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ON AN OPTIMAL INTERPOLATION FORMULA WITH DERIVATIVE IN A HILBERT SPACE

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This study is devoted to the development of a system of equations for determining the coefficients of optimal interpolation formulas that integrate derivative information within a Hilbert space framework. Conventional interpolation methods, which rely solely on pointwise function values, often prove inadequate for functions exhibiting complex or rapidly varying behavior. To overcome this limitation, the proposed approach incorporates derivative data into the interpolation process, enhancing both the stability and accuracy of the resulting formulas. By formulating the interpolation problem within a Hilbert space, we establish a robust framework for deriving these optimal coefficients. The core analytical contribution lies in the formulation of a system of equations, derived through variational principles and leveraging tools such as the Riesz representation theorem and convolution operations. Solving this system enables the explicit computation of the optimal coefficients, which are readily applicable to practical interpolation tasks. This methodology is particularly significant in numerical analysis, especially in scenarios where positional data, directional vectors, or object velocities are available, as the inclusion of derivative information is both intuitive and critical. Furthermore, the approach is applicable to data approximation, signal processing, and computational contexts requiring function reconstruction from sampled or noisy data.

Keywords: optimal coefficients, derivative-based interpolation, Hilbert space, error minimization, variational methods.

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1 Introduction

Interpolation is a fundamental technique in numerical analysis and applied mathematics, used to estimate unknown function values based on known data points. Traditional interpolation methods often rely solely on function values at specific nodes [1]. However, incorporating derivative information into interpolation formulas can significantly enhance accuracy and convergence rates, especially for functions exhibiting complex behaviors.

Hilbert spaces provide a natural framework for analyzing functions and their derivatives. In particular, the Hilbert space $W_2^{(2,1)}(0,1)$ consists of functions whose first and second derivatives are square-integrable. This space is instrumental in studying interpolation problems that involve both function values and their derivatives [2].

Determining coefficients for optimal interpolation formulas within $W_2^{(2,1)}(0,1)$ is crucial for minimizing interpolation errors. Optimal interpolation formulas aim to achieve the best possible approximation by appropriately weighting function values and their derivatives.

2 Preliminaries

This section introduces the necessary mathematical framework for constructing optimal interpolation formulas with derivatives in the space $W_2^{(2,1)}(0,1)$. At first, definition of Sobolev space. A Sobolev space $W_p^k(a,b)$ is defined as the set of functions whose weak derivatives up to order k belong to the Lebesgue space $L_p(a,b)$:

$$W_p^k(a,b) = \left\{\varphi \in L_p(a,b) \;\middle|\; \varphi^{(j)} \in L_p(a,b), \quad \forall j = 1,2,\ldots,k\right\},$$

where $\varphi^{(j)}$ denotes the weak derivative of order j, and $L_p(a,b)$ is the space of functions whose p – th power is integrable over (a,b). We consider the linear space defined as

$$W_2^{(2,1)}(0,1) = \left\{\varphi | \varphi : [0,1] \to \mathbb{R}, \quad \varphi' \text{ is absolutely continuous and } \varphi'' \in L_2(0,1)\right\}.$$

This space is a Hilbert space. In such spaces the norm is usually defined based on inner product, using the function and its derivatives [3].
Below is the definition of the inner product. If φ and ψ are elements of the space $W_2^{(2,1)}$, then the following inner product is defined

$$< \varphi, \psi >= \int_0^1 (\varphi''(x) + \varphi'(x))(\psi''(x) + \psi'(x))dx.$$

Therefore, for the function $\varphi \in W_2^{(2,1)}(0,1)$, the norm is defined as follows

$$\|\varphi\|_{W_2^{(2,1)}} = \sqrt{< \varphi, \varphi >} = \sqrt{\int_0^1 (\varphi''(x) + \varphi'(x))^2 dx}. \tag{1}$$

This norm accounts for different integrability conditions on φ' and φ'', distinguishing this space from the classical $W_2^2(0,1)$ Sobolev space [4].

3 Statement of the problem

Classical interpolation techniques rely solely on function values, which may not provide sufficient accuracy for complex functions. A more refined approach involves incorporating derivative information into the interpolation process, which enhances precision. However, determining optimal interpolation coefficients that minimize approximation errors while maintaining computational efficiency remains an open challenge.

Interpolation techniques are widely used in numerical analysis and applied mathematics to approximate unknown function values using given data points [5]. Classical interpolation methods primarily rely on function values, which may lead to significant errors, especially for functions with strong curvature or rapid changes. To improve accuracy, interpolation formulas incorporating derivatives can be developed within Sobolev spaces, providing a more robust framework.

Interpolation formulas with derivatives extend classical interpolation methods by incorporating derivative information into the approximation. This approach is particularly useful for functions that exhibit sharp changes, allowing for a more accurate reconstruction.

The general form of the interpolation function is given on the following approximation

$$\varphi(x) \cong P_\varphi(x) = \sum_{\beta=0}^{N} C_{\beta,0}\varphi(h\beta) + \sum_{\beta=0}^{N} C_{\beta,1}\varphi'(h\beta), \tag{2}$$

where φ is the function being approximated; the first summation term involves function values $\varphi(h\beta)$ at interpolation nodes $h\beta$, $C_{\beta,0}$ are known and they are the coefficients of the optimal interpolation formula in the space $L_2^{(1)}(0,1)$. They have the following form [6]

$$C_{\beta,0}(x) = \frac{1}{2h}(|x-h(\beta-1)| + |x-h(\beta+1)| - 2|x-h\beta|), \quad \beta = 0,1,\dots,N. \tag{3}$$

The coefficients (3) can be rewritten in the following form

$$C_{0,0}(x) = \begin{cases} \frac{h-x}{h}, & 0 \leqslant x \leqslant h, \\ 0, & h < x \leqslant 1, \end{cases} \tag{4}$$

$$C_{\beta,0}(x) = \begin{cases} \frac{x+h-h\beta}{h}, & h(\beta-1) < x \leqslant h\beta, \\ \frac{h-x+h\beta}{h}, & h\beta < x \leqslant h(\beta+1), \\ 0, & \text{otherwise}, \end{cases} \quad \beta = 1,2,\dots,N-1, \tag{5}$$

$$C_{N,0}(x) = \begin{cases} 0, & 0 \leqslant x \leqslant h(N-1), \\ \frac{h-1+x}{h}, & h(N-1) < x \leqslant 1. \end{cases} \tag{6}$$

The second summation term in (2) incorporates derivative values $\varphi'(h\beta)$, unknown coefficients $C_{\beta,1}$. $C_{\beta,1}$ must be determined to minimize the interpolation error. This formula ensures that both the function values and their derivatives are accounted in the interpolation, leading to greater accuracy.

The error associated with the approximate equality (2) takes the form of a difference expressed as

$$E_\varphi(x) = \varphi(x) - P_\varphi(x). \tag{7}$$

In the space $W_2^{(2,1)}(0,1)$, the error functional $R(x,z)$ at each fixed point $x = z$ in the interval $[0,1]$ takes the following form:

$$R(x,z) = \delta(x-z) - \sum_{\beta=0}^{N} C_{\beta,0}\delta(x-h\beta) + \sum_{\beta=0}^{N} C_{\beta,1}\delta'(x-h\beta). \tag{8}$$

In this expression, the Dirac-delta function $\delta(x-z)$ ensures exact interpolation at given nodes; the summation terms represent corrections made to the kernel function using function values and their derivatives; $C_{\beta,0}$ and $C_{\beta,1}$ coefficients control the weighting of the interpolation points.

According to equality (8), we obtain the following expression:

$$\begin{aligned} (R,\varphi) &= \int_{-\infty}^{\infty} R(x,z)\cdot\varphi(x)dx = \\ &= \int_{-\infty}^{\infty} (\delta(x-z) - \sum_{\beta=0}^{N} C_{\beta,0}\delta(x-h\beta) + \sum_{\beta=0}^{N} C_{\beta,1}\delta'(x-h\beta))\varphi(x)dx = \\ &= \varphi(z) - \sum_{\beta=0}^{N} C_{\beta,0}\varphi(h\beta) - \sum_{\beta=0}^{N} C_{\beta,1}\varphi'(h\beta). \end{aligned} \tag{9}$$

In this work, we consider the approximation of the form (2), we impose the condition that the class of functions that transforms this approximate equality into an exact equality in $W_2^{(2,1)}$ space should be any linear combination of constant term and exponential function $\exp(-x)$. If we take $\varphi_1(x) = 1$ and $\varphi_2(x) = \exp(-x)$ as the basis functions, the imposition of

$$(R, 1) = 0, \tag{10}$$

$$(R, \exp(-x)) = 0. \tag{11}$$

conditions on the error functional $R(x, z)$ is enough for the approximation formula (2) to be exact for $\text{span}\{1, \exp(-x)\}$.

To develop an optimal interpolation formula in the form (2), it is crucial to determine the norm $||R||_{W_2^{(2,1)*}}$ of the error functional given in (8). This requirement follows from the Cauchy-Schwarz inequality, which bounds the error (2) using the product of norms

$$|(R, \varphi)| \leqslant ||R||_{W_2^{(2,1)*}} \cdot ||\varphi||_{W_2^{(2,1)}}. \tag{12}$$

Clearly, the norm $||R||_{W_2^{(2,1)*}}$ depends on the coefficients $C_{\beta,1}$. Therefore, it is necessary to find the minimum value of $||R||_{W_2^{(2,1)*}}$ concerning $C_{\beta,1}$, meaning that the following quantity must be obtained [7]:

$$\inf_{C_{\beta,1}} ||R||_{W_2^{(2,1)*}}. \tag{13}$$

The coefficients $C_{\beta,1}$ that achieve this minimum value, as given in (13), are referred to as the optimal coefficients.

Thus, the optimization process involves:

- Computing the norm $||R||_{W_2^{(2,1)*}}$.
- Determining $C_{\beta,1}$ that achive the quantity (13).

4 Optimal coefficients for interpolation formulas

The **optimal coefficients** $C_{\beta,1}$ in this work are the values that minimize the interpolation error in the Hilbert space $W_2^{(2,1)}$. These coefficients ensure that the interpolation formula provides the best approximation while maintaining stability and smoothness. We find the quantity (13), taking into account (10) and (11), and get the system of linear equations for $C_{\beta,1}$.

4.1 The extremal function of the interpolation formula

The extremal function plays a crucial role in optimal interpolation theory as it helps determine the best approximation by minimizing the interpolation error in the given space [8]. In the Hilbert space $W_2^{(2,1)}$, interpolation is formulated as an optimization problem where we seek to minimize the norm of the interpolation error.

The extremal function helps achieve this by ensuring that

$$(R, \psi_R) = \|R\| \cdot \|\psi_R\|, \tag{14}$$

which guarantees that the interpolation function is constructed in the most optimal manner.

Interpolation formulas require choosing optimal coefficients $C_{\beta,1}$. To obtain these coefficients, we need an extremal function $\psi_R(x)$ that satisfies (14). The optimal coefficients $C_{\beta,1}$ are obtained by solving a system of equations that arises from the extremal function.

Since $W_2^{(2,1)}$ is the Hilbert space than by the Riesz representation theorem the extremal function is expressed with the help of the error functional $R(x,z)$ as follows (see, for example, [4, 9])

$$\psi_R(x) = R(x,z) * G_2(x) + p_0 + de^{-x},$$

where

$$G_2(x) = \frac{\operatorname{sgn}(x)}{2} \cdot \left(\frac{e^x - e^{-x}}{2} - x \right).$$

Furthermore, the following holds

$$\|R\|^2_{W_2^{(2,1)*}} = \|\psi_R\|^2_{W_2^{(2,1)}} = (R, \psi_R).$$

Now we compute the convolution in the expression of the extremal function

$$(R * G_2)(x) = \int_{-\infty}^{\infty} R(y) \cdot G_2(x-y)\,dy =$$

$$= \int_{-\infty}^{\infty} \left(\delta(y-z) - \sum_{\beta=0}^{N} C_{\beta,0}(z)\delta(y-x_\beta) + \sum_{\beta=0}^{N} C_{\beta,0}(z)\delta'(y-x_\beta) \right) \times$$

$$\times \left(\frac{\operatorname{sgn}(x-y)}{2} \cdot \left(\frac{e^{x-y} - e^{-(x-y)}}{2} - (x-y) \right) \right) dy =$$

$$= \frac{\operatorname{sgn}(x-z)}{2} \cdot \left(\frac{e^{x-z} - e^{-(x-z)}}{2} - (x-z) \right) -$$

$$- \sum_{\beta=0}^{N} C_{\beta,0} \frac{\operatorname{sgn}(x-h\beta)}{2} \cdot \left(\frac{e^{x-h\beta} - e^{-(x-h\beta)}}{2} - (x-h\beta) \right) +$$

$$+ \sum_{\beta=0}^{N} C_{\beta,1} \frac{\operatorname{sgn}(x-h\beta)}{2} \cdot \left(\frac{e^{x-h\beta} + e^{-(x-h\beta)}}{2} - 1 \right).$$

Thus, for the convolution we have

$$(R * G_2)(x) = G_2(x-z) - \sum_{\beta=0}^{N} C_{\beta,0} G_2(x-h\beta) + \sum_{\beta=0}^{N} C_{\beta,1} G_2'(x-h\beta). \tag{15}$$

Taking into account (15) the extremal function can be written in the form

$$\psi_R(x) = G_2(x-z) - \sum_{\beta=0}^{N} C_{\beta,0} G_2(x-h\beta) + \sum_{\beta=0}^{N} C_{\beta,1} G_2'(x-h\beta) + p_0 + de^{-x}. \tag{16}$$

With the extremal function, we can find the optimal coefficients and guarantee minimal interpolation error, making it a fundamental tool in this work.

4.2 Minimum value of the norm of the error functional

In numerical analysis and approximation theory, optimizing the norm of the error functional is a crucial step in constructing an interpolation formula. The error functional quantifies the deviation between the exact function and its interpolation, and minimizing its norm ensures the best possible approximation under given constraints [10].

For the function space $W_2^{(2,1)}(0,1)$, where smoothness conditions are imposed, the optimization process involves determining the optimal coefficients of the interpolation formula. This is typically achieved by formulating the error norm in an appropriate metric and solving a minimization problem using functional analysis techniques. By obtaining these optimal coefficients, we ensure that the interpolation error is minimized in the chosen norm, leading to a more precise and stable numerical method.

Thus, by applying (8) and (16) while considering (10) and (11), we obtain:

$$\|R\|^2_{W_2^{(2,1)*}} = (R, \psi_R) = \int_{-\infty}^{\infty} R(x) \cdot (R * G_2)(x) dx =$$
$$= \int_{-\infty}^{\infty} \left(\delta(x-z) - \sum_{\beta=0}^{N} C_{\beta,0}\delta(x-h\beta) + \sum_{\beta=0}^{N} C_{\beta,1}\delta'(x-h\beta) \right) \times$$
$$\times \left(\frac{\operatorname{sgn}(x-z)}{2} \cdot \left(\frac{e^{(x-z)} - e^{-(x-z)}}{2} - (x-z) \right) - \right.$$
$$- \sum_{\gamma=0}^{N} C_{\gamma,0} \frac{\operatorname{sgn}(x-h\gamma)}{2} \cdot \left(\frac{e^{(x-h\gamma)} - e^{-(x-h\gamma)}}{2} - (x-h\gamma) \right) +$$
$$\left. + \sum_{\gamma=0}^{N} C_{\gamma,1} \frac{\operatorname{sgn}(x-h\gamma)}{2} \cdot \left(\frac{e^{(x-h\gamma)} + e^{-(x-h\gamma)}}{2} - 1 \right) \right).$$

The simplification of the last expression is given in detail in [11]. Therefore, we get the following expression for the norm of the error functional of the interpolation formula (2)

$$\begin{aligned} \|R\|^2_{W_2^{(2,1)*}} = &- \sum_{\gamma=0}^{N} \sum_{\beta=0}^{N} C_{\beta,1} C_{\gamma,1} \frac{\operatorname{sgn}(h\beta - h\gamma)}{2} \cdot \left(\frac{e^{h\beta - h\gamma} - e^{-(h\beta - h\gamma)}}{2} \right) + \\ &+ 2 \sum_{\beta=0}^{N} C_{\beta,1} \frac{\operatorname{sgn}(z - h\beta)}{2} \cdot \left(\frac{e^{z-h\beta} + e^{-(z-h\beta)}}{2} - 1 \right) - \\ &- 2 \sum_{\beta=0}^{N} C_{\beta,0} \frac{\operatorname{sgn}(z-h\beta)}{2} \cdot \left(\frac{e^{z-h\beta} + e^{-(z-h\beta)}}{2} - (z-h\beta) \right) + \\ &+ \sum_{\beta=0}^{N} \sum_{\gamma=0}^{N} C_{\beta,0} C_{\gamma,0} \frac{\operatorname{sgn}(h\beta - h\gamma)}{2} \cdot \left(\frac{e^{h\beta-h\gamma} - e^{-(h\beta-h\gamma)}}{2} - (h\beta - h\gamma) \right) + \\ &+ 2 \sum_{\gamma=0}^{N} \sum_{\beta=0}^{N} C_{\beta,1} C_{\gamma,0} \frac{\operatorname{sgn}(h\beta - h\gamma)}{2} \cdot \left(\frac{e^{h\beta-h\gamma} + e^{-(h\beta-h\gamma)}}{2} - 1 \right). \end{aligned} \tag{17}$$

Now, to determine the optimal coefficients that minimize the error norm (17), we now introduce the Lagrange multiplier method [12]. Since the interpolation formula must satisfy the exactness conditions (10) and (11), we incorporate these constraints into the minimization problem via Lagrange multipliers, say λ_1 and λ_2.

We define the Lagrangian functional as

$$\mathcal{L}\big(C_{\beta,1}, \lambda_1, \lambda_2\big) = \|R\|^2_{W_2^{(2,1)}} + \lambda_1(R, 1) + \lambda_2(R, e^{-x}),$$

where $\|R\|^2_{W_2^{(2,1)}}$ is given by expression (17). Since this norm is a quadratic form in the unknown coefficients $C_{\beta,1}(z)$, the minimization of $\|R\|^2_{W_2^{(2,1)}}$ subject to the constraints can be carried out by requiring that the partial derivatives of $\mathcal{L}$ with respect to each $C_{\beta,1}$ vanish.

Taking the derivative with respect to a fixed coefficient $C_{\beta,1}$ yields

$$\frac{\partial\mathcal{L}}{\partial C_{\beta,1}}=\frac{\partial}{\partial C_{\beta,1}}\|R\|^2_{W_2^{(2,1)*}}+\lambda_1\frac{\partial}{\partial C_{\beta,1}}(R,1)+\lambda_2\frac{\partial}{\partial C_{\beta,1}}(R,e^{-x})=0.$$

Because $\|R\|^2_{W_2^{(2,1)*}}$ is explicitly given by a double summation (17), differentiating it with respect to $C_{\beta,1}$ leads to a linear system of the form

$$\sum_{\gamma=0}^{N}C_{\gamma,1}\frac{\operatorname{sgn}(h\beta-h\gamma)}{2}\cdot\left(\frac{e^{h\beta-h\gamma}-e^{-(h\beta-h\gamma)}}{2}\right)-\frac{\lambda_2}{2}e^{-h\beta}=$$
$$=\sum_{\gamma=0}^{N}C_{\gamma,0}\frac{\operatorname{sgn}(h\beta-h\gamma)}{2}\cdot\left(\frac{e^{h\beta-h\gamma}+e^{-(h\beta-h\gamma)}}{2}-1\right)+$$
$$+\frac{\operatorname{sgn}(z-h\beta)}{2}\cdot\left(\frac{e^{z-h\beta}+e^{-(z-h\beta)}}{2}-1\right),\quad \beta=0,1,\dots,N,$$

The first and second derivatives of the function $G_2(x)$ is given by

$$G_2'(x)=\frac{\operatorname{sgn}(x)}{2}\left(\frac{e^x+e^{-x}}{2}-1\right),\quad G_2''(x)=\frac{\operatorname{sgn}(x)}{2}\left(\frac{e^x-e^{-x}}{2}\right).$$

According to G_2, we can write the last equality in a compact form as

$$\sum_{\gamma=0}^{N}C_{\gamma,1}\,G_2''\big(h\beta-h\gamma\big)-\frac{\lambda_2}{2}\,e^{-h\beta}=\sum_{\gamma=0}^{N}C_{\gamma,0}\,G_2'\big(h\beta-h\gamma\big)+G_2'\big(z-h\beta\big),\qquad \beta=0,1,\dots,N. \tag{18}$$

This succinct form emphasizes how the kernels in the original equality correspond directly to the derivatives of G_2.

In addition, taking the derivative of $\mathcal{L}$ with respect to the multiplier λ_2 recovers the constraint

$$\frac{\partial\mathcal{L}}{\partial\lambda_2}=(R,e^{-x})=0. \tag{19}$$

Hence, the condition $(R,1)=0$ is automatically satisfied [7], so it can be omitted from the final system. Consequently, the complete system of equations simplifies to

$$\begin{cases}\sum\limits_{\gamma=0}^{N}C_{\gamma,1}\,G_2''\big(h\beta-h\gamma\big)-\frac{\lambda_2}{2}\,e^{-h\beta}=\sum\limits_{\gamma=0}^{N}C_{\gamma,0}\,G_2'\big(h\beta-h\gamma\big)+G_2'\big(z-h\beta\big),\beta=0,1,...,N,\\(R,e^{-x})=0.\end{cases}$$

From here, taking into account denotation $\lambda=-\frac{\lambda_2}{2}$, by expanding (R,e^{-x}), we obtain the following final system of linear equations

$$\begin{cases}\sum\limits_{\gamma=0}^{N}C_{\gamma,1}\,G_2''\big(h\beta-h\gamma\big)+\lambda\,e^{-h\beta}=\sum\limits_{\gamma=0}^{N}C_{\gamma,0}\,G_2'\big(h\beta-h\gamma\big)+G_2'\big(z-h\beta\big),\beta=0,1,...,N,\\\sum\limits_{\gamma=0}^{N}C_{\gamma,1}\,e^{-h\beta}=-e^{-z}+\sum\limits_{\gamma=0}^{N}C_{\gamma,0}\,e^{-h\beta}.\end{cases}$$

Solving this system provides the optimal coefficients $C_{\beta,1}$ that minimize the norm of the error functional, ensuring that the interpolation formula exactly reproduces the function e^{-x}.

5 Conclusion

The optimal coefficients guarantee that the interpolation formula not only reproduces the target functions exactly at the specified nodes but also minimizes the interpolation error in the space $W_2^{(2,1)}(0,1)$. Such a result is of paramount importance for achieving high accuracy in numerical approximations, particularly when dealing with functions that exhibit rapid variations or complex behavior.

Future work will focus on the numerical solution of the derived system and the investigation of the stability and convergence properties of the proposed method. Moreover, extensions of this approach to higher-dimensional problems and other function spaces could broaden its applicability in computational mathematics and engineering.

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ОПТИМАЛЬНАЯ ИНТЕРПОЛЯЦИОННАЯ ФОРМУЛА С ПРОИЗВОДНОЙ В ГИЛЬБЕРТОВОМ ПРОСТРАНСТВЕ

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Данное исследование посвящено разработке системы уравнений для определения коэффициентов оптимальных интерполяционных формул, включающих информацию о производных в рамках гильбертова пространства. Традиционные методы интерполяции, основанные исключительно на значениях функции в точках, часто оказываются недостаточными для функций с сложным или быстро меняющимся поведением. Для преодоления этого ограничения предложенный подход интегрирует данные о производных в процесс интерполяции, повышая устойчивость и точность получаемых формул. Формулировка задачи интерполяции в гильбертовом пространстве создаёт надёжную основу для вывода оптимальных коэффициентов. Основной аналитический вклад заключается в формировании системы уравнений, полученной с использованием вариационных принципов и таких инструментов, как теорема представления Рисса и операции свёртки. Решение этой системы позволяет явно вычислить оптимальные коэффициенты, которые легко применимы в практических задачах интерполяции. Данная методология особенно значима в числовом анализе, особенно в сценариях, где доступны данные о положении, направляющих векторах или скоростях объектов, поскольку включение информации о производных является интуитивно понятным и критически важным. Кроме того, подход применим в задачах аппроксимации данных, обработки сигналов и вычислительных контекстах, требующих восстановления функций из выборочных или зашумленных данных.

Ключевые слова: оптимальные коэффициенты, интерполяция с использованием производных, гильбертово пространство, минимизация ошибки, вариационные методы.

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