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CONSTRUCTION OF AN ALGEBRAIC-HYPERBOLIC NATURAL TENSION SPLINE OF EIGHTH ORDER

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This paper first demonstrates that an algebraic-hyperbolic spline of eighth order minimizes the norm within a Hilbert space framework. Subsequently, employing Sobolev's method, which involves constructing a discrete analogue of the differential operator, the spline function is developed. Unknown coefficients of the spline are computed according to predefined smoothness criteria and boundary conditions. As a result, the constructed spline exhibits exceptional smoothness, enhances interpolation accuracy, and precisely reproduces hyperbolic functions, linear polynomials, and constants. The findings indicate that this approach is highly effective for applications requiring smooth interpolation and accurate modeling of physical phenomena. Additionally, incorporating tension parameters enables precise adjustment of the spline's stiffness or flexibility.

Keywords: Hilbert space, generalized spline, algebraic-hyperbolic spline, convolution, discrete analogue.

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1 Introduction

The history of the spline functions goes back to the work of draftsmen, who often had to draw a smoothly turning curve between points on a drawing [1]. This process is called wrapping, and it can be accomplished with a number of special devices, such as a French curve, made of plastic and presenting the draftsmen with a choice of curves of varying curvature. Long wooden strips were also used, which were passed through control points by means of weights placed on the draftsman's table and attached to the strips. The weights were called ducks, and the wooden strips were called splines as early as 1891. The elasticity of the wooden strips allowed them to bend only slightly as they passed through the given points. In effect, the wood solved a differential equation and minimized the strain energy. The latter, as is known, is a simple function of curvature.

The mathematical theory of these curves owes much to early researchers especially Isaac Schoenberg in the 1940s and 1950s [2]. His original work involved numerical procedures for solving differential equations, where a sufficiently fundamental and detailed study was conducted, however, splines without the use of the term itself were studied earlier, for example V. Quade and L. Collatz studied periodic splines in 1938 (see historical notes in the monograph by L. Schumeiker [3, p. 10]).However, the intensive study of splines began only in the early 1960s. A rather simple work by J. Holliday [4], in by which he noted that piecewise cubic functions of class C^2 minimize the functional

$$\int f''(x)dx,$$

close to the deformation energy integral describing the profile of an elastic rod fixed in some set of points. Such functions immediately became the basis of the description apparatus and approximations of curves and surfaces, in which the urgent need arose by that time due to the emergence of the first computers. The much broader applications of splines to the areas of data fitting and computer-aided graphic design became evident with the widespread availability of computers.

The theory of splines was first systematically presented by Ahlberg, Nilson, Uolsh in [5]. In this work, we construct an eighth order algebraic-hyperbolic natural spline, which is a type of generalized spline. Now, we give the definition of generalized splines based on the book [5]. Let L be a linear differential operator

$$L \equiv a_n \frac{d^n}{dx^n} + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} + ... + a_0(x),$$

where, $a_j(x) \in C^j(a,b),\ j = 0, 1, ..., n$ and $a_n(x) \neq 0,\ x \in [a,b]$. The operator L^* is the conjugate to L and has the form:

$$L^* \equiv (-1)^n \frac{d^n}{dx^n}\{a_n(x)\cdot\} + (-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}}\{a_{n-1}(x)\cdot\} + ... + a_0(x).$$

Definition 1.1. Let a mesh $\Delta : a = x_0 < x_1 < ... < x_n = b$ be given in $[a,b]$. A generalized spline with defect k $(0 \leqslant k \leqslant n)$ on the mesh Δ is a function $S_\Delta(x)$ from the class $K^{(2n-k)}(a,b)$ and satisfying the following differential equation

$$L^* L S_\Delta(x) = 0,$$

on each interval (x_{i-1}, x_i), $(i = 1, 2, ..., n)$. We say that, $S_\Delta(x)$ spline has an order $2m$, when we need to specify the order of the operator L^*L defining $S_\Delta(x)$. Where, $K^{(2n-k)}$ is a class of functions defined on the interval $[a,b]$ that have an absolutely continuous $(2n-k-1)^{st}$ derivative and the $(2n-k)^{th}$ derivative belonging to the space $L_2(a,b)$. The defect is usually taken to be 1, and the $(2n-1)^{st}$ order derivative of the spline $S_\Delta(x)$ has a discontinuity on the mesh Δ, and the spline consists of smoothly connected piecewise functions.

We present some of the main results on splines. In [6], a greedy algorithm for exponential-polynomial splines is given. [7], is proposed an integro spline quasi-interpolant based on second order Uniform Algebraic Hyperbolic functions. The main tool of this approach is Marsden's identity. The advantage of this method is that it does not need any additional data and does not require the solution of any system of equations. This construction can be extended to derive fourth order approximating splines, but the expressions provided for the coefficients of the quasi-interpolants are very complex. The construction of certain types of L splines, which are also used in many practical areas, is given in [8], [9], [10], [11], because, an important task in the theory of L- splines is to construct them. In this work, we will consider the properties of eighth-order algebraic-hyperbolic natural tension spline and construct it. This work consists of the following sections: in the second section, we give a definition of eighth-order algebraic-hyperbolic natural tension spline, then in the third section, the properties of the minimum norm for the spline are studied, in the next section, we will find the form of the spline and obtain a system of equations for its coefficients, sections 5, 6, and 7 consist of an algorithm for solving a system of equations, calculating the coefficients of the interpolation spline, and a conclusion, respectively.

2 Eighth order algebraic-hyperbolic tension natural spline

Now, we give a definition of the eighth order algebraic-hyperbolic spline by following the definition given [5].

Let L_4 be a linear operator given by the formula

$$L_4 \equiv \frac{d^4}{dx^4} - v^2 \frac{d^2}{dx^2}. \qquad (1)$$

Then L_4^* the operator conjugate to L_4 has the form:

$$L_4^* \equiv \frac{d^4}{dx^4} - v^2 \frac{d^2}{dx^2}. \qquad (2)$$

Definition 2.1. Eighth order algebraic-hyperbolic tension spline of defect 1 relative to the mesh: $a = x_0 < x_1 < ... < x_N = b$ on the segment $[a, b]$ is the function $S_\Delta(x)$ from the class $K^{(7)}(a, b)$ satisfying the differential equation

$$L_4^* L_4 S = 0, \qquad (3)$$

on each open interval (x_{i-1}, x_i), $i = 1, 2, ..., N$.
Class $K_{4,v}$ is a factorized Hilbert space where the inner product is introduced as follows:

$$< f, g >_{K_{4,v}} = \int_a^b (f^{(4)}(x) - v^2 f^{(2)}(x))(g^{(4)}(x) - v^2 g^{(2)}(x))dx, \qquad (4)$$

the norm is defined using the inner product as follows:

$$\|f\|_{K_{4,v}} = \sqrt{< f, f >}, \qquad (5)$$

for convenience, we take the interval $[0, 1]$ instead of $[a, b]$ and $0 = x_0 < x_1 < ... < x_N = 1$. Let us be given the corresponding values $Y : y_0,\ y_1, ...,\ y_N$ at the nodes $x_0 < x_1 < ... < x_N$. We construct a spline that satisfies the following interpolation condition:

$$S_\Delta(Y, x_j) = y_j,\ j = 0, 1, ..., N.$$

In the next section, we consider that under what conditions the spline $S_\Delta(Y, x_j)$ gives a minimum to the norm in the space $K_{4,v}$

3 The first integral identity for the eighth order algebraic hyperbolic spline

Now, we consider the following problem in the space $K_{4,v}$.

Problem 1. Find the function $f(x) \in K_{4,v}$ which gives minimum to the semi-norm (5) and satisfies the interpolation condition

$$f(x_j) = y_j,\ j = 0, 1, ..., N,\ x_j \in [0, 1].$$

Now we will show that among the functions $f(x)$, an algebraic hyperbolic spline of the eighth order minimizes the norm in the space $K_{4,v}$

Theorem 3.1. If the function $f(x)$ belongs to the space $K_{4,v}$ then the spline $S_\Delta(f; x)$ interpolates it on the mesh $0 = x_0 < x_1 < ... < x_N = 1$ and one of the following conditions fulfilled:

1. at the end points of the mesh Δ for the eighth order algebraic-hyperbolic tension spline $S_\Delta(f, x)$ the equalities $(L_4 S_\Delta(f; 1))^\alpha = (L_4 S_\Delta(f; 0))^\alpha = 0$, $\alpha = 0, 1, 2$ are valid;

2. The function $f(x)$ and eighth order algebraic-hyperbolic tension spline $S_\Delta(f,x)$ satisfy the boundary conditions $f^{(k)}(0)=S_\Delta^{(k)}(f,0),\ f^{(k)}(1)=S_\Delta^{(k)}(f,1)\ k=1,2,3.$
3. The function $f(x)$ and eighth order algebraic-hyperbolic tension spline $S_\Delta(f,x)$ are periodic;

Then the integral identity holds

$$\int_0^1 \{Lf(x)\}^2dx=\int_0^1 \{LS_\Delta(f;x)\}^2dx+\int_0^1 \{L[f(x)-S_\Delta(f;x)]\}^2dx. \tag{6}$$

It should be noted that a eighth order algebraic-hyperbolic tension spline is called:

1. $S_\Delta(f,x)$ is called natural interpolation spline, when condition a) satisfied.
2. $S_\Delta(f,x)$ is called clamped interpolation spline, when condition b) satisfied.
3. $S_\Delta(f,x)$ is called periodic interpolation spline, when condition c) satisfied.

Prove. Let $u(x)$ and $v(x)$ be functions from the Hilbert space $K_{4,v}$. Integrating by parts the indefinite integral of the product $L_4u(x)v(x)$ we obtain the following identity.

$$\int L_4u(x)v(x)dx=\sum_{j=0}^{3}u^{(3-j)}\sum_{k=0}^{j}(-1)^k\{a_{4-j+k}(x)v(x)\}^{(k)}+\int u(x)L_4^*v(x)dx, \tag{7}$$

where a_k are the coefficients of the operator L_4 and in our case, they are

$$a_4=1,\ a_3=0,\ a_2=-v^2, a_1=0,\ a_0=0. \tag{8}$$

We differentiate both sides of equation (7) and then we put $u(x)=f(x)-S_\Delta(f,x)$ and $v(x)=L_4S_\Delta(f,x)$ on it. Then we obtain the following equality by integrating the resulting equality over $[x_{i-1},x_i]$, $i=1,2,...,N$ interval

$$\int_{x_{i-1}}^{x_i} L_4\{f(x)-S_\Delta(f,x)\}L_4S_\Delta(f,x)dx=$$

$$=\sum_{j=0}^{3}[f(x)-S_\Delta(f,x)]^{(3-j)}\sum_{k=0}^{j}(-1)^{(k)}\{a_{4-j+k}(x)L_4S_\Delta(f,x)\}^{(k)}\Bigg|_{x_{i-1}}^{x_i}+$$

$$+\int_{x_{i-1}}^{x_i}\{f(x)-S_\Delta(f,x)\}L_4^*L_4S_\Delta(f,x)dx.$$

From here, taking into account equation (3), we come to the following

$$\int_{x_{i-1}}^{x_i} L_4\{f(x)-S_\Delta(f,x)\}L_4S_\Delta(x)dx=$$

$$=(\sum_{j=0}^{3}[f(x)-S_\Delta(f,x)]^{(3-j)}\sum_{k=0}^{j}(-1)^k\{a_{4-j+k}(x)L_4S_\Delta(f,x)\}^{(k)})\Bigg|_{x_{i-1}}^{x_i}. \tag{9}$$

Next we consider the following identity.

$$\int_0^1\{L_4[f(x)-S_\Delta(f,x)]\}^2dx=$$

$$= \int_0^1 \{L_4 f(x)\}^2 dx - \int_0^1 \{L_4 S_\Delta(f,x)\}^2 dx - 2\int_0^1 L_4[f(x) - S_\Delta(f,x)] L_4 S_\Delta(f,x) dx.$$

From here, taking into account (9), we obtain the following.

$$\int_0^1 \{L_4[f(x) - S_\Delta(f,x)]\}^2 dx = \int_0^1 \{L_4 f(x)\}^2 dx - \int_0^1 \{L_4 S_\Delta(f,x)\}^2 dx -$$

$$-2\sum_{i=1}^{N} \left(\sum_{j=0}^{3}[f(x) - S_\Delta(f,x)]^{(3-j)} \sum_{k=0}^{j}(-1)^k \{a_{4-j+k}(x) L_4 S_\Delta(f,x)\}^{(k)}\right)\Bigg|_{x_{i-1}}^{x_i}. \tag{10}$$

Now, we study the conditions on the function $f(x)$ and on the generalized spline $S_\Delta(f,x)$ that the last sum becomes zero in (10).
Let us consider the last sum in identity (10) and denote it as follows:

$$A(f,S_\Delta) = \sum_{i=1}^{N} \left(\sum_{j=0}^{3}[f(x) - S_\Delta(f,x)]^{(3-j)} \sum_{k=0}^{j}(-1)^k \{a_{4-j+k}(x) L_4 S_\Delta(f,x)\}^{(k)}\right)\Bigg|_{x_{i-1}}^{x_i},$$

for $A(f,S_\Delta)$, expanding the integral sum, we have

$$\begin{aligned}A(f,S_\Delta) = &\sum_{i=1}^{N} \{[f'''(x) - S'''_\Delta(f,x)] \cdot a_4(x) L_4 S_\Delta(f,x)\}|_{x_{i-1}}^{x_i} + \\ &+\sum_{i=1}^{N} \{[f''(x) - S''_\Delta(f,x)] \cdot (a_3(x) L_4 S_\Delta(f,x) - (a_4(x) L_4 S_\Delta(f,x))')\}|_{x_{i-1}}^{x_i} + \\ &+\sum_{i=1}^{N} \{[f'(x) - S'_\Delta(f,x)] \cdot (a_2(x) L_4 S_\Delta(f,x) - (a_3 L_4 S_\Delta(f,x))' + (a_4 L_4 S_\Delta(f,x))'')\}|_{x_{i-1}}^{x_i} + \\ &+\sum_{i=1}^{N} [f(x) - S_\Delta(f,x)] \cdot (a_1(x) L_4 S_\Delta(f,x) - (a_2 L_4 S_\Delta(f,x))' + \\ &+ (a_3 L_4 S_\Delta(f,x))'' - (a_4 L_4 S_\Delta(f,x))''')\}|_{x_{i-1}}^{x_i}.\end{aligned} \tag{11}$$

From Theorem 3.1, it is known that the spline $S_\Delta(f,x)$ interpolates the function $f(x)$ on the mesh Δ in $[0,1]$, that is

$$S_\Delta(f,x_j) = f(x_j),\ j = 0,1,...,N.$$

Therefore, the last sum in the expression (11) becomes zero and taking into account expression (8), we obtain the following expression

$$\begin{aligned}A(f,S_\Delta) = &\sum_{i=1}^{N} \{[f'''(x) - S'''_\Delta(f,x)] \cdot L_4 S_\Delta(f,x)\}|_{x_{i-1}}^{x_i} + \\ &+\sum_{i=1}^{N} \{[f''(x) - S''_\Delta(x)] \cdot (-L_4 S'_\Delta(f,x))\}|_{x_{i-1}}^{x_i} + \\ &+\sum_{i=1}^{N} \{[f'(x) - S'_\Delta(f,x)] \cdot (-v^2 L_4 S_\Delta(f,x) + L_4 S''_\Delta(f,x))\}\Big|_{x_{i-1}}^{x_i}.\end{aligned} \tag{12}$$

When we expand the sums in (12), leaving only the first and last terms and shortening the remaining middle terms, we get the following expression

$$\begin{aligned}
A(f,S_\Delta) = \sum_{i=1}^{N} \{[f'''(x_N) - S'''_\Delta(x_N)] \cdot L_4 S_\Delta(f,x_N)\}\Big|_{x_{i-1}}^{x_i} - \\
- \sum_{i=1}^{N} \{[f'''(x_0) - S'''_\Delta(x_0)] \cdot L_4 S_\Delta(f,x_0)\}\Big|_{x_{i-1}}^{x_i} + \\
+ \sum_{i=1}^{N} \{[f''(x_N) - S''_\Delta(x_N)] \cdot (-L_4 S'_\Delta(f,x_N))\}\Big|_{x_{i-1}}^{x_i} - \\
- \sum_{i=1}^{N} \{[f''(x_0) - S''_\Delta(x_0)] \cdot (-L_4 S'_\Delta(f,x_0))\}\Big|_{x_{i-1}}^{x_i} + \\
+ \sum_{i=1}^{N} \{[f'(x_N) - S'_\Delta(x_N)] \cdot (-v^2 L_4 S_\Delta(f,x_N) + L_4 S''_\Delta(f,x_N))\}\Big|_{x_{i-1}}^{x_i} - \\
- \sum_{i=1}^{N} \{[f'(x_0) - S'_\Delta(f,x_0)] \cdot (-v^2 L_4 S_\Delta(f,x_0) + L_4 S''_\Delta(f,x_0))\}\Big|_{x_{i-1}}^{x_i}.
\end{aligned} \tag{13}$$

For expression (13) to be equal to zero, it is sufficient to satisfy one of the conditions of Theorem 3.1. This result is based on the general result given in [5] and (6) integral identity is called as the first integral identity. Since, $A(f,S_\Delta) = 0$, the first integral identity holds. Consequently, the function minimizing the norm in the space $K_{4,v}$ is the spline $S_\Delta(f,x)$ satisfying one of the conditions of Theorem 3.1.
In this work, we construct a natural spline satisfying only a) condition from the conditions of Theorem 3.1.

4 System of equations for the coefficients of the spline

From now on, instead of the term eighth-order algebraic hyperbolic natural tension spline, we use the term of natural spline. We know that, the kernel of the operator L_4 consists of $\sinh vx$,$\cosh vx$,x, 1 functions and the kernel of the operator $L_4^* L_4$ consists of $\sinh vx$, $\cosh vx$,$vx\sinh v$,$vx\cosh vx$,x^3, x^2, x, 1 functions. Then the natural spline $S_\Delta(f,x)$ is uniquely determined by the following conditions.

1. $S_\Delta(f,x)$ consists of a linear combinations of functions $\sinh vx$, $\cosh vx$, $vx\sinh v$, $vx\cosh vx$, x^3, x^2, x, 1 in each (x_i, x_{i+1}), $i = 0,1,...,N-1$ interval
2. $S_\Delta(f,x)$ is a linear combination of $\sinh vx$,$\cosh vx$,x, 1 in intervals $(-\infty, 0)$ and $(1, \infty,)$
3. $S_\Delta(f,x)$ satisfies the following continuity and natural spline conditions

$$S_\Delta^{(\alpha)}(f, x_i - 0) = S_\Delta^{(\alpha)}(f, x_i + 0), \;\; \alpha = 0,1,2,3,4,5,6, \quad i = 1,2,...,N-1,$$

$$(L_4 S_\Delta(f,0))^{(k)} = (L_4 S_\Delta(f,1))^{(k)} = 0, \;\; k = 0,1,2.$$

4. The function $S_\Delta(f,x)$ satisfies interpolation conditions

$$S_\Delta(f,x_i) = y_i, \; i = 0,1,...,N.$$

Using the above, we present the following theorem about the form of the natural spline, which is one of the solutions to Problem 1.

Theorem 4.1. The natural spline, minimizing the norm in the space $K_{4,v}$, has the following form:

$$S_\Delta(f,x) = \sum_{i=0}^{N} C_i G_4(x-x_i) + d_1 \sinh(\nu x) + d_2 \cosh(\nu x) + p_0 + p_1 x, \tag{14}$$

where C_i, $i = 0,1,...,N, d_1, d_2$, p_0, p_1 $-$ are coefficients of the spline (14) , $G(x)$ is the fundamental solution of the operator $L_4^* L_4$ and satisfies equation $L_4^* L_4 G_4(x) = \delta(x)$ and has the following form

$$G_4(x) = \frac{sign(x)}{4v^7}(vx\cosh(vx) - 5\sinh(vx) + 2\sum_{k=1}^{2}\frac{(3-k)(vx)^{2k-1}}{(2k-1)!}), \tag{15}$$

here, $\delta(x)-$ is the Dirac delta-function.
The Coefficients of the natural spline satisfies the following system of linear equations

$$\sum_{i=0}^{N} C_i G_4(x_j - x_i) + d_1 \sinh(vx_j) + d_2 \cosh(vx_j) + p_0 + p_1 x = y_j,\ j = 0,1,...,N, \tag{16}$$

$$\sum_{i=0}^{N} C_i \sinh(vx_i) = 0, \tag{17}$$

$$\sum_{i=0}^{N} C_i \cosh(vx_i) = 0, \tag{18}$$

$$\sum_{i=0}^{N} C_i x_i = 0, \tag{19}$$

$$\sum_{i=0}^{N} C_i = 0. \tag{20}$$

It is known that (see, for example [17]) the solution $S_\Delta(f,x)$ of the form (14) exists only for $N \geqslant 3$.
Proof. $G_4(x-x_\gamma)$ has until sixth order continuity and its seventh order derivative has a discontinuity of the first type at the point x_γ and the discontinuity is equal to $G_4(x - - x_\gamma^+) - G_4(x - x_\gamma^-) = 1$. Suppose that the function $p_\gamma(x)$ overlaps with $S_\Delta(f,x)$ in the interval $(x_\gamma, x_{\gamma+1})$, i.e., $p_\gamma(x) := p_{\gamma-1}(x) + C_\gamma G_4(x-x_\gamma),\ x \in (x_\gamma, x_{\gamma+1})$, where C_γ is the jump of the function $S_\Delta(f,x)$ at x_γ:

$$C_\gamma = S_\Delta^{(7)}(f, x_\gamma^+) - S_\Delta^{(7)}(f, x_\gamma^-).$$

Then the spline $S_\Delta(f,x)$ can be written in the following form

$$S_\Delta(f,x) = \sum_{\gamma=0}^{N} C_\gamma G_4(x-x_\gamma) + p_{-1}(x), \tag{21}$$

where,

$$p_{-1}(x) = d_1 \sinh(vx) + d_2 \cosh(vx) + p_0 + p_1 x. \tag{22}$$

We obtain equation (16) from (21), (22) and the condition (IV).
Furthermore, the function $S_\Delta(f,x)$ satisfies the condition (II) and therefore it leads to the following conditions for C_γ,

$$\sum_{\gamma=0}^{N} C_\gamma \sinh(vx_\gamma) = 0,\ \sum_{\gamma=0}^{N} C_\gamma \cosh(vx_\gamma) = 0,\ \sum_{\gamma=0}^{N} C_\gamma x = 0\,,\ \sum_{\gamma=0}^{N} C_\gamma = 0. \qquad (23)$$

Conditions (23) are said orthogonality conditions and we obtain system of equations (16)-(20). Theorem 4.1 is proved.

A system of equations of the form (16)-(20) is called discrete systems of the Winer-Hopf [18]. Such systems are ill-conditioned and standard methods require a large amount of computation to solve them. Therefore S.L.Sobolev proposed a special method for solving such systems. This method allows to obtain an analytical solution of such systems. This method is based on constructing a discrete analogue of the differential operator $L_4^* L_4$. In the next section we give an algorithm for solving a system of (16)-(20) equations.

5 Algorithm for solving a system of equations

In this section, we give an algorithm for finding the coefficients of the spline (14). We assume that the nodes x_β are equally spaced, i.e., $x_\beta = h\beta$, $h = 1/N$, $N = 1, 2, ...$ Here we use a similar method proposed by S.L.Sobolev [14,15] for finding the coefficients of optimal quadrature formulas. We use mainly the concept of discrete argument functions and operations on them. The theory of discrete argument functions is given, for example, in [15,16].

The convolution of two discrete argument functions is defined as.

$$\varphi(h\beta) * \psi(h\beta) = [\varphi(h\beta), \psi(h\beta - h\gamma)] = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma) \cdot \psi(h\beta - h\gamma). \qquad (24)$$

Suppose that $C_\beta = 0$ when $\beta < 0$ and $\beta > N$. Using convolution, we rewrite equalities (16)-(20) as follows:

$$G_4(h\beta) * C_\beta + d_1 \sinh(vh\beta) + d_2 \cosh(vh\beta) + p_0 + p_1 h\beta = f(h\beta),\ \beta = 0, 1, ...N, \qquad (25)$$

$$\sum_{\beta=0}^{N} C_\beta \sinh(vh\beta) = 0, \qquad (26)$$

$$\sum_{\beta=0}^{N} C_\beta \cosh(vh\beta) = 0, \qquad (27)$$

$$\sum_{\beta=0}^{N} C_\beta h\beta = 0, \qquad (28)$$

$$\sum_{\beta=0}^{N} C_\beta = 0, \qquad (29)$$

where $G_4(h\beta)$ is a function of discrete argument corresponding to the function $G_4(x)$.
Thus, we have the following problem.
Problem 2. Find the coefficients C_β, $\beta = 0, 1, ..., N$ and the constants d_1, d_2, p_0, p_1 which satisfy the system (25)-(29).

Further we investigate Problem 2 which is equivalent to Problem 1. Namely, instead of C_β we introduce the following functions

$$v_4(h\beta) = G_4(h\beta) * C_\beta, \tag{30}$$

$$u_4(h\beta) = v_4(h\beta) + d_1 \sinh(vh\beta) + d_2 \cosh(vh\beta) + p_0 + p_1 h\beta. \tag{31}$$

In such a statement it is necessary to express the coefficients C_β by the function $u_4(h\beta)$. For this we have to construct such an operator $D_4(h\beta)$ which satisfies the equality

$$D_4(h\beta) * G_4(h\beta) = \delta(h\beta),$$

where $\delta(h\beta) = \begin{cases} 0, & \beta \neq 0 \\ 1, & \beta = 0 \end{cases}$ is the discrete delta-function.

The construction of the discrete analogue $D_4(h\beta)$ of the differential operator $\frac{d^8}{dx^8} - 2v^2\frac{d^6}{dx^6} + v^4\frac{d^4}{dx^4}$ is given in [12] .

Following [12] we have:

Theorem 5.1 The discrete analogue of the differential operator $\frac{d^8}{dx^8} - 2v^2\frac{d^6}{dx^6} + v^4\frac{d^4}{dx^4}$ has the form

$$D_4(h\beta) = \frac{2v^7}{p_6}\begin{cases} \sum_{k=1}^{3} A_k \lambda_k{}^{|\beta|-1}, & |\beta| \geqslant 2, \\ 1 + \sum_{k=1}^{3} A_k, & |\beta| = 1, \\ C_4 + \sum_{k=1}^{3} \frac{A_k}{\lambda_k}, & |\beta| = 0, \end{cases} \tag{32}$$

where

$$\begin{gathered} p_6 = hv\cosh(hv) - 5\sinh(vh) + 2\sum_{k=1}^{2} \frac{(3-k)}{(2k-1)!}, \\ p_5 = -20hv\cosh(hv) + 20sinh(hv) + 5\sinh(2hv) - 10hv - \tfrac{4}{3}\cosh(hv))(hv)^3 + \tfrac{4}{3}(hv)^3, \\ C_4 = -(4 + 4\cosh(vh)) - \frac{p_5}{p_6}, \\ A_k = \frac{(1-\lambda_k)^4(\lambda_k^2 + 1 - 2\lambda_k \cosh h)^2 p_6}{\lambda_k\, P_6{}'(\lambda_k)}, \end{gathered} \tag{33}$$

here, $\lambda_k,\ k = 1, 2, 3$ are zeros of the following polynomial

$$P_6(\lambda) = (1-\lambda)^4[(vh\cosh(vh) - 5\sinh(vh))\lambda^2 + [5\sinh(2vh) - 2vh]\lambda +$$

$$+(vh\cosh(vh) - 5\sinh(vh)]) + 2(\lambda^2 + 1 - 2\lambda\cosh(vh))^2 \sum_{k=1}^{2} \frac{(3-k)(vh)^{2k-1}E_{2k-2}}{(2k-1)!},$$

and $|\lambda_k| < 1$,

where E_{2k-2} is Euler-Frobenius polynomial.

Theorem 5.2 Discrete analogue $D_4(h\beta)$ of the differential operator $\frac{d^8}{dx^8} - 2v^2\frac{d^6}{dx^6} + v^4\frac{d^4}{dx^4}$ satisfies the following equalities:

1) $D_4(h\beta) * \sinh(vh\beta) = 0,$
2) $D_4(h\beta) * \cosh(vh\beta) = 0,$
3) $D_4(h\beta) * (vh\beta)\sinh(vh\beta) = 0,$
4) $D_4(h\beta) * (vh\beta)\cosh(vh\beta) = 0,$
5) $D_4(h\beta) * G_4(h\beta) = \delta(h\beta),$
6) $D_4(h\beta) * (h\beta) = 0,$
7) $D_4(h\beta) * 1 = 0\,.$

This properties were proved in [14]. Then, taking into account (32) and Theorem 5.2 for optimal coefficients we have

$$C_\beta = D_4(h\beta) * u_4(h\beta). \tag{34}$$

Thus, if we find the function $u_4(h\beta)$ then the coefficients C_β can be obtained from equality (34). In order to calculate the convolution (34) we need a representation of the function $u_4(h\beta)$ for all integer values of β. From equality (25) we get that $u_4(h\beta) = f(h\beta)$ when $h\beta \in [0,1]$. Now we need to find a representation of the function $u_4(h\beta)$ when $\beta < 0$ and $\beta > N$.
Since $C_\beta = 0$ when $h\beta \notin [0,1]$ then $C_\beta = D_4(h\beta) * u_4(h\beta) = 0$. We calculate now the convolution $v_4(h\beta) = G_4(h\beta) * C_\beta$ when $\beta \leqslant 0$ and $\beta \geqslant N$.
Supposing $\beta \leqslant 0$ and taking into account equalities (15), (26)-(29) we have

$$v_4(h\beta) = \sum_{\gamma=-\infty}^{\infty} C_\gamma G_4(h\beta - h\gamma) = \sum_{\gamma=-\infty}^{\infty} C_\gamma \frac{sign(h\beta - h\gamma)}{4v^7}\{(vh\beta - vh\gamma)\cosh(vh\beta - vh\gamma) -$$

$$-5\sinh(vh\beta - vh\gamma) + 2\sum_{k=1}^{2} \frac{(3-k)(vh\beta - vh\gamma)^{2k-1}}{(2k-1)!}\} =$$

$$= \cosh(vh\beta)\frac{1}{4v^7}\sum_{\gamma=0}^{N} C_\gamma (vh\gamma)\cosh(vh\gamma) - \sinh(vh\beta)\frac{1}{4v^7}\sum_{\gamma=0}^{N} C_\gamma (vh\gamma)\sinh(vh\gamma) -$$

$$-\frac{1}{4v^7}(vh\beta)\sum_{\gamma=0}^{N} C_\gamma (vh\gamma)^2 + \frac{1}{12v^7}\sum_{\gamma=o}^{N} C_\gamma (vh\gamma)^3.$$

Denoting

$$D_1 = \frac{1}{4v^7}\sum_{\gamma=0}^{N} C_\gamma (vh\gamma)\sinh(vh\gamma), \quad D_2 = \frac{1}{4v^7}\sum_{\gamma=0}^{N} C_\gamma (vh\gamma)\cosh(vh\gamma),$$

$$Q_0 = \frac{1}{12v^7}\sum_{\gamma=0}^{N} C_\gamma (v)^3 (h\gamma)^3, \ Q_1 = \frac{1}{4v^6}\sum_{\gamma=0}^{N} C_\gamma (v)^3 (h\gamma)^2,$$

we get for $\beta \leqslant 0$

$$v_4(h\beta) = -D_1\sinh(vh\beta) + D_2\cosh(vh\beta) + Q_0 - Q_1 h\beta.$$

And for $\beta \geqslant N$

$$v_4(h\beta) = +D_1\sinh(vh\beta) - D_2\cosh(vh\beta) - Q_0 + Q_1 h\beta.$$

Now, setting

$$d_1^- = d_1 - D_1, \ d_2^- = d_2 + D_2, \ p_0^- = p_0 + Q_0, \ \ p_1^- = p_1 - Q_1,$$

$$d_1^+ = d_1 + D_1, \ d_2^+ = d_2 - D_2, \ p_0^+ = p_0 - Q_0, \ p_1^+ = p_1 + Q_1,$$

We formulate the following problem:

Problem 3 Find the solution of the equation

$$D_4(h\beta) * u_4(h\beta) = 0, \ \ h\beta \notin [0,1]. \tag{35}$$

In the form:

$$u_4(h\beta) = \begin{cases} d_1^- \sinh(vh\beta) + d_2^- \cosh(vh\beta) + p_0^- + p_1^- h\beta, & \beta \leqslant 0, \\ f(h\beta), & 0 \leqslant \beta \leqslant N, \\ d_1^+ \sinh(vh\beta) + d_2^+ \cosh(vh\beta) + p_0^+ + p_1^+ h\beta, & \beta \geqslant N, \end{cases} \quad (36)$$

where $d_1^-,\ d_2^-,\ p_0^-,\ p_1^-,\ d_1^+,\ d_2^+,\ p_0^+,\ p_1^+$ are unknowns.
It is clear that

$$d_i = \frac{1}{2}(d_i^+ + d_1^-),\ i = 1.2,\ p_i = \frac{1}{2}(p_i^+ + p_i^-),\ i = 0,1. \quad (37)$$

These unknowns $d_1^-,\ d_2^-,\ p_0^-,\ p_1^-,\ d_1^+,\ d_2^+,\ p_0^+,\ p_1^+$ can be found from equation (35), using the function $D_4(h\beta)$. Then the explicit from of the function $u_4(h\beta)$ and coefficients $C_\beta,\ d_1,\ d_2,\ p_0, p_1$ can be found. Thus, Problem 3 and respectively Problems 2 and 1 can be solved.

In the next section we realize this algorithm for computing the coefficients $C_\beta, \beta = 0, 1, .., N,\ d_1,\ d_2,\ p_0,\ p_1$ of the interpolation spline (14).

6 Computing of the coefficients of the interpolation spline

In this section using the algorithm from the previous section we obtain explicit formulae for coefficients of natural spline (14) which is the solution of Problem 1.
The following holds.
Theorem 6.1 Coefficients of natural spline (14) which minimizes the semi norm in that space $K_{4,v}$ with equally spaced nodes in the space $K_2(P_3)$ have the following form:

$$C_0 = \frac{2v^7}{p_6}\{-d_1^- \sinh(vh) + d_2^- \cosh(vh) + p_0^- - p_1^- h + C_4 f(0) + f(h) +$$
$$+ \sum_{k=1}^{3} \frac{A_k}{\lambda_k}[M_k + \sum_{\gamma=0}^{N} \lambda_k^\gamma f(h\gamma) + \lambda_k^N N_k]\},$$
$$C_\beta = \frac{2v^7}{p_6}\{f(h(\beta-1)) + Cf(h\beta) + f(h(\beta+1)) +$$
$$+ \sum_{k=1}^{3} \frac{A_k}{\lambda_k}[\lambda_k^\beta M_k + \sum_{\gamma=0}^{N} \lambda_k^{|\beta-\gamma|} f(h\gamma) + \lambda_k^{N-\beta} N_k]\}, \quad \beta = 1, 2..., N-1,$$
$$C_N = \frac{2v^7}{p_6}\{d_1^+ \sinh(v(h+1)) + d_2^+ \cosh(v(h+1)) + p_0^+ + p_1^+ \cdot (h+1) + f(1-h) +$$
$$+ C_4 f(1) + \sum_{k=1}^{3} \frac{A_k}{\lambda_k}[\lambda_k^N M_k + \sum_{\gamma=0}^{N} \lambda_k^{N-\gamma} f(h\gamma) + N_k]\},$$
$$d_i = \frac{1}{2}(d_i^+ + d_1^-),\ i = 1.2,\ p_i = \frac{1}{2}(p_i^+ + p_i^-),\ i = 0,1,$$

where

$$M_k = -d_1^- \frac{\lambda_k \sinh(vh)}{1 + {\lambda_k}^2 - 2\lambda_k \cosh(vh)} + d_2^- \frac{\lambda_k(\cosh(vh) - \lambda_k)}{1 + {\lambda_k}^2 - 2\lambda_k \cosh(vh)} +$$
$$+ p_0^- \frac{\lambda_k}{1-\lambda_k} - hp_1^- \frac{\lambda_k}{(1-\lambda_k)^2}, \quad (38)$$
$$N_k = d_1^+ \frac{\lambda_k(\sinh(v(h+1)) - \lambda_k \sinh v)}{1 + {\lambda_k}^2 - 2\lambda_k \cosh(vh)} + d_2^+ \frac{\lambda_k(\cosh(v(h+1)) - \lambda_k \cosh v)}{1 + {\lambda_k}^2 - 2\lambda_k \cosh(vh)} +$$

$$+(p_0^+ + p_1^+)\frac{\lambda_k}{1-\lambda_k} + hp_1^+\frac{\lambda_k}{(1-\lambda_k)^2}, \tag{39}$$

and p_6, A_k, C_4, λ_k are given in (33) and d_1^-, d_2^-, p_0^-, p_1^-, d_1^+, d_2^+, p_0^+, p_1^+ are defined by (40), (44).

Proof. First we find the expression for d_2^- and d_2^+. When $\beta = 0$ and $\beta = N$, from (36) we get

$$d_2^- = f(0) - p_0^-,\ d_2^+ = \frac{f(1) - d_1^+\sinh(v) - p_0^+ - p_1^-}{\cosh(v)}. \tag{40}$$

Now we find other four unknowns d_1^-, p_0^-, p_1^-, d_1^+, p_0^+, p_1^+ which can be found from (35) when $\beta = -1, -2, -3, N+1, N+2, N+3$. Taking into account (36) and from (35) we have:

$$\sum_{\gamma=1}^{\infty} D_4(h\beta+h\gamma)(-d_1^-\sinh(vh\gamma) + d_2^-\cosh(vh\gamma) + p_0^- - p_1^-\cdot(h\gamma))+$$

$$+\sum_{\gamma=0}^{N} D_4(h\beta-h\gamma)f(h\gamma) + \sum_{\gamma=1}^{\infty} D_4(h(N+\gamma-\beta))(d_1^+\sinh(v(h\gamma+1))+$$

$$+d_2^+\cosh(v(h\gamma+1)) + p_0^+ + p_1^+(h\gamma+1))) = 0.$$

Now, we use (40) and for $\beta = -1, -2, -3, N+1, N+2, N+3$ we get the following system of linear equations for d_1^-, p_0^-, p_1^-, d_1^+, p_0^+, p_1^+

$$-d_1^-\sum_{\gamma=1}^{\infty} D_4(h\gamma+h\beta)\sinh(vh\gamma) + p_0^-\sum_{\gamma=1}^{\infty} D_4(h\gamma+h\beta)(1-\cosh(vh\gamma))-$$

$$-hp_1^-\sum_{\gamma=1}^{\infty} D_4(h\gamma+h\beta)\cdot\gamma + \frac{d_1^+}{\cosh v}\sum_{\gamma=1}^{\infty} D_4(h\,(\gamma+N-\beta))\sinh(vh\gamma)+$$

$$+p_0^+\sum_{\gamma=1}^{\infty} D_4(h(\gamma+N-\beta))\left(1-\frac{\cosh\left(v(h\gamma+1)\right)}{\cosh v}\right)+$$

$$+p_1^+\sum_{\gamma=1}^{\infty} D_4(h\,(\gamma+N-\beta))(h\gamma+1-\frac{\cosh(v(h\gamma+1))}{\cosh v}) =$$

$$= -\sum_{\gamma=0}^{N} D_4(h\beta-h\gamma)f(h\gamma) - f(0)\sum_{\gamma=1}^{\infty} D_4(h\beta+h\gamma)\cosh(vh\gamma)-$$

$$-\frac{f(1)}{\cosh v}\sum_{\gamma=1}^{\infty} D_4(h(\gamma+N-\beta))\cosh(v(h\gamma+1)). \tag{41}$$

Now, we consider the cases $\beta = -1, -2, -3$. In (41) , we replace β with $-\beta$ and write it in the following form:

$$-d_1^- B_{d_1^-,\beta} + p_0^- B_{p_0^-,\beta} + p_1^- B_{p_1^-,\beta} + d_1^+ B_{d_1^+,\beta} + p_0^+ B_{p_0^+,\beta} + p_1^+ B_{p_1^+,\beta} = T_\beta,\ \beta = 1,2,3, \tag{42}$$

where

$$B_{d_1^-,\beta} = -\frac{2v^7}{p_6}\left[\sinh(vh(\beta-1)) + C_4\sinh\left(vh\beta\right) + \sinh\left(vh(\beta+1)\right)+\right.$$

$$+\sum_{k=1}^{3}\frac{A_k}{\lambda_k}\sum_{\gamma=1}^{\infty}\lambda_k^{|\beta-\gamma|}\sinh(vh\gamma)\big],$$

$$B_{p_0^-,\beta}=\frac{2v^7}{p_6}\big[(1-\cosh(vh(\beta-1)))+C_4(1-\cosh(vh\beta))+1-\cosh(vh(\beta+1))+$$

$$+\sum_{k=1}^{3}\frac{A_k}{\lambda_k}\sum_{\gamma=1}^{\infty}\lambda_k^{|\beta-\gamma|}(1-\cosh(vh\gamma\big],$$

$$B_{p_1^-,\beta}=\frac{2v^7}{p_6}[2\beta+C_4\,\beta+\sum_{k=1}^{3}\frac{A_k}{\lambda_k}\sum_{\gamma=1}^{\infty}\lambda_k^{|\beta-\gamma|}\gamma],$$

$$B_{d_1^+,\beta}=\frac{2v^7}{p_6\ \cosh v}\sum_{k=1}^{3}\frac{A_k{\lambda_k}^{N+\beta}\sinh(vh)}{1+\lambda_k^2-2\lambda_k\cosh(vh)},$$

$$B_{p_0^+,\beta}=\frac{2v^7}{p_6\ \cosh v}\sum_{k=1}^{3}A_k{\lambda_k}^{N+\beta}(\frac{1}{1-\lambda_k}-\frac{\cosh(v(h+1))-\lambda_k\cosh v}{\cosh v(1+\lambda_k^2-2\lambda_k\cosh(vh))}),$$

$$B_{p_1^+,\beta}=\frac{2v^7}{p_6\ \cosh v}\sum_{k=1}^{3}A_k{\lambda_k}^{N+\beta}(h\frac{1}{(1-\lambda_k)^2}+\frac{1}{1-\lambda_k}-\frac{\cosh(v(h+1))-\lambda_k\cosh v}{\cosh v(1+\lambda_k^2-2\lambda_k\cosh(vh))}),$$

$$T_\beta=-\frac{2v^7}{p_6}\Bigg[f(0)\cdot(\cosh(vh(\beta-1))+C_4\cosh(vh\beta)+\cosh(vh(\beta+1))+$$

$$+\sum_{k=1}^{3}\frac{A_k}{\lambda_k}\sum_{\gamma=1}^{\infty}\cosh(vh\gamma)+\sum_{k=1}^{3}A_k\sum_{\gamma=0}^{N}\lambda_k^{\gamma}f(h\gamma)+$$

$$+\frac{f(1)}{\cosh(v)}\cdot\sum_{k=1}^{3}A_k\lambda_k^{N+\beta}\frac{\cosh(v(h+1))-\lambda_k\cosh v}{1+\lambda_k^2-2\lambda_k\cosh(vh)}\Bigg].$$

Further, in (42), we consider the cases $N+1, N+2, N+3$. From (42) by substituting β with $N+\beta$ and using (32), we obtain the following system of 3 linear equations.

$$-d_1^-A_{d_1^-,\beta}+p_0^-A_{p_0^-,\beta}+p_1^-A_{p_1^-,\beta}+d_1^+A_{d_1^+,\beta}+p_0^+A_{p_0^+,\beta}+p_1^+A_{p_1^+,\beta}=S_\beta,\ \beta=1,2,3 \quad (43)$$

Where

$$A_{d_1^-,\beta}=-\frac{2v^7}{p_6}\sum_{k=1}^{3}\frac{A_k{\lambda_k}^{N+\beta}\sinh(vh)}{1+\lambda_k^2-2\lambda_k\cosh(vh)},$$

$$A_{p_0^-,\beta}=\frac{2v^7}{p_6}\sum_{k=1}^{3}A_k{\lambda_k}^{N+\beta}(\frac{1}{1-\lambda_k}-\frac{\cosh(vh)-\lambda_k}{\cosh v(1+\lambda_k^2-2\lambda_k\cosh(vh))}),$$

$$A_{p_1^-,\beta}=-\frac{2v^7h}{p_6}\sum_{k=1}^{3}\frac{A_k{\lambda_k}^{N+\beta}}{(1-\lambda_k)^2},$$

$$A_{d_1^+,\beta}=\frac{2v^7}{p_6\cosh(v)}\Big[\sinh(vh(\beta-1))+C_4\sinh(vh\beta)+\sinh(vh(\beta+1))+$$

$$+\sum_{k=1}^{3}\frac{A_k}{\lambda_k}\sum_{\gamma=1}^{\infty}\lambda_k^{|\beta-\gamma|}\sinh(vh\gamma)\Big],$$

$$A_{p_0^+,\beta} = \frac{2v^7}{p_6}\Bigg[\sum_{k=1}^{3}\frac{A_k}{\lambda_k}\sum_{\gamma=1}^{\infty}\lambda_k^{|\beta-\gamma|} + 2 + C_4 - \frac{1}{\cosh v}(\sum_{k=1}^{3}\frac{A_k}{\lambda_k}\sum_{\gamma=1}^{\infty}\lambda_k^{|\beta-\gamma|}\cosh(v(h\lambda+1))+$$

$$+\cosh(v(h(\beta-1)+1)) + \cosh(v(h\beta+1)) + \cosh(v(h(\beta+1)+1)))\Bigg],$$

$$A_{p_1^+,\beta} = \frac{2v^7}{p_6}\Bigg[h\{\sum_{k=1}^{3}\frac{A_k}{\lambda_k}\sum_{\gamma=1}^{\infty}\lambda_k^{|\beta-\gamma|} + 2\beta + C_4\beta\} + \sum_{k=1}^{3}\frac{A_k}{\lambda_k}\sum_{\gamma=1}^{\infty}\lambda_k^{|\beta-\gamma|} + 2 + C_4 -$$

$$-\frac{1}{\cosh v}\sum_{k=1}^{3}\frac{A_k}{\lambda_k}\sum_{\gamma=1}^{\infty}\lambda_k^{|\beta-\gamma|}\cosh(v(h\lambda+1)) + \cosh(v(h(\beta-1)+1))+$$

$$+\cosh(v(h\beta+1)) + \cosh(v(h(\beta+1)+1)))\Bigg],$$

$$S_\beta = -\frac{2v^7}{p_6}\Bigg[f(0)\cdot\sum_{k=1}^{3}A_k\lambda_k^{N+\beta}\frac{\cosh(vh)-\lambda_k}{1+\lambda_k^2-2\lambda_k\cosh(vh)} + \sum_{k=1}^{3}\frac{A_k}{\lambda_K}\sum_{\gamma=0}^{N}\lambda_k^{N+\beta-\gamma}f(h\gamma))+$$

$$+\frac{f(1)}{\cosh(v)}\cdot(\cosh(v(h(\beta-1)+1)) + C_4\cosh(v(h\beta+1))+$$

$$+\cosh(v(h(\beta+1)+1)) + \sum_{k=1}^{3}\frac{A_k}{\lambda_k}\cdot\sum_{\gamma=1}^{\infty}\cosh(v(h\gamma+1)))\Bigg].$$

Since $|\lambda_k| < 1,\ k = 1, 2$, the series in the previous system of equations are convergent. (42) and (43) together, give a system of 6 equations with 6 unknowns,

$$\begin{pmatrix} B_{d_1^-1} & B_{p_0^-,1} & B_{p_1^-1} & B_{d_1^+1} & B_{p_0^+,1} & B_{p_1^+1} \\ B_{d_1^-2} & B_{p_0^-,2} & B_{p_1^-2} & B_{d_1^+2} & B_{p_0^+,2} & B_{p_1^+2} \\ B_{d_1^-3} & B_{p_0^-,3} & B_{p_1^-3} & B_{d_1^+3} & B_{p_0^+,3} & B_{p_1^+3} \\ A_{d_1^-1} & A_{p_0^-,1} & A_{p_1^-1} & A_{d_1^+1} & A_{p_0^+,1} & A_{p_1^+1} \\ A_{d_1^-2} & A_{p_0^-,2} & A_{p_1^-2} & A_{d_1^+2} & A_{p_0^+,2} & A_{p_1^+2} \\ A_{d_1^-3} & A_{p_0^-,3} & A_{p_1^-3} & A_{d_1^+3} & A_{p_0^+,3} & A_{p_1^+3} \end{pmatrix}\begin{pmatrix} d_1^- \\ p_0^- \\ p_1^- \\ d_1^+ \\ p_0^+ \\ p_1^+ \end{pmatrix} = \begin{matrix} T_1 \\ T_2 \\ T_3 \\ A_1 \\ A_2 \\ A_3. \end{matrix} \tag{44}$$

and d_1^-, p_0^-, $p_1^-, d_1^+, p_0^+, p_1^+$, have a single solution, and we find these solutions using Cramer's method.

We obtain d_1^-, d_2^-, d_3^-, d_1^+, d_2^+, d_3^+ by combining (40) and (44). Then we obtain d_1, d_2, p_0, p_1 by using (37).

Now, We calculate the coefficients C_β, $\beta = 0, 1, 2..., N$. Taking into account (32) from (34) for C_β we get

$$C_\beta = D_4(h\beta) * u_4(h\beta) = \sum_{\gamma=1}^{\infty}D_4(h\beta-h\gamma)u_4(h\gamma) = \sum_{\gamma=0}^{N}D_4(h\beta-h\gamma)f(h\gamma)+$$

$$+\sum_{\gamma=1}^{\infty}D_4(h\beta+h\gamma)\Big(-d_1^-\sinh(vh\gamma) + d_2^-\cosh(vh\gamma) + p_0^- - p_1^-\cdot(h\gamma)\Big)+$$

$$+\sum_{\gamma=1}^{\infty} D_4(h(N+\gamma-\beta))\Big(d_1^+\sinh(v(h\gamma+1))+d_2^+\cosh(v(h\gamma+1))+p_0^++p_1^+(h\gamma+1)\Big). \quad (45)$$

From which, using (34) and taking into account notations (38), (39), when $\beta = = 0, 1, ..., N$, for C_β we obtain the expression given in Theorem 6.1

7 Conclusion

In this work, we constructed an eighth order algebraic-hyperbolic tension natural spline. To solve this problem, we used the Sobolev method and obtain a spline function for the approximate calculation of the unknown function. We first presented the interpolation spline function under which conditions gives a minimum to the norm in a certain Hilbert space. To find the coefficients of this spline, we created a system of equations based on certain conditions. We used Sobolev method and gave the algorithm to solve equations system. When we found the coefficients of the eighth order algebraic-hyperbolic interpolation natural spline, we obtain the exact expression of this spline.

References

[1] Cheney W., Kincaid D. 2013. *Numerical Mathematics and Computing . Brooks Cole, USA, Seventh edition* – 700 p.

[2] Schoenberg I.J. 1946. *J. Contributions to the problem of approximation of equidistant data by analytic functions. Quart. Appl. Math.* – Vol. 4. – P. 112–141.

[3] Schumaker L.L. 1981. *Spline functions: basic theory. New York: Wiley* – 553 p.

[4] Holladay J.C.. 1957. *A smoothest curve approximation. Math. Tables Aids Comput,* – Vol. 11, – no. 60. – P. 233–243.

[5] Alberg J., Nilson E., Wolsh J. 1967. *The theory of splines and their applications.* Mathematics in Science and Engineering, New York: Academic Press.

[6] Campagna R., Marchi S.D. Perracchione E.E., Santin G. 2021. *Greedy algorithms for learning via exponential-polynomial splines. Numerical Analysis.* – No2. – P. 1–17.

[7] Domingo R., Salah E., Abdellah L. 2020. *Uniform algebraic-hyperbolic spline quasi-interpolant based on mean integral values. Article in Computational and Mathematical Methods*, doi: `http://dx.doi.org/10.1002/cmm4.1123`.

[8] Kholmat M. Shadimetov, Abdulla R. Hayotov. 2013. *Construction of interopolation splines minimizing semi-norm in* $W_2^{(m,m-1)}(0,1)$ *space. BIT Numerical Mathematics*, 53, – P. 545–563.

[9] Cabada A., Hayotov A.R., Shadimetov KH.M. 2014. *Construction of* D^m *splines in* $L_2^{(m)}(0,1)$ *space by Sobolev method. Applied Mathematics and computation .* – P. 543–551.

[10] Hayotov A. R, Qurbonnazarov A.I. 2023. *An optimal quadrature formula for the approximate calculation of fourier integrals in the space* $K_2^3(0,1)$*. Problems of computational and applied mathematics* , no3/1. – P. 1–17.

[11] Shadimetov Kh.M., Ahmadaliyev G.N. 2024. *Optimal coefficients of the quadrature formulkas in the space* $K_{3,\omega}$*. AIP Conf. Proc.* 3004. – P. 1–14.

[12] Ahmadaliyev G.N., Hayotov A.R.. 2017. *A discrete analogue of the differential operator* $\frac{d^{2m}}{dx^{2m}} - 2\omega^2\frac{d^{2m-2}}{dx^{2m-2}} + \omega^4\frac{d^{2m-4}}{dx^{2m-4}}$*. Uzbek mathematical journa*Tashkent no. 3, – P. 10–22.

[13] Qurbonnazarov A.I. 2023. *Properties of Discrete Analogue of the Differential Operator. Problems of computational an applied mathematics* – P. 20–30.

[14] Sobolev S.L. 2006. *The coefficients of optimal quadrate formulas, in Selected Work of S.L.Sobolev. Springer*, – P. 561–566.

[15] Sobolev S.L., Vaskevich V.L. 1997. *The Theory of Cubature Formulas. Mathematics and Its Applications* (MAIA, volume 415)).

[16] Sobolev S.L. 1974. *Introduction to the theory of Cubature Formulas. Nauka, Moscow, (in Russian).*

[17] Vasilenko V.A. 1983. *Spline functions: Theory, Algorithms, Programs. Nauka, Novosibirsk, (in Russian).*

[18] Sobolev S.L. 2006. *Selected Works of S.L. Sobolev. Volume I Equations of Mathematical Physics, Computational Mathematics, and Cubature Formulas, Springer, – 426 p.* doi: http://dx.doi.org/10.1007/978-0-387-34149-1.

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ПОСТРОЕНИЕ АЛГЕБРАИЧЕСКИ-ГИПЕРБОЛИЧЕСКОГО СПЛАЙНА ЕСТЕСТВЕННОГО НАТЯЖЕНИЯ ВОСЬМОГО ПОРЯДКА

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В статье обосновывается то, что алгебраически-гиперболический сплайн восьмого порядка минимизирует норму в гильбертовом пространстве. Затем, применяя метод Соболева, основанный на построении дискретного аналога дифференциального оператора, строится функция сплайна. Неизвестные коэффициенты сплайна вычисляются с учетом заданных условий гладкости и граничных условий. В результате построенный сплайн обладает высокой степенью гладкости, повышает точность интерполяции и точно воспроизводит гиперболические функции, линейные полиномы и константы. Полученные результаты свидетельствуют о высокой эффективности подхода для задач, требующих гладкой интерполяции и точного моделирования физических процессов. Кроме того, использование параметров натяжения позволяет точно регулировать жесткость или гибкость сплайна.

Ключевые слова: Гильбертово пространство, обобщенный сплайн, алгебраически-гиперболический сплайн, свёртка, дискретный аналог.

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