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NUMERICAL STUDY OF LYAPUNOV STABILITY OF AN UPWIND DIFFERENCE SCHEME FOR A QUASILINEAR HYPERBOLIC SYSTEM

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This study addresses a mixed problem for a quasilinear system of hyperbolic equations expressed in Riemann invariants, incorporating dissipative nonlinear boundary conditions. A numerical approach is developed through an initial-boundary difference problem utilizing an upwind difference scheme. The stability of nonlinear difference schemes is investigated, with a focus on establishing a sufficient stability criterion based on Lyapunov vector functions. The proposed criterion extends prior theoretical work, where a discrete Lyapunov function was formulated to demonstrate the exponential stability of the steady state for the quasilinear system. Numerical computations for a model problem validate these theoretical findings. The research highlights the potential of adapting the direct Lyapunov method to analyze the stability of nonlinear hyperbolic systems by constructing a positive definite function that exhibits monotonic decay along system solutions.

Keywords: exponential stability, hyperbolic system, mixed problem, difference scheme, Lyapunov function.

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1 Introduction

The Lyapunov function method has found application in the stability theory of ordinary difference equations (see Martynyuk, 1972; Evtushenko and Zhadan, 1975). There are generalizations of the direct Lyapunov method to partial differential equations (see Zubov, 1957; Banda and Herti, 2013; Gottlich and Shillen, 2017; Veldegiyorgis, 2017; Mapunidi and Gediyon, 2020). However, it should be noted that the method of energy inequalities, widely used in the theory of linear partial differential equations, is essentially a special case of the direct Lyapunov method (in this case, the Lyapunov function is constructed as a certain quadratic form from the solution to the problem). All this allows us to hope for a generalization of the Lyapunov function method to nonlinear difference schemes. Note that the works of Aloev et al. (2021) and Aloev et al. (2022) are devoted to the construction of the Lyapunov function for linear difference schemes.

The known methods of stability analysis according to Lyapunov are based on the qualitative theory of ordinary differential equations. Research on this basis is necessary for the theory and practice of automatic regulation, control and monitoring, and super-operational control. As a rule, stability analysis is carried out either a priori, before the creation of a control system, or a posteriori, based on the results of operation. However, stability control is important for the current state of the system.

The need to study the stability of motion or a certain state arises at all stages of designing or studying physical systems. For the first time, a strict mathematical definition of stability and precise methods for solving the stability problem for a fairly wide class of systems were given by A.M. Lyapunov in his famous work Lyapunov, 1950. This work was the logical conclusion of the entire previous stage of the development of stability theory. With its appearance, stability theory reached the level of an independent discipline, taking a worthy place among other mathematical disciplines. A.M. Lyapunov proposed two methods for analyzing the stability of solutions to ordinary differential equations. The first method consists of constructing solutions to the differential equations of disturbed motions themselves in the form of certain series. Based on the subsequent qualitative study of these solutions, conclusions are made about stability or instability. The second method consists of finding some auxiliary function, the properties of which determine the stability or instability of the solution. At present, these functions are called Lyapunov functions, and the method is called the Lyapunov function method, the second Lyapunov method, or the direct Lyapunov method.

Lyapunov's works became the starting point for research of this kind. His ideas are developed and deepened in many directions. New theorems have been established that expand these methods, many questions of the existence of Lyapunov functions and their effective construction have been solved, questions of the stability of unsteady and periodic motions, the stability of the first approximation, in critical cases, under constantly acting disturbances, and many others have been studied.

The development of the theory of stability as applied to automatic control and regulation systems is the theory of motion stabilization, which studies such modes of system control in which some programmed motion (unperturbed motion) of the system will be stable in one sense or another. In many cases, along with the requirement for stability of unperturbed motion, additional requirements are imposed on both the nature of transient processes and control actions. Often these requirements can be expressed as a minimum of some integral functional. Stabilization problems with these additional requirements are called problems of optimal stabilization or analytical design of regulators.

2 Results and discussions

2.1 Linear mixed problem

Let us consider a linear system of hyperbolic equations:

$$\begin{cases} \dfrac{\partial \xi}{\partial t} + \bar{\varphi}\dfrac{\partial \xi}{\partial x} = 0, \\ \dfrac{\partial \eta}{\partial t} - \bar{\psi}\dfrac{\partial \eta}{\partial x} = 0, \end{cases} \tag{1}$$

where $\bar{\varphi} > 0, \;\; \bar{\psi} > 0$,
with boundary conditions:

$$\begin{cases} \xi\left(t,0\right) = \varkappa_0 \eta\left(t,0\right), \\ \eta\left(t,L\right) = \varkappa_L \xi\left(t,L\right), \end{cases} \tag{2}$$

and with initial data

$$\begin{cases} \xi\left(0,x\right) = \xi_0\left(x\right), \\ \eta\left(0,x\right) = \eta_0\left(x\right), \end{cases} \qquad 0 < x < L\,. \tag{3}$$

Let us assume that the parameters of the boundary conditions (2) $\varkappa_0$, $\varkappa_L$ satisfy the inequality $0 < |\varkappa_0 \varkappa_L| < 1$.

2.2 Exponential stability of the numerical solution

In this section, an explicit upwind difference scheme for solving the discussed linear problems is presented. Then, the results obtained earlier by the authors of this article in previous works are presented, namely, a theorem for testing the exponential stability of the numerical solution.

2.3 Exponential stability of the numerical solution of a linear problem

Let us consider the linear hyperbolic system (1). For the numerical solution of the linear hyperbolic system (1), an explicit upwind difference scheme is proposed:

$$\begin{cases} \frac{\xi_j^{k+1}-\xi_j^k}{\Delta t} + \bar{\varphi}\frac{\xi_j^k-\xi_{j-1}^k}{\Delta x} = 0, \quad k=\overline{0,K-1}, \quad j=\overline{1,J}, \\ \frac{\eta_j^{k+1}-\eta_j^k}{\Delta t} - \bar{\psi}\frac{\eta_{j+1}^k-\eta_j^k}{\Delta x} = 0, \quad k=\overline{0,K-1}, \; j=\overline{0,J-1}. \end{cases} \tag{4}$$

Let us introduce the notation for the Courant number $C_\varphi = \bar{\varphi}\frac{\Delta t}{\Delta x}$, $C_\psi = \bar{\psi}\frac{\Delta t}{\Delta x}$, $C = \max(C_\varphi, C_\psi)$. We will choose the steps of the difference grid Δt, Δx so that they satisfy the Courant-Friedrichs-Lewy condition:

$$C < 1. \tag{5}$$

Let us rewrite the system of difference equations (4) in the form (6)

$$\begin{cases} \xi_j^{k+1} = (1-C_\varphi)\,\xi_j^k + C_\varphi \xi_{j-1}^k, \; k=\overline{0,K-1}, \; j=\overline{1,J}, \\ \eta_j^{k+1} = (1-C_\psi)\,\eta_j^k + C_\psi \eta_{j+1}^k, \; k=\overline{0,K-1}, \; j=\overline{0,J-1}, \end{cases} \tag{6}$$

As a discrete Lyapunov function for system (6), we consider the function:

$$L^k = L_1^k + L_2^k. \tag{7}$$

Where

$$L_1^k = \frac{\mathrm{A}}{\bar{\varphi}}\Delta x \sum_{j=1}^{J} \left(\xi_j^k\right)^2 \exp\left(-\frac{m}{\bar{\varphi}}x_{j-1}\right), \;\; L_2^k = \frac{\mathrm{B}}{\bar{\psi}}\Delta x \sum_{j=0}^{J-1} \left(\eta_j^k\right)^2 \exp\left(\frac{m}{\bar{\psi}}x_{j+1}\right),$$

with positive coefficients A, B, m.
The difference time derivative of the discrete Lyapunov function (7) along the solution of system (6) is equal to

$$\frac{L^{k+1}-L^k}{\Delta t} = \frac{L_1^{k+1}-L_1^k}{\Delta t} + \frac{L_2^{k+1}-L_2^k}{\Delta t}. \tag{8}$$

Let us calculate the difference derivatives $\frac{L_1^{k+1}-L_1^k}{\Delta t}$ and $\frac{L_2^{k+1}-L_2^k}{\Delta t}$ of the right-hand side of equality (8) separately on the solutions of system (6).

Lemma 2.1. For grid functions ξ_j^k satisfying the first difference equations (6), the following inequality holds:

$$\frac{L_1^{k+1}-L_1^k}{\Delta t} \leqslant -mL_1^k - \mathrm{A}\left[\left(\xi_j^k\right)^2 \exp\left(-\frac{m}{\bar{\varphi}}x_j\right)\right]\Bigg|_0^J. \tag{9}$$

Proof of Lemma 2.1.
Substituting the value of the expression for ξ_j^{k+1} from (6) into the expression $\frac{L_1^{k+1}-L_1^k}{\Delta t}$ we obtain:

$$\frac{L_1^{k+1}-L_1^k}{\Delta t}=\frac{\mathrm{A}}{\bar{\varphi}}\Delta x\sum_{j=1}^{J}\left[\frac{\left(\xi_j^{k+1}\right)^2-\left(\xi_j^k\right)^2}{\Delta t}\right]\exp\left(-\frac{m}{\bar{\varphi}}x_{j-1}\right)=$$
$$=\frac{\mathrm{A}}{C_\varphi}\sum_{j=1}^{J}\left[\left(\xi_j^{k+1}\right)^2-\left(\xi_j^k\right)^2\right]\exp\left(-\frac{m}{\bar{\varphi}}x_{j-1}\right)= \tag{10}$$
$$=\frac{\mathrm{A}}{C_\varphi}\sum_{j=1}^{J}\left[\left\{(1-C_\varphi)\,\xi_j^k+C_\varphi\xi_{j-1}^k\right\}^2-\left(\xi_j^k\right)^2\right]\exp\left(-\frac{m}{\bar{\varphi}}x_{j-1}\right).$$

According to Jensen's inequality, for convex mappings $y\to y^2$ the following inequality holds:

$$\left[q_1y_1+q_2y_2\right]^2\leqslant q_1\left(y_1\right)^2+q_2\left(y_2\right)^2, \tag{11}$$

where $q_1,\,q_2>0$ and $q_1+q_2=1$. Therefore, using Jensen's inequality (11) to estimate the expression $\left\{(1-C_\varphi)\,\xi_j^k+C_\varphi\xi_{j-1}^k\right\}^2$ from above, we have

$$\frac{L_1^{k+1}-L_1^k}{\Delta t}\leqslant\frac{\mathrm{A}}{C_\varphi}\sum_{j=1}^{J}\left[(1-C_\varphi)\left(\xi_j^k\right)^2+C_\varphi\left(\xi_{j-1}^k\right)^2-\left(\xi_j^k\right)^2\right]\exp\left(-\frac{m}{\bar{\varphi}}x_{j-1}\right)=$$
$$=\mathrm{A}\sum_{j=1}^{J}\left[\left(\xi_{j-1}^k\right)^2-\left(\xi_j^k\right)^2\right]\exp\left(-\frac{m}{\bar{\varphi}}x_{j-1}\right). \tag{12}$$

The following equality is true with an accuracy of $O\left(\Delta x^2\right)$

$$\left[\exp\left(-\frac{m}{\bar{\varphi}}x_j\right)-\exp\left(-\frac{m}{\bar{\varphi}}x_{j-1}\right)\right]=\exp\left(-\frac{m}{\bar{\varphi}}x_{j-1}\right)\left(\exp\left(-\frac{m}{\bar{\varphi}}\Delta x\right)-1\right)=$$
$$=\exp\left(-\frac{m}{\bar{\varphi}}x_{j-1}\right)\left[-\frac{m}{\bar{\varphi}}\Delta x+O\left(\Delta x^2\right)\right]. \tag{13}$$

Using the formula of difference differentiation

$$\left(u_j-u_{j-1}\right)v_{j-1}=\left(u_jv_j-u_{j-1}v_{j-1}\right)-u_j\left(v_j-v_{j-1}\right) \tag{14}$$

and taking into account equality (13) with accuracy $O\left(\Delta x\right)$ from inequality (12) we obtain

$$\frac{L_1^{k+1}-L_1^k}{\Delta t}\leqslant-\mathrm{A}\sum_{j=1}^{J}\left[\left(\xi_j^k\right)^2\exp\left(-\frac{m}{\bar{\varphi}}x_j\right)-\left(\xi_{j-1}^k\right)^2\exp\left(-\frac{m}{\bar{\varphi}}x_{j-1}\right)\right]+$$
$$+\mathrm{A}\sum_{j=1}^{J}\left(\xi_j^k\right)^2\left[\exp\left(-\frac{m}{\bar{\varphi}}x_j\right)-\exp\left(-\frac{m}{\bar{\varphi}}x_{j-1}\right)\right]=$$
$$=-\mathrm{A}\left(\xi_J^k\right)^2\exp\left(-\frac{m}{\bar{\varphi}}x_J\right)+\mathrm{A}\left(\xi_0^k\right)^2-m\frac{\mathrm{A}}{\bar{\varphi}}\Delta x\sum_{j=1}^{J}\left(\xi_j^k\right)^2\exp\left(-\frac{m}{\bar{\varphi}}x_{j-1}\right)=$$
$$=-mL_1^k-\mathrm{A}\left[\left(\xi_j^k\right)^2\exp\left(-\frac{m}{\bar{\varphi}}x_j\right)\right]_0^J.$$

Lemma 2.1 is proven.

Lemma 2.2 For grid functions η_j^k satisfying the difference equations (6), the following inequality is valid:

$$\frac{L_2^{k+1} - L_2^k}{\Delta t} \leqslant -mL_2^k + \mathrm{B}\left[\left(\eta_j^k\right)^2 \exp\left(\frac{m}{\bar{\psi}} x_j\right)\right]\Bigg|_0^J. \tag{15}$$

Proof of Lemma 2.2.
Substituting the value of the expression for η_j^{k+1} from (6) into the expression $\frac{L_2^{k+1}-L_2^k}{\Delta t}$, we obtain

$$\begin{aligned} \frac{L_2^{k+1} - L_2^k}{\Delta t} &= \frac{\mathrm{B}}{\bar{\psi}} \Delta x \sum_{j=0}^{J-1} \left[\frac{\left(\eta_j^{k+1}\right)^2 - \left(\eta_j^k\right)^2}{\Delta t}\right] \exp\left(\frac{m}{\bar{\psi}} x_{j+1}\right) = \\ &= \frac{\mathrm{B}}{C_\psi} \sum_{j=0}^{J-1} \left[\left(\eta_j^{k+1}\right)^2 - \left(\eta_j^k\right)^2\right] \exp\left(\frac{m}{\bar{\psi}} x_{j+1}\right) = \\ &= \frac{\mathrm{B}}{C_\psi} \sum_{j=0}^{J-1} \left[\left\{(1 - C_\psi)\,\eta_j^k + C_\psi \eta_{j+1}^k\right\}^2 - \left(\eta_j^k\right)^2\right] \exp\left(\frac{m}{\bar{\psi}} x_{j+1}\right). \end{aligned} \tag{16}$$

Using Jensen's inequality (11) to estimate the expression $\left\{(1 - C_\varphi)\,\xi_j^k + C_\varphi \xi_{j-1}^k\right\}^2$ from above in (16), instead of equality we have the inequality

$$\frac{L_2^{k+1} - L_2^k}{\Delta t} \leqslant \mathrm{B} \sum_{j=0}^{J-1} \left[\left(\eta_{j+1}^k\right)^2 - \left(\eta_j^k\right)^2\right] \exp\left(\frac{m}{\bar{\psi}} x_{j+1}\right). \tag{17}$$

The following equality is true to $O\left(\Delta x^2\right)$ accuracy.

$$\begin{aligned} \left[\exp\left(\frac{m}{\bar{\psi}} x_{j+1}\right) - \exp\left(\frac{m}{\bar{\psi}} x_j\right)\right] &= \exp\left(\frac{m}{\bar{\psi}} x_{j+1}\right)\left(1 - \exp\left(-\frac{m}{\bar{\psi}} \Delta x\right)\right) = \\ &= \exp\left(\frac{m}{\bar{\psi}} x_{j+1}\right)\left[\frac{m}{\bar{\psi}} \Delta x + O\left(\Delta x^2\right)\right]. \end{aligned} \tag{18}$$

Using the formula of difference differentiation

$$\left(u_{j+1} - u_j\right) v_{j+1} = \left(u_{j+1} v_{j+1} - u_j v_j\right) - u_j \left(v_{j+1} - v_j\right) \tag{19}$$

and taking into account equality (18) with accuracy $O\left(\Delta x\right)$ from inequality (17) we obtain

$$\begin{aligned} l\frac{L_2^{k+1} - L_2^k}{\Delta t} &\leqslant \mathrm{B} \sum_{j=0}^{J-1} \left[\left(\eta_{j+1}^k\right)^2 \exp\left(\frac{m}{\bar{\psi}} x_{j+1}\right) - \left(\eta_j^k\right)^2 \exp\left(\frac{m}{\bar{\psi}} x_j\right)\right] - \\ &\quad - \mathrm{B} \sum_{j=0}^{J-1} \left(\eta_j^k\right)^2 \left[\exp\left(\frac{m}{\bar{\psi}} x_{j+1}\right) - \exp\left(\frac{m}{\bar{\psi}} x_j\right)\right] = \\ &= \mathrm{B}\left(\eta_J^k\right)^2 \exp\left(\frac{m}{\bar{\psi}} x_J\right) - \mathrm{B}\left(\eta_0^k\right)^2 - m\frac{\mathrm{B}}{\bar{\psi}} \Delta x \sum_{j=0}^{J-1} \left(\eta_j^k\right)^2 \exp\left(\frac{m}{\bar{\psi}} x_{j+1}\right) = \\ &= -mL_2^k + \mathrm{B}\left[\left(\eta_j^k\right)^2 \exp\left(\frac{m}{\bar{\psi}} x_j\right)\right]_0^J. \end{aligned}$$

Lemma 2.2 is proved.
Taking into account Lemma 2.1-2.2 from equality (8) we obtain

$$\frac{L^{k+1}-L^k}{\Delta t}=\frac{L_1^{k+1}-L_1^k}{\Delta t}+\frac{L_2^{k+1}-L_2^k}{\Delta t}\leqslant$$

$$\leqslant -mL_1^k-\mathrm{A}\left[\left(\xi_j^k\right)^2\exp\left(-\frac{m}{\bar{\varphi}}x_j\right)\right]\Bigg|_0^J-mL_2^k+\mathrm{B}\left[\left(\eta_j^k\right)^2\exp\left(\frac{m}{\bar{\psi}}x_j\right)\right]\Bigg|_0^J=$$

$$=-mL^k-\mathrm{A}\left[\left(\xi_j^k\right)^2\exp\left(-\frac{m}{\bar{\varphi}}x_j\right)\right]\Bigg|_0^J+\mathrm{B}\left[\left(\eta_j^k\right)^2\exp\left(\frac{m}{\bar{\psi}}x_j\right)\right]\Bigg|_0^J,$$

from which it follows that

$$\begin{aligned}\frac{L^{k+1}-L^k}{\Delta t}= & -mL^k-\left[\mathrm{A}\left(\xi_J^k\right)^2\exp\left(-\frac{m}{\bar{\varphi}}L\right)-\mathrm{A}\left(\xi_0^k\right)^2\right]- \\ & -\left[\mathrm{B}\left(\eta_0^k\right)^2-\mathrm{B}\left(\eta_J^k\right)^2\exp\left(\frac{m}{\bar{\psi}}L\right)\right].\end{aligned} \tag{20}$$

We approximate the boundary conditions (2) as follows:

$$\xi_0^k=\varkappa_0\eta_0^k,\qquad \eta_J^k=\varkappa_L\xi_J^k.$$

Then the difference time derivative of the discrete Lyapunov function on the solutions of system (6) satisfies the inequality

$$\begin{aligned}\frac{L^{k+1}-L^k}{\Delta t}= & -mL^k-\left[\mathrm{A}\left(\xi_J^k\right)^2\exp\left(-\frac{m}{\bar{\varphi}}L\right)-\mathrm{A}\varkappa_0^2\left(\eta_0^k\right)^2\right]- \\ & -\left[\mathrm{B}\left(\eta_0^k\right)^2-\mathrm{B}\exp\left(\frac{m}{\bar{\psi}}L\right)\varkappa_L^2\left(\xi_J^k\right)^2\right]=\end{aligned}$$

$$-mL^k+\left[\mathrm{B}\exp\left(\frac{m}{\bar{\psi}}L\right)\varkappa_L^2-\mathrm{A}\exp\left(-\frac{m}{\bar{\varphi}}L\right)\right]\left(\xi_J^k\right)^2+\left[\mathrm{A}\varkappa_0^2-\mathrm{B}\right]\left(\eta_0^k\right)^2.$$

Then, similarly to the differential problem (see [9]), the A,B can be chosen so that:

$$\mathrm{A}\varkappa_0^2-\mathrm{B}<0\quad and\quad \mathrm{B}\exp\left(\frac{m}{\bar{\psi}}L\right)\varkappa_L^2-\mathrm{A}\exp\left(-\frac{m}{\bar{\varphi}}L\right)<0. \tag{21}$$

Then it is obvious that $\frac{L^{k+1}-L^k}{\Delta t}\leqslant -mL^k$ along the solution of system (6) and that

$\frac{L^{k+1}-L^k}{\Delta t}=0$ if and only if $\xi_j^k=\eta_j^k=0$ (i.e. in the equilibrium state of the system).

2.4 Exponential stability of the numerical solution of a nonlinear initial-boundary difference problem.

Statement of the quasilinear mixed problem. According to the work of Corona et al. (2007), in the domain $\bar{\omega}\triangleq\{(t,x):0\leqslant t\leqslant T,0\leqslant x\leqslant L\}$ a mixed problem is considered for the following quasilinear hyperbolic system:

$$\left\{\begin{array}{l} \xi_t+\varphi\left(\xi,\eta\right)\xi_x=0,\quad \gamma_t+\varphi\left(\xi,\eta\right)\gamma_x+\gamma f=0,\quad \rho_t+\varphi\left(\xi,\eta\right)\rho_x+\rho f+\gamma f_x=0, \\ \eta_t-\psi\left(\xi,\eta\right)\eta_x=0,\quad \delta_t-\psi\left(\xi,\eta\right)\delta_x-\delta p=0,\quad \theta_t-\psi\left(\xi,\eta\right)\theta_x-\theta p-\delta p_x=0, \end{array}\right. \tag{22}$$
$$0<t\leqslant T,\ 0<x<L,$$

with boundary conditions at $x=0,L$:
at $x=0:$

$$\left\{\begin{array}{l} \xi\left(t,0\right)=a\left(\eta\left(t,0\right)\right),\quad \varphi\left(t,0\right)\gamma\left(t,0\right)=-a'\left(\eta\left(t,0\right)\right)\psi\left(t,0\right)\delta\left(t,0\right), \\ \varphi\left(t,0\right)\rho\left(t,0\right)+\gamma\left(t,0\right)f\left(t,0\right)=-e'\left(t\right)\delta\left(t,0\right)-\ e\left(t\right)\left[\begin{array}{c}\psi\left(t,0\right)\theta\left(t,0\right)+ \\ +\delta\left(t,0\right)p\left(t,0\right)\end{array}\right], \end{array}\right. \tag{23}$$

at $x=L:$

$$\left\{\begin{array}{l} \eta\left(t,L\right)=b\left(\xi\left(t,L\right)\right),\quad \psi\left(t,L\right)\delta\left(t,L\right)=-b'\left(\xi\left(t,L\right)\right)\varphi\left(t,L\right)\gamma\left(t,L\right), \\ \psi\left(t,L\right)\theta\left(t,L\right)+\delta\left(t,L\right)p\left(t,L\right)=-h'\left(t\right)\gamma\left(t,L\right)+\ h\left(t\right)\left[\begin{array}{c}\varphi\left(t,L\right)\rho\left(t,L\right)+ \\ +\gamma\left(t,L\right)f\left(t,L\right)\end{array}\right], \end{array}\right. \tag{24}$$

and with initial data

$$\left\{\begin{array}{l} \xi\left(0,x\right)=\xi_0\left(x\right),\quad \gamma\left(0,x\right)=\xi_0'\left(x\right),\quad \rho\left(0,x\right)=\xi_0''\left(x\right), \\ \eta\left(0,x\right)=\eta_0\left(x\right),\quad \delta\left(0,x\right)=\eta_0'\left(x\right),\quad \theta\left(0,x\right)=\eta_0''\left(x\right), \end{array}\right. \qquad 0<x<L. \tag{25}$$

Here
$\xi=\xi\left(t,x\right)$, $\eta=\eta\left(t,x\right)$, $\gamma=\gamma\left(t,x\right)=\xi_x$, $\delta=\delta\left(t,x\right)=\eta_x$, $\rho=\rho\left(t,x\right)=\gamma_x$, $\theta=\theta\left(t,x\right)=\delta_x$, unknowns to be determined, and $\varphi=\varphi\left(\xi,\eta\right)$, $\psi=\psi\left(\xi,\eta\right)$ given functions that have continuous derivatives of the second order inclusive. Suppose that $a,b\in C^2\left(\mathrm{R}\right)$.

$$f=\gamma\frac{\partial\varphi}{\partial\xi}+\delta\frac{\partial\varphi}{\partial\eta},\quad p=\gamma\frac{\partial\psi}{\partial\xi}+\delta\frac{\partial\psi}{\partial\eta},$$

here

$$\varphi\left(t,\ 0\right)=\varphi\left(\xi\left(t,\ 0\right)\ ,\eta\left(t,\ 0\right)\right),\qquad \psi\left(t,\ 0\right)=\psi\left(\xi\left(t,\ 0\right)\ ,\eta\left(t,\ 0\right)\right),$$
$$\varphi\left(t,L\right)=\varphi\left(\xi\left(t,\ L\right)\ ,\eta\left(t,\ L\right)\right),\qquad \psi\left(t,\ L\right)=\psi\left(\xi\left(t,\ L\right)\ ,\eta\left(t,\ L\right)\right),$$

$$f\left(t,0\right)=f\left(\xi\left(t,0\right),\ \eta\left(t,0\right),\ \gamma\left(t,0\right),\ \delta\left(t,0\right)\right),$$
$$p\left(t,0\right)=p\left(\xi\left(t,0\right),\ \eta\left(t,0\right),\ \gamma\left(t,0\right),\ \delta\left(t,0\right)\right),$$
$$f\left(t,L\right)=f\left(\xi\left(t,L\right),\ \eta\left(t,L\right),\ \gamma\left(t,L\right),\ \delta\left(t,L\right)\right),$$
$$p\left(t,L\right)=p\left(\xi\left(t,L\right),\ \eta\left(t,L\right),\ \gamma\left(t,L\right),\ \delta\left(t,L\right)\right).$$

The functions $e\left(t\right)$ and $h\left(t\right)$ are defined as

$$e\left(t\right):=\frac{a'\left(\eta\left(t,0\right)\right)\psi\left(t,0\right)}{\varphi\left(t,0\right)},\quad h\left(t\right):=\frac{b'\left(\xi\left(t,L\right)\right)\varphi\left(t,L\right)}{\psi\left(t,L\right)}.$$

In this section we present the results on the exponential stability of the numerical solution of the initial boundary difference problem for the mixed problem (22), (23), (24) (25) obtained by the authors in other works.

To obtain the initial-boundary difference problem, we will use the upwind difference scheme for the numerical calculation of system (1). For this, we will cover the spatial region [0, 1] with a uniform grid $\Omega_{\Delta x}=\left\{x_j=j\cdot\Delta x,\ \ j=\overline{0,J}\right\}$, Δx step by x.

For the numerical solution of the mixed problem (22), (23), (24), (25) we propose the following upwind explicit difference scheme

$$\begin{cases} \xi_j^{k+1} = \left(1 - [C_\varphi]_{j-1}^k\right) \xi_j^k + [C_\varphi]_{j-1}^k \, \xi_{j-1}^k, \; k = \overline{0, K-1}, \; j = \overline{1, J}, \\ \eta_j^{k+1} = \left(1 - [C_\psi]_{j+1}^k\right) \eta_j^k + [C_\psi]_{j+1}^k \, \eta_{j+1}^k, \; k = \overline{0, K-1}, \; j = \overline{0, J-1}, \\ \gamma_j^{k+1} = \left(1 - [C_\varphi]_{j-1}^k\right) \gamma_j^k + [C_\varphi]_{j-1}^k \, \gamma_{j-1}^k - \Delta t \cdot \gamma_j^k f_j^k, \; k = \overline{0, K-1}, \; j = \overline{1, J}, \\ \delta_j^{k+1} = \left(1 - [C_\psi]_{j+1}^k\right) \delta_j^k + [C_\psi]_{j+1}^k \, \delta_{j+1}^k + \Delta t \cdot \delta_j^k p_j^k, \; k = \overline{0, K-1}, \; j = \overline{0, J-1}, \\ \rho_j^{k+1} = \left(1 - [C_\varphi]_{j-1}^k\right) \rho_j^k + [C_\varphi]_{j-1}^k \, \rho_{j-1}^k - \Delta t \cdot \left[\rho_j^k f_j^k + \gamma_j^k \left(\frac{\partial f}{\partial x}\right)_j^k\right], \; j = \overline{1, J}, \\ \theta_j^{k+1} = \left(1 - [C_\psi]_{j+1}^k\right) \theta_j^k + [C_\psi]_{j+1}^k \, \theta_{j+1}^k + \Delta t \cdot \left[\theta_j^k p_j^k + \delta_j^k \left(\frac{\partial p}{\partial x}\right)_j^k\right], \; j = \overline{0, J-1}, \end{cases} \tag{26}$$

with boundary conditions

$$\begin{cases} \xi_0^k = a\left(\eta_0^k\right), \; \varphi_0^k \gamma_0^k = -a'\left(\eta_0^k\right) \psi_0^k \delta_0^k, \; \varphi_0^k \rho_0^k + \gamma_0^k f_0^k = -\left(e'(t)\right)^k \delta_0^k - e^k \left[\psi_0^k \theta_0^k + \delta_0^k p_0^k\right], \\ \eta_J^k = b\left(\xi_J^k\right), \; \psi_J^k \delta_J^k = -b'\left(\xi_J^k\right) \varphi_J^k \gamma_J^k, \; \psi_J^k \theta_J^k + \delta_J^k p_J^k = -\left(h'(t)\right)^k \gamma_J^k + h^k \left[\varphi_J^k \rho_J^k + \gamma_J^k p_J^k\right], \end{cases} \tag{27}$$

and with initial data

$$\begin{gathered} \xi_j^0 = \xi_0(x_j), \; \eta_j^0 = \eta_0(x_j), \; \gamma_j^0 = \xi_0'(x_j), \\ \delta_j^0 = \eta_0'(x_j), \; \rho_j^0 = \xi_0''(x_j), \; \theta_j^0 = \eta_0''(x_j), \; j \in \{0, 1, 2, \cdots, J\}, \end{gathered} \tag{28}$$

here

$$[C_\varphi]_j^k = \varphi_j^k \frac{\Delta t}{\Delta x}, \; [C_\psi]_j^k = \psi_j^k \frac{\Delta t}{\Delta x}, \; C_j^k = \max\left([C_\varphi]_j^k, [C_\psi]_j^k\right).$$

Let us introduce the following vectors into consideration

$$\begin{gathered} \boldsymbol{\xi} = (\xi, \gamma, \rho), \quad \boldsymbol{\eta} = (\eta, \delta, \theta), \; \boldsymbol{\xi}^* = (\xi^*, \gamma^*, \rho^*), \\ \boldsymbol{\eta}^* = (\eta^*, \delta^*, \theta^*), \quad \boldsymbol{\varphi}_1 = (\xi_0, {\xi'}_0, \xi''), \; \boldsymbol{\varphi}_2 = (\eta_0, {\eta'}_0, \eta''). \end{gathered}$$

and the following matrices:

$$\mathbf{U}^k \triangleq diag\left(\boldsymbol{\eta}_0^k, \boldsymbol{\xi}_1^k, \boldsymbol{\eta}_1^k, \cdots\cdots, \boldsymbol{\xi}_{J-1}^k, \boldsymbol{\eta}_{J-1}^k, \boldsymbol{\xi}_J^k\right), \quad \mathbf{U}^* \triangleq diag\left(\boldsymbol{\eta}^*, \overbrace{\boldsymbol{\xi}^*, \boldsymbol{\eta}^*, \cdots\cdots, \boldsymbol{\xi}^*, \boldsymbol{\eta}^*}^{6J}, \boldsymbol{\xi}^*\right),$$

$$\mathbf{U}^0 \triangleq diag\left(\boldsymbol{\varphi}_2(x_0), \boldsymbol{\varphi}_1(x_1), \boldsymbol{\varphi}_2(x_1), \cdots\cdots, \boldsymbol{\varphi}_1(x_{J-1}), \boldsymbol{\varphi}_2(x_{J-1}), \boldsymbol{\varphi}_1(x_J)\right).$$

Definition 2.1. The equilibrium state $\mathbf{U}^*$ of the boundary difference problem (26), (27) is stable in the l^2-norm if there exist positive real constants $n_1 > 0$, $n_2 > 0$ such that for any initial vector function $\mathbf{\Phi}$ the solution $\mathbf{U^k}$, $k \in \{1, 2, \cdots, K\}$ of the boundary difference problem (26), (27) satisfies the inequality

$$\left\|\mathbf{U^k} - \mathbf{U}^*\right\|_{l^2} \leqslant n_2 e^{-n_1 t^k} \left\|\mathbf{\Phi} - \mathbf{U}^*\right\|_{l^2}, \quad k \in \{1, 2, \cdots\}, \tag{29}$$

where

$$\mathbf{U}^k \triangleq \left(\boldsymbol{\eta}_0^k, \boldsymbol{\xi}_1^k, \boldsymbol{\eta}_1^k, \cdots\cdots, \boldsymbol{\xi}_{J-1}^k, \boldsymbol{\eta}_{J-1}^k, \boldsymbol{\xi}_J^k\right)^T, \qquad \mathbf{U}^* \triangleq \overbrace{\left(\boldsymbol{\eta}^*, \boldsymbol{\xi}^*, \boldsymbol{\eta}^*, \cdots\cdots, \boldsymbol{\xi}^*, \boldsymbol{\eta}^*, \boldsymbol{\xi}^*\right)^T}^{6J},$$

$$\mathbf{\Phi} \triangleq \left(\boldsymbol{\varphi}_2(x_0), \boldsymbol{\varphi}_1(x_1), \boldsymbol{\varphi}_2(x_1), \cdots\cdots, \boldsymbol{\varphi}_1(x_{J-1}), \boldsymbol{\varphi}_2(x_{J-1}), \boldsymbol{\varphi}_1(x_J)\right).$$

and

$$\left\|\mathbf{U}^k-\mathbf{U}^*\right\|_{l^2}^2 \triangleq \Delta x\left(\left[\boldsymbol{\eta}_0^k-\boldsymbol{\eta}^*\right]^T,\left[\boldsymbol{\eta}_0^k-\boldsymbol{\eta}^*\right]^T\right)+h\left(\left[\boldsymbol{\xi}_J^k-\boldsymbol{\xi}^*\right]^T,\left[\boldsymbol{\xi}_J^k-\boldsymbol{\xi}^*\right]^T\right)+$$
$$+\Delta x \sum_{j=1}^{J-1}\left\{\left(\left[\boldsymbol{\xi}_j^k-\boldsymbol{\xi}^*\right]^T,\left[\boldsymbol{\xi}_j^k-\boldsymbol{\xi}^*\right]^T\right)+\left(\left[\boldsymbol{\eta}_j^k-\boldsymbol{\eta}^*\right]^T,\left[\boldsymbol{\eta}_j^k-\boldsymbol{\eta}^*\right]^T\right)\right\},$$

$$\left\|\boldsymbol{\Phi}-\mathbf{U}^*\right\|_{l^2} \triangleq \Delta x\left(\left[\boldsymbol{\varphi}_2\left(x_0\right)-\boldsymbol{\eta}^*\right]^T,\left[\boldsymbol{\varphi}_2\left(x_0\right)-\boldsymbol{\eta}^*\right]^T\right)+$$
$$+\Delta x\left(\left[\boldsymbol{\varphi}_1\left(x_J\right)-\boldsymbol{\xi}^*\right]^T,\left[\boldsymbol{\varphi}_1\left(x_J\right)-\boldsymbol{\xi}^*\right]^T\right)+$$
$$+\Delta x \sum_{j=1}^{J-1}\left\{\left(\left[\boldsymbol{\varphi}_1\left(x_j\right)-\boldsymbol{\xi}^*\right]^T,\left[\boldsymbol{\varphi}_1\left(x_j\right)-\boldsymbol{\xi}^*\right]^T\right)+\left(\left[\boldsymbol{\varphi}_2\left(x_j\right)-\boldsymbol{\eta}^*\right]^T,\left[\boldsymbol{\varphi}_2\left(x_j\right)-\boldsymbol{\eta}^*\right]^T\right)\right\},$$
$$k \in\{0,1, \cdots\}.$$

Definition 2.2. (Discrete Lyapunov function). The function $L^k: \mathrm{R}^{6 \times J} \rightarrow \mathrm{R}_0^{+}$ is said to be a discrete Lyapunov function of the boundary-value difference problem (27), (28) if

1. there exist constants $h_1>0$, $h_2>0$ such that for all $k \in\{0,1, \cdots, K\}$ the inequality
$$h_1\left\|\mathbf{U}^{\mathbf{k}}-\mathbf{U}^*\right\|_{l^2}^2 \leqslant \mathbf{L}^k\left(\mathbf{U}^{\mathbf{k}}\right) \leqslant h_2\left\|\mathbf{U}^{\mathbf{k}}-\mathbf{U}^*\right\|_{l^2}^2, \tag{30}$$
2. there exists a positive constant $n>0$ such that for all $k \in\{0,1, \cdots, K\}$ the inequality
$$\frac{\mathbf{L}^k\left(\mathbf{U}^{k+1}\right)-\mathbf{L}^k\left(\mathbf{U}^k\right)}{\Delta t} \leqslant-n \mathbf{L}^k\left(\mathbf{U}^k\right). \tag{31}$$

In order to simplify the notation, we introduce into consideration a sequence of discrete quantities $\mathcal{L}^k$ as $\mathrm{L}^k=L^k\left(\mathbf{U}^k\right), \quad k \in\{0,1, \cdots, K\}$,
Where $\mathbf{U}^k$ is a given solution of the initial-boundary value problem (26), (27), (28).

It should be noted that the presence of a discrete Lyapunov function ensures the stability of the equilibrium state.

Let us assume that the boundary condition functions (27) $a, b: \mathrm{R} \rightarrow \mathrm{R}$ satisfy the inequalities:
$$\begin{gathered}\max_{0 \leqslant k \leqslant K}\left|a\left(\eta_0^k\right)\right|<+\infty, \quad \max_{0 \leqslant k \leqslant K}\left|a'\left(\eta_0^k\right)\right|<+\infty, \\ \max_{0 \leqslant k \leqslant K}\left|b\left(\xi_J^k\right)\right|<+\infty, \quad \max_{0 \leqslant k \leqslant K}\left|b'\left(\xi_J^k\right)\right|<+\infty.\end{gathered} \tag{32}$$
and denote
$$\varkappa_0=a'(0), \quad \varkappa_L=b'(0).$$
Let us assume that $\varkappa_0$, $\varkappa_L$, m, A, B satisfy the inequalities $\left|\varkappa_0 \varkappa_L\right|<1$, $\mathrm{A} \varkappa_0^2-\mathrm{B}<0$ and $\mathrm{B} \exp \left(\frac{m}{\bar{\psi}} L\right) \varkappa_L^2-\mathrm{A} \exp \left(-\frac{m}{\bar{\varphi}} L\right)<0$.

Theorem 2.1.(On exponential stability in the case of $\mathbf{U}^* \geqslant 0$). Suppose that the necessary Courant-Friedrichs-Lewy (CFL) stability condition of type
$$C=\max \left(\max_{\substack{0 \leqslant k \leqslant K \\ 0 \leqslant j \leqslant J}}\left|\left[C_{\varphi}\right]_j^k\right|, \max_{\substack{0 \leqslant k \leqslant K \\ 0 \leqslant j \leqslant J}}\left|\left[C_{\psi}\right]_j^k\right|\right)<1,$$

is satisfied on the solutions of the difference scheme (26), (27), (28). For each equilibrium state $\mathbf{U}^*$ satisfying the inequality $\mathbf{U}^* \geqslant 0$, and each $\varkappa_0, \varkappa_L$ satisfying the inequality $0 < |\varkappa_0 \varkappa_L| < 1$, and each constant $\varepsilon > 0$ and for each initial vector function $\mathbf{\Phi}$ satisfying the matrix inequality $\mathbf{U}^0 \geqslant 0$, and inequality

$$\|\mathbf{\Phi} - \mathbf{U}^*\|_{l^2} < \varepsilon, \tag{33}$$

the solution $\mathbf{U}^k$ of the initial-boundary difference problem (26), (27), (28) satisfies the matrix inequality $\mathrm{U}^k \geqslant 0, \quad k \in \{0, 1, \cdots, K\}$ and the equilibrium state $\mathbf{U}^*$ of the boundary difference problem (27), (28 is stable in the l^2-norm.
For any $\mathbf{U}^k \in \mathrm{R}^{6\times J}$, as a candidate for the discrete Lyapunov function for the initial-boundary difference problem (26), (27), (28) we consider the following function.

$$\mathbf{L}\left(\mathbf{U}^k\right) = \Delta x \sum_{j=1}^{J} \left[\frac{\mathrm{A}}{\bar{\varphi}}\left(\xi_j^k\right)^2 + \bar{\varphi}\mathrm{A}\left(\gamma_j^k\right)^2 + \bar{\varphi}^3\mathrm{A}\left(\rho_j^k\right)^2\right] \exp\left(-\frac{m}{\bar{\varphi}} x_{j-1}\right) +$$
$$+\Delta x \sum_{j=0}^{J-1} \left[\frac{\mathrm{B}}{\bar{\psi}}\left(\eta_j^k\right)^2 + \bar{\psi}\mathrm{B}\left(\delta_j^k\right)^2 + \bar{\psi}^3\mathrm{B}\left(\theta_j^k\right)^2\right] \exp\left(\frac{m}{\bar{\psi}} x_{j+1}\right).$$

2.5 Numerical example for a quasilinear problem

In the domain $\bar{\omega} \triangleq \{(t,x) : 0 \leqslant t \leqslant 1, 0 \leqslant x \leqslant 1\}$ we consider a mixed problem for the following quasilinear hyperbolic system

$$\begin{cases} \xi_t + \varphi(\xi,\eta)\,\xi_x = 0, \\ \eta_t - \psi(\xi,\eta)\,\eta_x = 0, \end{cases} \qquad 0 < t \leqslant 1, 0 < x < 1. \tag{34}$$

Example 3.1. As an example, consider the case $\varphi(\xi,\eta) = \xi, \quad \psi(\xi,\eta) = \eta$. Then system (34) can be rewritten as follows

$$\begin{cases} \xi_t + \xi\xi_x = 0, \\ \eta_t - \eta\eta_x = 0, \end{cases} \qquad 0 < t \leqslant 1, \;\; 0 < x < 1. \tag{35}$$

Let the initial and boundary conditions for the above problem be given as follows. with boundary conditions at $x = 0, L$

$$\begin{cases} \xi(t,0) = \varkappa_0 \cdot \eta(t,0), \\ \eta(t,1) = \varkappa_L \cdot \xi(t,1), \end{cases} \qquad 0 < t \leqslant 1, \tag{36}$$

and with initial data.

$$\begin{cases} \xi(0,x) = \xi_0(x), \\ \eta(0,x) = \eta_0(x), \end{cases} \qquad 0 < x < 1\,. \tag{37}$$

Computing experience
We conduct computational experiments to demonstrate the stability of the numerical solution of a quasilinear hyperbolic system with initial dissipative boundary conditions (35), (36), (37), obtained using the upwind difference scheme in the Lyapunov sense and in the l^2-norm, when the conditions of Theorem 3.1 are met.
We take the following steps:
Step 1. We set the initial parameters.

$$L := 1, \quad T := 1\,, J := 55, \quad K := 100, \quad \Delta x := \frac{1}{J}, \quad \Delta t := \frac{1}{K}.$$

Step 2. We verify if the data meets the CFL condition.

$$[C_\varphi]_j^k = \varphi_j^k \frac{\Delta t}{\Delta x}, \ \ [C_\psi]_j^k = \psi_j^k \frac{\Delta t}{\Delta x}, \ \ C_j^k = \max\left([C_\varphi]_j^k, [C_\psi]_j^k\right).$$

$$C = \max\left(\max_{\substack{0 \leqslant k \leqslant K \\ 0 \leqslant j \leqslant J}} \left|[C_\varphi]_j^k\right|, \max_{\substack{0 \leqslant k \leqslant K \\ 0 \leqslant j \leqslant J}} \left|[C_\psi]_j^k\right| \right) < 1.$$

Step 3. Introducing the initial conditions:

$$\xi(0,x) := \xi_0(x), \quad \eta(0,x) := \eta_0(x).$$

Step 4. We present the boundary condition parameters:

$$\varkappa_0 = \frac{1}{2}, \varkappa_L = \frac{1}{5}.$$

Step 5. We confirm the stability condition as stated in Theorem 1.

$$|\varkappa_0 \cdot \varkappa_L| = \frac{1}{2} \cdot \frac{1}{5} = \frac{1}{10} < 1.$$

Step 6. $For \ \ j = 0, ..., J$ set $\begin{matrix} \xi_{0,j} = \xi_0(x_j). \\ \eta_{0,j} = \eta_0(x_j). \end{matrix}$

Step 7. We compute in time intervals.

$$For \ \ k = 0, ..., 9K-1,$$

$$\begin{matrix} for \ \ j = 1, ..., J & \xi_{k+1,j} = \xi_{k,j} - C\xi_{k,j-1}(\xi_{k,j} - \xi_{k,j-1}). \\ for \ \ j = 0, ..., J-1 & \eta_{k+1,j} = \eta_{k,j} - C\eta_{k,j+1}(\eta_{k,j} - \eta_{k,j+1}). \end{matrix}$$

Step 8. We calculate the boundary conditions

$$c\xi_{k+1,0} = \varkappa_0 \cdot \eta_{k+1,0},$$
$$\eta_{k+1,J} = \varkappa_L \cdot \xi_{k+1,J.}$$

Step 9. We calculate the l^2-norm of the solution using the discrete Lyapunov function.

$$\mathbf{L}\left(\mathbf{U}^k\right) = \Delta x \sum_{j=1}^{J} \left[\frac{\mathrm{A}}{\bar{\varphi}} \left(\xi_j^k\right)^2 + \bar{\varphi}\mathrm{A}\left(\gamma_j^k\right)^2 + \bar{\varphi}^3 \mathrm{A}\left(\rho_j^k\right)^2 \right] \exp\left(-\frac{m}{\bar{\varphi}} x_{j-1}\right) +$$
$$+\Delta x \sum_{j=0}^{J-1} \left[\frac{\mathrm{B}}{\bar{\psi}} \left(\eta_j^k\right)^2 + \bar{\psi}\mathrm{B}\left(\delta_j^k\right)^2 + \bar{\psi}^3 \mathrm{B}\left(\theta_j^k\right)^2 \right] \exp\left(\frac{m}{\bar{\psi}} x_{j+1}\right).$$

$$L_k = \sum_{j=1}^{J} \left[\Delta x \cdot (\xi_{k+1,j})^2 \cdot e^{-(j-1)\cdot\Delta x}\right] + \sum_{j=0}^{J-1} \left[\Delta x \cdot (\eta_{k+1,j})^2 \cdot e^{(j+1)\cdot\Delta x}\right].$$

OUTPUT(L)

Step 10. STOP. (Theprocedure is complete.

We present the initial conditions as follows.

No	Solution	$\xi_0(x)$ – initial condition	$\eta_0(x)$ – initial condition
1	$U1_k$	$\xi_0(x) = x + 1$	$\eta_0(x) = \frac{3-x}{5}$
2	$U2_k$	$\xi_0(x) = \frac{1}{\pi} \cdot \cos(0,1 \cdot x)$	$\eta_0(x) = e^{-0.5 \cdot x^2 - x}$
3	$U3_k$	$\xi_0(x) = \sin(\pi \cdot x) \cdot e^{-x}$	$\eta_0(x) = \frac{1}{0.2+\sqrt{x}}$

The graph demonstrates that the solution is stable in l^2-norm for arbitrary initial conditions, as long as the theorem's conditions are satisfied.

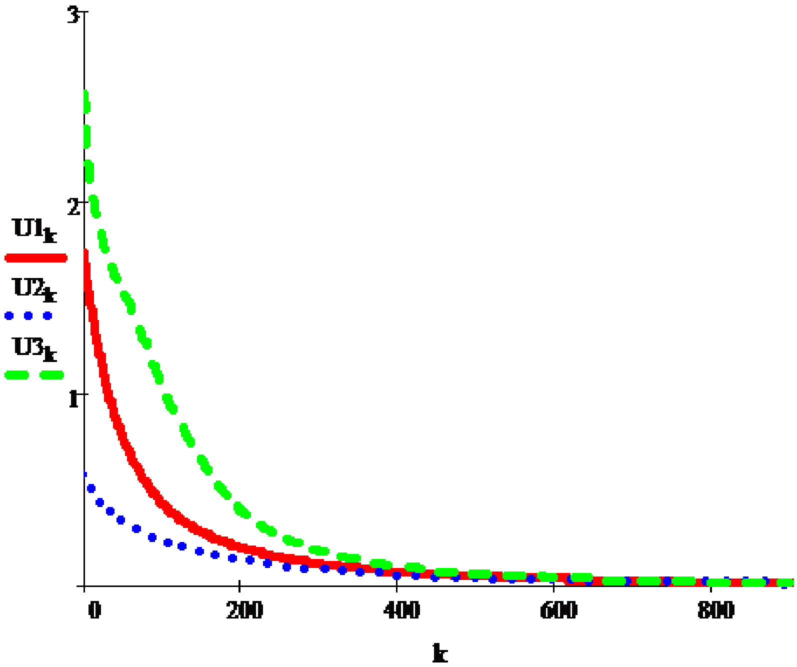

Figure 1 The exponential stability

It follows that, according to Theorem 2.1, the numerical solution $\mathbf{U}^k$ of the initial boundary difference problem is exponentially stable in the l^2-norm. (See Figure 1). Above is a table and graph of the values of the k dependent numerical solution in the l^2-norm, confirming its exponential stability.

3 Conclusion

Thus, in this work, the problem of exponential stability of the numerical solution of the upwind difference scheme for a quasilinear hyperbolic system with dissipative boundary conditions is numerically investigated. An upwind difference scheme is constructed to numerically solve the initial boundary value problem. The exponential stability of the numerical solution to the equilibrium state of the initial-boundary difference problem is defined. A numerical experiment was conducted to construct the discrete Lyapunov function, and the theorem on the exponential stability of the equilibrium state of the initial-boundary difference problem for a quasilinear hyperbolic system was numerically substantiated.

Conflict of interest

The authors have no conflicts of interest to declare.

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ЧИСЛЕННОЕ ИССЛЕДОВАНИЕ УСТОЙЧИВОСТИ ПО ЛЯПУНОВУ ПРОТИВОПОТОЧНОЙ РАЗНОСТНОЙ СХЕМЫ ДЛЯ КВАЗИЛИНЕЙНОЙ ГИПЕРБОЛИЧЕСКОЙ СИСТЕМЫ

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В данной работе рассматривается смешанная задача для квазилинейной системы гиперболических уравнений, выраженной в инвариантах Римана, с учётом диссипативных нелинейных граничных условий. Для численного решения задачи предложена начально-граничная разностная проблема, основанная на разностной схеме против потока. Исследуется устойчивость нелинейных разностных схем с акцентом на установление достаточного критерия устойчивости, основанного на векторных функциях Ляпунова. Предложенный критерий развивает предыдущие теоретические результаты, в которых была построена дискретная функция Ляпунова для доказательства экспоненциальной устойчивости стационарного состояния квазилинейной системы. Численные расчёты для модельной задачи подтверждают эти теоретические выводы. Исследование подчёркивает перспективность адаптации прямого метода Ляпунова для анализа устойчивости нелинейных гиперболических систем путём построения положительно определённой функции, монотонно убывающей вдоль решений системы.

Ключевые слова: экспоненциальная устойчивость, гиперболическая система, смешанная задача, разностная схема, функция Ляпунова.

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