem, using the Airy function and the condition of minimizing the strain energy, was solved in the work of Timoshenko and Goodier [4].

In our case, this problem is formulated as boundary value problems A (5-7) and B (10-12) with corresponding boundary (8) and additional (9) boundary conditions. Boundary (8) in the case of a tensile parabolic load has the

$$\begin{aligned} \text{for : } x = \pm a : \sigma_{xx} &= S_0 \left(1 - \frac{y^2}{a^2} \right), \sigma_{xy} = 0, \\ \text{for : } y = \pm b : \sigma_{yy} &= 0, \sigma_{xy} = 0. \end{aligned} \quad (37)$$

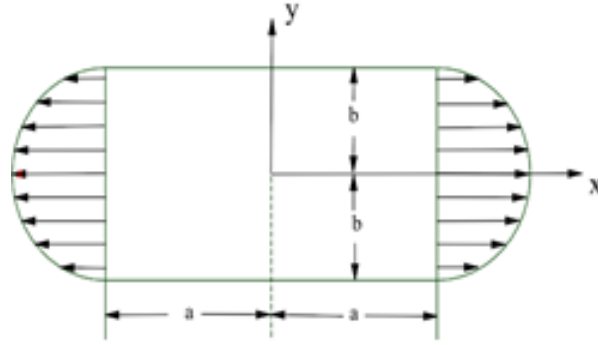


Figure 1 Rectangular plate under parabolic load

In this case, the initial data had the following values $\lambda = 0.78$, $\mu = 0.5$, $a = 1$, $b = 1$, $h_1 = h_2 = 0.2$.

The first two rows of Table 1 show the values of the stresses σ_{xx} at the section $x = 0$, obtained by solving problems A [12] and B by the iterative method. The third row of Table 1 shows the numerical results of problem B obtained by the method of alternating directions within the framework of the marching method. The fourth row shows the solutions found by Timoshenko-Goodier [4] by the variational method. The closeness of the results obtained shows the validity of the formulated boundary value problems A and B in stresses, as well as the methods of their numerical solution. Taking into account the symmetry conditions, the table shows the numerical results for one quarter of the rectangle.

Table 1. Values of the stresses

<i>Problems $x=0$;</i>	<i>$y=-1$</i>	<i>$y=-0.8$</i>	<i>$y=-0.6$</i>	<i>$y=-0.4$</i>	<i>$y=-0.2$</i>	<i>$y=0$</i>
<i>Problem A [12]</i>	0.3371	0.3371	0.5992	0.7865	0.8988	0.9363
<i>Problem B</i>	0.3221	0.3257	0.5726	0.8589	0.8589	0.8947
<i>The marching method</i>	0.3404	0.5166	0.6536	0.7915	0.8302	0.8798
<i>Timoshenko [4]</i>	0.3404	0.5166	0.6536	0.7515	0.8102	0.8298

6 Conclusion

Within the framework of the Beltrami-Michell equations, two planar boundary value problems (A and B) of the theory of elasticity in stresses are formulated. The first boundary value problem A consists of two equilibrium equations, and one Beltrami-Michell equation with the corresponding boundary and additional boundary conditions. The additional boundary conditions are obtained on the basis of the equilibrium equations. In the case of the planar boundary value problem B, the differentiated equilibrium equations are used in the formulation of the boundary value problem, respectively with respect to x and y . The discrete equations are composed by the finite-difference method and solved by the iterative method and the successive application of the marching method.

The numerical solution of the problem of the equilibrium of a rectangular plate under the action of a parabolic load applied to the opposite sides is solved. The validity of the formulated problems and the reliability of the numerical results are ensured by comparison with the known solutions of Timoshenko-Goodier.

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СРАВНИТЕЛЬНЫЙ АНАЛИЗ ЧИСЛЕННЫХ МЕТОДОВ РЕШЕНИЯ ЗАДАЧ ТЕОРИИ УПРУГОСТИ В НАПРЯЖЕНИЯХ

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Обычно, решение плоской задачи теории упругости в напряжениях сводится к решению бигармонического уравнения относительно функции напряжений Эри. В данной работе, сформулированы два (А и В) варианта плоских краевых задач теории упругости непосредственно относительно напряжений. В первом случае (А) краевая задача состоит из двух уравнений равновесия, и одного уравнения Бельтрами-Мичелла с соответствующими граничными и дополнительными граничными условиями. При формулировке второй плоской краевой задачи (В), в отличие от первой использованы продифференцированные по x и y , соответственно уравнения равновесия. Построены симметричные конечно-разностные уравнения и для сравнения решена известная задача Тимошенко-Гудьера о растяжении прямоугольной пластины параболической нагрузкой. Дискретные аналоги краевых задач А и В составлены конечно-разностным методом и для решения применены итерационный метод и метод прогонки. Сравнением численных результатов краевых задач А и В, полученных двумя методами, обеспечивается справедливость сформулированных краевых задач и достоверность полученных результатов.

Ключевые слова: термоупругость, условие совместности Сен-Венана, деформация, конечно-разностный метод, явная и неявная схемы, метод прогонки, краевая задача.

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Содержание

<i>Халдусигитов А., Адамбаев У., Тилозов О., Рахмонова Р., Махмадиерова М.</i> Сравнительный анализ численных методов решения задач теории упругости в напряжениях	8
<i>Нуралиев Ф.М., Тохиров Б.Н.</i> Комплексное математическое моделирование термо-электро-магнито-упругих процессов в анизотропных тонких пластинах сложной формы на основе ме- тода RFM	17
<i>Нормуродов Ч.Б., Зиякулова Ш.А.</i> Численное моделирование изгиба тонкой пластины с применением дискрет- ного варианта метода предварительного интегрирования	26
<i>Равшанов Н., Журабоева О., Боборахимов Б., Шарипов Х.</i> Моделирование распространения загрязняющих веществ в атмосфере с уче- том рельефа и метеорологических условий	38
<i>Саидов У., Жураев И., Туракулов Ж.</i> Моделирование процесса фильтрования малоконцентрированного раствора через пористую среду	47
<i>Муминов С.Ю.</i> Построение автомодельного решения системы нелинейных дифференциаль- ных уравнений, представляющих задачи взаимной диффузии.	56
<i>Ахмедов Д.М., Буваширов Д.С.</i> Оптимальная квадратурная формула для гиперсингулярных интегралов ти- па Адамара с высокой осцилляцией в пространстве Соболева	65
<i>Алоев Р.Д., Алимова В.</i> Исследование экспоненциальной устойчивости численного решения гипербо- лической системы с отрицательными нелокальными характеристическими скоростями	75
<i>Шадиметов Х.М., Нуралиев Ф.А., Едилбекова Р.М.</i> Система для нахождения оптимальных коэффициентов квадратурных фор- мул типа Эрмита с производными третьего порядка	88
<i>Нормуродов Ч.Б., Джураева Н.Т., Норматова М.М.</i> Исследование динамики производных дифференциального уравнения чет- вертого порядка с малым параметром при старшей производной	97
<i>Шадиметов Х.М., Нуралиев Ф.А., Миркомиллов Д.М.</i> Оптимальные квадратурные формулы для приближенного вычисления быст- роосциллирующих интегралов	110
<i>Игнатъев Н.А., Рамазонов Ш.Ш.</i> Отношение связанности в метрических алгоритмах классификации и анализ его свойств	122

Contents

<i>Khaldjigitov A., Adambaev U., Tilovov O., Rakhmonova R., Makhmadiygorova M.</i>	
Numerical solution of plane problems of the theory of elasticity directly in stresses	8
<i>Nuraliyev F.M., Tokhirov B.N.</i>	
Comprehensive mathematical modeling of thermo-electro-magneto-elastic processes in anisotropic thin plates of complex shape based on the RFM method	17
<i>Normurodov Ch.B., Ziyakulova Sh.A.</i>	
Numerical modeling of thin plate bending using a discrete version of the pre-integration method	26
<i>Ravshanov N., Juraboeva O., Boborakhimov B., Sharipov Kh.</i>	
Modeling the dispersion of pollutants in the atmosphere, accounting for terrain and meteorological conditions	38
<i>Saidov U., Juraev I., Turakulov J.</i>	
Modeling the process of filtering a low-concentration solution through a porous medium	47
<i>Muminov S.Y.</i>	
Construction of a self-similar solution to mutual diffusion problems.	56
<i>Akhmedov D.M., Buvashero D.S.</i>	
An optimal quadrature formula for Hadamard-type hypersingular integrals with high oscillation in the Sobolev space	65
<i>Aloev R.D., Alimova V.</i>	
Investigation of the exponential stability of the numerical solution of a hyperbolic system with negative nonlocal characteristic velocities	75
<i>Shadimetov Kh.M., Nuraliev F.A., Edilbekova R.M.</i>	
System for finding optimal coefficients of Hermite-type quadrature formulas with third-order derivatives	88
<i>Normurodov Ch.B., Juraeva N.T., Normatova M.M.</i>	
Study of the dynamics of derivatives of a fourth-order differential equation with a small parameter at the highest derivative	97
<i>Shadimetov X.M., Nuraliyev F.A., Mirkomilov D.M.</i>	
Optimal quadrature formulas for approximate calculation of fast oscillating integral	110
<i>Ignatiev N.A., Ramazonov Sh.Sh.</i>	
Relationship in metric classification algorithms and analysis of its properties	122