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DEGENERATE LOTKA-VOLTERRA MAPPINGS AND THEIR CORRESPONDING BIGRAPHS AS A DISCRETE MODEL OF THE EVOLUTION OF THE INTERACTION OF TWO VIRUSES

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This paper proposes a discrete model of the interaction between two airborne viruses, based on an operator acting in a four-dimensional simplex. The model describes the progression of an epidemic in a closed population, divided into five compartments: susceptible individuals, individuals in the latent stage of the first virus, those infected with the first virus, those infected with the second virus, and individuals who have recovered from the first virus. The mathematical structure of the model captures complex transitions between states and interactions between strains, including cases of co-infection. Special attention is given to the analysis of the sets of initial and final states of the disease, defined by systems of inequalities. Depending on the model parameters, these sets may lie on different faces of the simplex, representing various scenarios of epidemic onset and resolution. Two main epidemiological scenarios are considered: one involving complete recovery after infection with the first virus, and another involving progression to co-infection without full recovery. The model is applicable to the analysis of tuberculosis co-infection with viral hepatitis B and C and allows assessment of the influence of various parameters on patient survival during multi-drug therapy. Finally, a numerical experiment is conducted, presenting trajectories, phase portraits, and 30-day dynamics of disease spread, illustrating system behavior under different initial conditions and parameter settings.

Keywords: Lotka-Volterra mapping, simplex, skew-symmetric matrix, convex hull, trajectory, partially oriented graph, bigraph.

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1 Introduction

The Lotka-Volterra mapping is uniquely determined by specifying a skew-symmetric matrix $A = (a_{ij}),\ a_{ij} = -a_{ji},\ i,\ j = \overline{1,m}$ and acts on the simplex $S^{m-1} = \left\{x \in R^m : \sum\limits_{i=1}^{m} x_i = 1,\ x_i \geqslant 0\right\}$ according to the formulas [1]

$$x_k^{'} = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right),\ k = 1, ..., m, \tag{1}$$

on condition $|a_{ki}| \leqslant 1$. Let $e_k = (0, ... 0, 1,\ 0, ..., 0)$, where "1" is at the k-th place, then $S^{m-1} = co\{e_1, ..., e_m\}$, i.e. the simplex is a convex hull of points e_k, which are called vertices of the simplex of the S^{m-1}. If $\gamma \subset I = \{1, 2, ..., m\}$ is a nonempty subset, then the $\Gamma_\gamma = co\{e_k : \ k \in \gamma\}$ is called the $|\gamma| - 1$-dimensional face of the simplex. Let $V : S^{m-1} \to S^{m-1}$ be the Lotka-Volterra mapping defined by equality (1). Obviously, any face of the simplex is invariant with regard to V, and the narrowing of $V$to this face is also a Lotka-Volterra mapping. Therefore, it is possible to limit the study of dynamic properties only to the interior of the simplex. As usual, for $x^0 \in S^{m-1}$ the $\left\{x^{(n)}\right\}$ trajectory is determined by the recurrence relation

$$x^{(n+1)} = Vx^{(n)}, \ n = 0, 1, ... \tag{2}$$

For any $x^0 \in S^{m-1}$, let's put $\omega(x^0) = \left\{x^0, x^{(1)}, ...\right\}'$ – the set of limit points of the positive trajectory and $\alpha(x^0) = \left\{x^0, x^{(-1)}, x^{(-2)}, ...\right\}'$ – the set of limit points of the negative trajectory. Since the simplex is compact, then $\omega(x^0) \neq \varnothing$, $\alpha(x^0) \neq \varnothing$, and they are closed and invariant with regarding to the V mapping . It is also known that from $x^0 \in S^{m-1}$ and $x^0 \neq Vx^0$ should $\omega(x^0) \subset \partial S^{m-1}$, i.e. $\omega(x^0)$ belongs to the boundary of the simplex. Next, we will need a theorem from

Theorem 1. [1–4]. Let $A = (a_{ki})$ be a skew-symmetric matrix, in this case $P = \{x \in S^{m-1} : Ax \geqslant 0\} \neq \varnothing$, $\ Q = \{x \in S^{m-1} : Ax \leqslant 0\} \neq \varnothing$ consist of from fixed points.

Definition 1. [4–6]. A graph with vertices 1, 2, ..., m is called a partially oriented graph in which any two vertices are connected by a directed edge is called a tournament. It is clear that with each skew-symmetric matrix, a partially oriented graph corresponding to it can be considered. For example, the matrix

$$A = \begin{pmatrix} 0 & a & 0 & -b \\ -a & 0 & c & d \\ 0 & -c & 0 & 0 \\ b & -d & 0 & 0 \end{pmatrix},$$

where $a, \ b, \ c, \ d > 0$ corresponds to a partially oriented graph [4]:

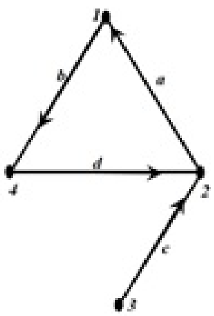

Figure 1 A partially oriented graph corresponding to the matrix A

Definition 2. [6, 7]. A skew-symmetric matrix is called a general position matrix if all major minors of even order are positive.

It is known [7] that the general position matrices form an open and everywhere dense subset in the set of all skew-symmetric matrices, moreover, the corresponding partially oriented graph is a tournament In general [6, 7], it is proved that $\omega(x^0)$ either consists of a single point or infinitely. It is also known [3] that in the case of strong tournaments, as a rule, infinitely, moreover, the Cesaro averages of positive trajectories do not converge. The main purpose of the work is an analytical analysis of those Lotka-Volterra mappings that describe the course of airborne diseases, i.e. to find for them sets of limit points of a positive and negative trajectory. In epidemiology, a set of limit points of a positive trajectory means the focus of the disease, and a set of limit points of a negative trajectory means the area of the end of the diseased population.

2 Statement of the problem and results

The discrete variants of the models studied in the works are considered below [8–10]. It should be noted that the dynamic behavior of trajectories in discrete models differs significantly from the dynamics of continuous models. Let

$$A = \begin{pmatrix} 0 & -a & 0 & b \\ a & 0 & -c & 0 \\ 0 & c & 0 & d \\ -b & 0 & -d & 0 \end{pmatrix},$$

where $0 < a,\ b,\ c,\ d \leqslant 1$. Obviously, $\det A = (ad + bc)^2 > 0$ and therefore $KerA \cap S^3 = \varnothing$. Solving inequalities $P = \{x \in S^2 :\ Ax \geqslant 0\}$ and $Q = \{x \in S^2 :\ Ax \leqslant 0\}$ we get

$$P = \left\{ \left(0,\ \frac{(1-\lambda)\,a + b}{a+b},\ 0,\ \frac{\lambda a}{a+b} \right) \right\},\ Q = \left\{ \left(\frac{\lambda c}{a+c},\ 0,\ \frac{a + (1-\lambda)\,c}{a+c},\ 0 \right) \right\}, \qquad (3)$$

where $0 \leqslant \lambda \leqslant 1$. In this case, the $V_1 : S^3 \to S^3$ defined by the A matrix has the form

$$V_1 : \begin{cases} x_1' = x_1(1 - ax_2 + bx_4), \\ x_2' = x_2(1 + ax_1 - cx_3), \\ x_3' = x_3(1 + cx_2 + dx_4), \\ x_4' = x_4(1 - bx_1 - dx_3). \end{cases} \qquad (4)$$

Theorem 2. If x^0 is the inner point of the simplex, then $\alpha(x^0) \subset P$ and $\omega(x^0) \subset Q$, with both positive and negative trajectories converging.

Proof. According to (4) we have

$$x_3^{(n+1)} = x_3^{(n)} \cdot \left(1 + cx_2^{(n)} + dx_4^{(n)} \right),\ n = 0,\ 1,\ ... \qquad (5)$$

Clearly, $\left\{ x_3^{(n)} \right\}$ converges. Since $\left\{ x_3^{(n)} \right\}$ is limited, then $\lim\limits_{n\to+\infty} x_2^{(n)} = \lim\limits_{n\to+\infty} x_4^{(n)} = 0$. Therefore, $\omega(x^0) \subset \Gamma_{13} = co\left\{e_1,\, e_3\right\}$. As is known [11], the convergence of the series follows from (5) $\sum\limits_{n=0}^{\infty} x_2^{(n)}$ and $\sum\limits_{n=0}^{\infty} x_4^{(n)}$, which ensures the convergence of the sequence $x_1^{(n)}$, with $\lim\limits_{n\to+\infty} x_1^{(n)} > 0$. Obviously, $\lim\limits_{n\to+\infty} x_4^{(n)} < 1$. Calculating the Jacobian of fixed points on the face Γ_{13} we get

$$J\left(V\left(x\right)\right) = \left(1-\lambda\right)^2 \left(1 + ax_1 - cx_3 - \lambda\right)\left(1 - bx_1 - dx_3 - \lambda\right).$$

Hence the eigenvalues of the Jacobian $\lambda_1 = \lambda_2 = 1;\ \lambda_3 = 1 + ax_1 - cx_3, \lambda_4 = 1 - bx_1 - dx_3$, and $\lambda_4 = 1 - bx_1 - dx_3 < 1$ for all points from Γ_{13}. If $x_1 < \frac{c}{a+c}$ is on the edge of Γ_{13} , then

$\lambda_3 < 1$, with $x_1 > \frac{c}{a+c}$, the corresponding fixed point is a saddle point, which means it cannot belong to the set of Q. Hence, $\omega(x^0) \subset Q$. For negative trajectories, the sequence of $\left\{x_4^{(-n)}\right\}$ at $n \to +\infty$ is increasing and limited. Therefore, $\lim\limits_{n\to\infty} x_1^{(-n)} = \lim\limits_{n\to\infty} x_3^{(-n)} = 0$, and the series $\sum\limits_{n=0}^{\infty} x_1^{(-n)}$ and $\sum\limits_{n=0}^{\infty} x_3^{(-n)}$ converge. Repeating the previous arguments , we get $\alpha(x^0) \subset P$. Theorem 2 is proved.

Definition 4. The points $p \in P$ and $q \in Q$ form a $(p,\, q)$ pair if there is a $x^0 \in S^3$ such that $\omega(x^0) = q$, $\alpha(x^0) = p$.

Remark 1. Since, in this example, λ_3 and λ_4 are simple eigenvalues of multiplicity $''1''$, then the correspondence of $p \leftrightarrow q$ is mutually unambiguous.

Let

$$A = \begin{pmatrix} 0 & -a & 0 & b \\ a & 0 & -c & 0 \\ 0 & c & 0 & -d \\ -b & 0 & d & 0 \end{pmatrix},$$

where $0 < a,\ b,\ c,\ d \leqslant 1$. Obviously, $\det A = (ad - bc)^2$.

1) If $ad = bc$, then the calculations show that $KerA \cap S^3 = P = Q = [M_1,\, M_2]$, where $M_1 = \left(\frac{c}{a+c},\ 0,\ \frac{a}{a+c}\,,\ 0\right)$ and $M_2 = \left(0,\ \frac{d}{c+d},\ 0,\ \frac{c}{c+d}\right)$ - the ends of the segment $[M_1,\, M_2]$. Here the $V_2 : S^3 \to S^3$ mapping is set by the equalities:

$$V: \begin{cases} x_1' = x_1(1 - ax_2 + bx_4), \\ x_2' = x_2(1 + ax_1 - cx_3), \\ x_3' = x_3(1 + cx_2 - dx_4), \\ x_4' = x_4(1 - bx_1 + dx_3). \end{cases} \tag{6}$$

Theorem 3. If x^0 is the inner point of the simplex, and $x^0 \neq Vx^0$, the positive trajectory does not converge and $\omega(x^0) \subset \partial S^3$, and the negative trajectory converges, and $\alpha(x^0) \subset [M_1,\, M_2]$.

Proof. Let's assume that $\lim\limits_{n\to+\infty} x^{(n)} = x^*$. Then x^* must be a fixed point for V. All points of the three segments are fixed $[M_1,\, M_2],\ [e_1,\, e_3]\,,\ [e_2,\, e_4]$. On the segment $[e_1,\, e_3]$, the Jacobian eigenvalues are found from the equation

$$(1-\lambda)^2\,(1 + ax_1 - cx_3 - \lambda)\,(1 - bx_1 + dx_3 - \lambda) = 0.$$

Note that $(ax_1 - cx_3)\,(-bx_1 - dx_3) = -abx_1^2 + adx_1x_3 - cbx_1x_3 - cdx_3^2 < 0$, as $ad = bc$. Hence, of the numbers $\lambda_3 = 1 + ax_1 - cx_3$ and $\lambda_4 = 1 - bx_1 + dx_3$, one is greater than $''1''$ and the other is less than $''1''$. Thus, any fixed point from $[e_1,\, e_3]$ is a saddle point. Therefore, $x^* \notin [e_1,\, e_3]$. We also get that $x^* \notin [e_2,\, e_4]$. The points of the segment of the $[M_1,\, M_2]$, with the exception of the ends, are internal fixed points. Therefore, the spectrum of the Jacobian

$$\begin{vmatrix} 1-\lambda & -ax_1 & 0 & bx_1 \\ ax_2 & 1-\lambda & -cx_2 & 0 \\ 0 & cx_3 & 1-\lambda & -dx_2 \\ -bx_4 & 0 & dx_4 & 1-\lambda \end{vmatrix} = 0.$$

Having calculated the value of the determinant, we get:

$$(1-\lambda)^4 + \left(a^2x_1x_2 + b^2x_1x_4 + c^2x_2x_3 + d^2x_3x_4\right)(1-\lambda)^2 + (ad - bc)^2x_1x_2x_3x_4 = 0.$$

Because $ad = bc$, then $\lambda_1 = \lambda_2 = 1$ and λ_3, λ_4 are complex numbers, and $|\lambda_3| > 1$, $|\lambda_4| > 1$. Therefore, $\omega(x^0)$ is not contained in the $[M_1, M_2]$ segment. Thus, the positive trajectory does not converge, $\omega(x^0) \subset \partial S^3$ and it is infinite. The convergence of the negative trajectory and $\alpha(x^0) \subset [M_1, M_2]$ follows from the general statement. Theorem 3 is proved.

Remark 2. If $x^0 \in \partial S^2$ and $x^0 \neq Vx^0$, then it is easy to prove that the trajectory converges, moreover, $\alpha(x^0)$ and $\omega(x^0)$ belong to either $\Gamma_{13} = [e_1, e_3]$ or $\Gamma_{24} = [e_2, e_4]$. In this example, at the boundary of the simplex, the trajectories converge, but inside the simplex, the positive trajectories do not converge.

2) Consider the case:$ad > bc$. Obviously, $KerA \cap S^3 = \varnothing$, just like $\det A = (ad - bc)^2 \neq 0$. Next, solving linear inequalities, we get $P = \{x \in S^3 : Ax \geqslant 0\} = [B_1, B_2] \subset \Gamma_{13}$, Where $B_1 = \left(\frac{c}{a+c}, 0, \frac{a}{a+c}, 0\right)$ and $B_2 = \left(\frac{d}{b+d}, 0, \frac{b}{b+d}, 0\right)$. $Q = \{x \in S^3 : Ax \leqslant 0\} = [C_1, C_2] \subset \Gamma_{24}$, where $C_1 = \left(0, \frac{b}{a+b}, 0, \frac{a}{a+b}, 0\right)$ and $C_2 = \left(0, \frac{d}{c+d}, 0, \frac{c}{c+d}, 0\right)$. Since $ad > bc$, then $\frac{c}{a+c} < \frac{d}{b+d}$ and $\frac{b}{a+b} < \frac{d}{c+d}$. Therefore, $P \neq \varnothing$ and $Q \neq \varnothing$ (Fig. 2).

Figure 2 The intervals in which the start and end points of the trajectory are located

3) Let $ad < bc$. Solving the inequalities $Ax \geqslant 0$ and $Ax \leqslant 0$ on the S^3 simplex, taking into account that $ad < bc$ we obtain: $P = [K_1, K_2] \subset \Gamma_{24}$, where $K_1 = \left(0, \frac{d}{c+d}, 0, \frac{c}{c+d}\right)$, $K_2 = \left(0, \frac{b}{a+b}, 0, \frac{a}{a+b}\right)$, $Q = [L_1, L_2] \subset \Gamma_{13}$, where $L_1 = \left(\frac{d}{b+d}, 0, \frac{b}{b+d}, 0\right)$, $L_2 = \left(\frac{c}{a+c}, 0, \frac{a}{a+c}, 0\right)$.

Theorem 4. For any inner point x^0 of the simplex S^3, the trajectories both positive and negative converge, with $\alpha(x^0) \subset P$ and $\omega(x^0) \subset Q$.

Proof. Let $p = (p_1, 0, p_3, 0) \in P$. On the simplex S^3, consider the function $\varphi(x) = x_1^{p_1} \cdot x_3^{p_3}$. Obviously, $\max\limits_{x \in S^3} \varphi(x) = p_1^{p_1} \cdot p_3^{p_3} = c$, moreover, the maximum is achieved only at one point, and $\max\limits_{x \in S^3} \varphi(x) = 0$ at $x \in \partial S^3$. It is easy to verify that for any $0 < l \leqslant c$, the set $\{x \in S^3; \varphi(x) \leqslant l\}$ is a nonempty convex closed set. Applying Young's inequality for $\varphi(Vx)$we get [1], [12–14]

$$\varphi(Vx) = (x_1 (1 - ax_2 + bx_4))^{p_1} \cdot (x_3 (1 + cx_2 - dx_4))^{p_3} =$$

$$= x_1^{p_1} \cdot x_3^{p_3} \cdot (1 - ax_2 + bx_4)^{p_1} \cdot (1 + cx_2 - dx_4)^{p_3} = \varphi(x) \cdot (1 - ax_2 + bx_4)^{p_1} \cdot (1 + cx_2 - dx_4)^{p_3} \leqslant$$

$$\leqslant \varphi(x) \cdot (p_1 - ap_1x_2 + bp_1x_4 + p_3 + cp_3x_2 - dp_3x_4) =$$

$$= \varphi(x) \cdot (1 - (ap_1 - cp_3)\, x_2 - (dp_3 - bp_1)\, x_4). \tag{7}$$

If $p = (p_1, 0, p_3, 0)$ belongs to the interior of the segment $[B_1, B_2]$, then we get, $ap_1 - cp_3 > 0$ and $dp_3 - bp_1 > 0$. Given that x^0 is the inner starting point of (7), we find

$$\varphi(Vx^0) = \varphi(x^0). \tag{8}$$

According to (8) the $\varphi(x)$ function decreases along any positive trajectory and increases along a negative trajectory, i.e. it is a Lyapunov function for a discrete dynamical system (6) Lotka-Volterra under the condition $ad > bc$. Therefore, with $x^0 = \{x \in S^3 : \varphi(x) \geqslant l\}$, the entire negative trajectory is contained in this set, i.e. $\alpha(x^0) = \{x \in S^3 : \varphi(x) \geqslant l\}$, and the positive trajectory starting from some number does not belong to this set. So, we get that $\alpha(x^0) \subset P$. Calculating the eigenvalues of the Jacobian at points from Q, we find $\lambda_1 = \lambda_2 = 1,\ \lambda_3 = 1 + cx_2 - dx_4,\ \lambda_4 = 1 - ax_2 + bx_4$. Within the $[C_1,\ C_2]$ segment , the $cx_2 - dx_4 < 0$ and $-ax_2 + bx_4 < 0$ inequalities are true . Therefore, $\lambda_3 < 1$ and $\lambda_4 < 1$, i.e. the inner points of the $[C_1,\ C_2]$ segment are attractive for positive trajectories. Therefore, $\omega(x^0)$ consists of one point, and $\omega(x^0) \subset Q = [C_1,\ C_2]$. Theorem 4 is proved.

Theorem 5. Any trajectory converges, and $\alpha(x^0) \subset P$, $\omega(x^0) \subset Q$.
The proof is carried out in the same way as in Theorem 4.

Now let's move on to the simplex an order of magnitude higher, i.e. consider the operator acting in S^4. Let the skew-symmetric matrix and the partially oriented graph corresponding to it have the following form:

$$A = \begin{pmatrix} 0 & 0 & c & -b & a \\ 0 & 0 & -f & e & -d \\ -c & f & 0 & 0 & 0 \\ b & -e & 0 & 0 & 0 \\ -a & d & 0 & 0 & 0 \end{pmatrix},$$

$$V:\ S^4 \to S^4 : \begin{cases} x_1' = x_1(1 + cx_3 - bx_4 + ax_5), \\ x_2' = x_2(1 - fx_3 + ex_4 - dx_5), \\ x_3' = x_3(1 - cx_1 + fx_2), \\ x_4' = x_4(1 + bx_1 - ex_2), \\ x_5' = x_5(1 - ax_1 + dx_2), \end{cases} \tag{9}$$

where $0 < a, b, c, d, e, f < 1$.

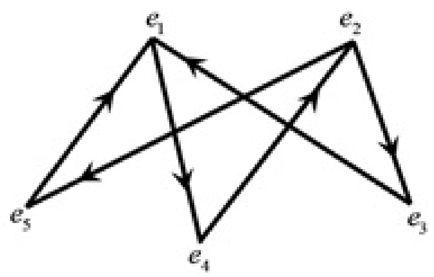

Figure 3 A fully oriented bigraph, corresponding to the $V:\ S^4 \to S^4$ mapping and the A matrix

Theorem 6. For the mapping defined by the equalities (8), the following are conditions:

–by the condition $af \geqslant cd$ and $bd \geqslant ae$ the set consists P of an edge segment Γ_{12}, i.e. $P = AB \subset \Gamma_{12}$;

–by the condition $af \leqslant cd$ and $fb \geqslant ce$ the set consists P of an edge segment Γ_{12}, i.e. $P = B \subset \Gamma_{12}$.

Proof. According to Theorem 1, we find the set $P = \{x \in S^4 : \ Ax \geqslant 0\}$, solving the system of inequalities

$$\begin{cases} cx_3 - bx_4 + ax_5 \geqslant 0, \\ -fx_3 + ex_4 - dx_5 \geqslant 0, \\ -cx_1 + fx_2 \geqslant 0, \\ bx_1 - ex_2 \geqslant 0, \\ -ax_1 + dx_2 \geqslant 0, \end{cases} \tag{10}$$

and we will find the following solution, i.e. we will get the segment :

$$\begin{cases} x_1 \leqslant \frac{f}{c+f}, \ x_2 \geqslant \frac{c}{c+f}, \\ x_1 \geqslant \frac{e}{b+e}, \ x_2 \leqslant \frac{b}{b+e}, \\ x_1 \leqslant \frac{d}{a+d}, \ x_2 \geqslant \frac{a}{a+d}. \end{cases}$$

The location of the segment depends on the coefficients. Let $af \geqslant cd$ and $bd \geqslant ae$. Then $A = \left(\frac{d}{a+d}, \ \frac{a}{a+d}, 0, \ 0, \ 0\right)$, $B = \left(\frac{e}{b+e}, \ \frac{b}{b+e}, 0, \ 0, \ 0\right)$. This means that $P = AB \subset \Gamma_{12}$. For the conditions $af \leqslant cd$ and $fb \geqslant ce$ it is proved similarly. Theorem 6 has been proved.

Theorem 7. For the mapping defined by the equalities (9) under the condition $bf \leqslant ce$ and $bd \leqslant ae$, the set consists of a part of the face Γ_{345}, i.e. $P = \Delta DEe_5 \cap \Delta MNe_3 \subset \Gamma_{345}$.

Proof. To do this, we also solve the system (10), on the face of Γ_{345}, for this we apply the condition $x_1 = x_2 = 0$, then we get

$$\begin{cases} cx_3 - bx_4 + ax_5 \geqslant 0, \\ -fx_3 + ex_4 - dx_5 \geqslant 0. \end{cases} \tag{11}$$

Here, to solve it, you need to consider several cases, i.e. the section of the face Γ_{345} :

1) Consider an edge Γ_{34} : Here $x_5 = 0$ and $x_3 + x_4 = 1$,$x_3 + x_4 = 1$, then the system (11) has the form

$$\begin{cases} cx_3 - b(1 - x_3) \geqslant 0, \\ -fx_3 + e(1 - x_3) \geqslant 0. \end{cases}$$

The solution of that system under the $bf \leqslant ce$ condition consists of a DE segment , i.e. $D = \left(0, \ 0, \ \frac{b}{c+b}, \ \frac{c}{c+b}, 0\right)$, $E = \left(0, \ 0, \ \frac{e}{f+e}, \ \frac{f}{f+e}, 0\right)$.

2) Consider an edge Γ_{35} : Here $x_4 = 0$ and $x_3 + x_5 = 1$, then the system (11) has the form

$$\begin{cases} cx_3 + a(1 - x_3) \geqslant 0, \\ -fx_3 - d(1 - x_3) \geqslant 0. \end{cases}$$

This system has no solution at $0 < a, \ c, \ f, \ d < 1$.

3) Consider an edge Γ_{45} : Here $x_3 = 0$ and $x_4 + x_5 = 1$, then the system (11) has the form

$$\begin{cases} -bx_4 + a(1 - x_4) \geqslant 0, \\ ex_4 - d(1 - x_4) \geqslant 0. \end{cases}$$

Solution of the system $\frac{d}{e+d} \leqslant x_4 \leqslant \frac{a}{a+b}$, only it makes sense if $bd < ae$, that is, it is a segment MN: $M = \left(0, \ 0, \ 0, \ \frac{a}{a+b}, \ \frac{b}{a+b}\right)$, $N = \left(0, \ 0, \ 0, \ \frac{d}{e+d}, \ \frac{e}{e+d}\right)$. Combining all three cases, under the condition $bf \leqslant ce$ and $bd \leqslant ae$, we get the set $P = \Delta DEe_5 \cap \Delta MNe_3 \subset \Gamma_{345}$ (see Figure 4).

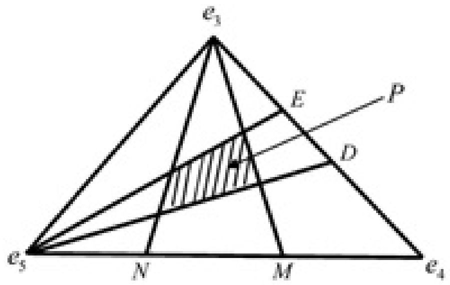

Figure 4 The set $P = \Delta DEe_5 \cap \Delta MNe_3 \subset \Gamma_{345}$

Now let's prove the theorem for the locations of the set.

Theorem 8. For the mapping defined by the equalities (9), the set Q consists of the following:

– by the condition $ce \geqslant fb$ and $af \geqslant cd$ the set consists Q of an edge segment Γ_{12}, i.e. $Q = B \subset \Gamma_{12}$;

– by the condition $cd \geqslant af$ and $bd \leqslant ae$ the set consists Q of an edge segment Γ_{12}, i.e. $Q = BC \subset \Gamma_{12}$.

Proof. To do this, we solve the $Q = \{x \in S^4 : \; Ax \leqslant 0\}$ system from the theorem 1:

$$\begin{cases} cx_3 - bx_4 + ax_5 \leqslant 0, \\ -fx_3 + ex_4 - dx_5 \leqslant 0, \\ -cx_1 + fx_2 \leqslant 0, \\ bx_1 - ex_2 \leqslant 0, \\ -ax_1 + dx_2 \leqslant 0. \end{cases} \tag{12}$$

We get a solution in the form of

$$\begin{cases} x_1 \leqslant \frac{f}{c+f}, \;\; x_2 \geqslant \frac{c}{c+f}, \\ x_1 \geqslant \frac{e}{b+e}, \;\; x_2 \leqslant \frac{b}{b+e}, \\ x_1 \leqslant \frac{d}{a+d}, \;\; x_2 \geqslant \frac{a}{a+d}. \end{cases}$$

Here, under the condition of $ce \geqslant fb$ and $af \geqslant cd$, there are many $Q = B \subset \Gamma_{12}$. The second part of the theorem is proved similarly.

Theorem 9. For the mapping defined by the equalities (9) under the condition $bf \geqslant ce$ and $bd \leqslant ae$ the set consists of a part of the face Γ_{345}, i.e. $Q = \Delta EDe_5 \cap \Delta NMe_3 \subset \Gamma_{345}$.

Proof. The theorem is proved similarly to Theorem 7.

3 A discrete model of the interaction of two viruses

In this part of the paper, we will consider the application of the operator in question in epidemiology. In the previous part of the paper, we investigated the dynamics of an operator operating in a four-dimensional simplex. This operator can fully serve as a discrete compartmental model describing the evolution of the interaction of two airborne viruses. A four-dimensional simplex means a closed population. The population is divided into five groups, such as: S is the part of the population that is susceptible to infection, but has not yet been infected; E is the part of the population that has the latent form of the first virus; I_1 is the part of the population infected with the first virus; I_2 is the part of the population infected with the second virus; R is the part of the population that got rid of the first virus. Then our model looks like:

$$\begin{cases} S^{(n+1)} = S^{(n)}(1 + cI_2^{(n)} - bE^{(n)} + aR^{(n)}), \\ I_1^{(n+1)} = I_1^{(n)}(1 - fI_2^{(n)} + eE^{(n)} - dR^{(n)}), \\ I_2^{(n+1)} = I_2^{(n)}(1 - cS^{(n)} + fI_1^{(n)}), \\ E^{(n+1)} = E^{(n)}(1 + bS^{(n)} - eI_1^{(n)}), \\ R^{(n+1)} = R^{(n)}(1 - aS^{(n)} + dI_1^{(n)}). \end{cases}$$

The epidemiological system corresponding to this model is depicted in Figure 5.

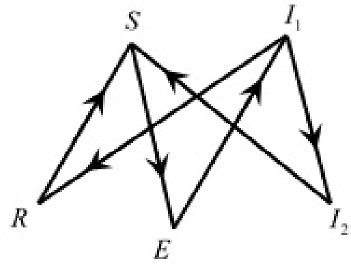

Figure 5 The spread trajectory of two-symptom viruses

This model describes two types of virus flow:

$$1) S \to E \to I_1 \to R \to S,$$

$$2) S \to E \to I_1 \to I_2 \to S.$$

In both routes, individuals pass through the latency period of the first virus. But strangely enough, the picture is different in both cases:
– in the first route, individuals will become infected only with the first virus, and then they recover and the recovered individuals return to the healthy part of the population;
– in the second route, individuals after infection with the first virus have contact with patients with the second type of virus. This type of patient does not have the opportunity to fully recover, and they, having the interaction of two viruses, return to a healthy part of the population.

This model describes a very complex disease structure. For example, the epidemiology of tuberculosis and its co-infections (viral hepatitis B and C). Despite the study of the epidemiology of tuberculosis and its co-infection (viral hepatitis B and C) over the past decades, a number of questions remain, including those related to the impact of co-infection on survival, depending on the chosen treatment regimen for tuberculosis, the likelihood of adverse outcomes in the form of gastrointestinal bleeding and cirrhosis of the liver and their connections with the therapy of the underlying disease.

Our main goal is to evaluate the survival rate of tuberculosis patients with co-infection (viral hepatitis B and C) and receiving multicomponent chemotherapy. The presence of chronic viral hepatitis B and/or C in patients with tuberculosis did not affect mortality from all causes and regardless of the type of virus during a long follow-up period. Patients who did not receive treatment for viral hepatitis and who had tuberculosis had a higher risk of death from all causes. The set of P means the beginning of the disease, and the set of Q means the end, that is, getting rid of the virus. Below we present a numerical analysis, consider the numerical dynamics and phase portrait (see Figure 6 and Table 1).

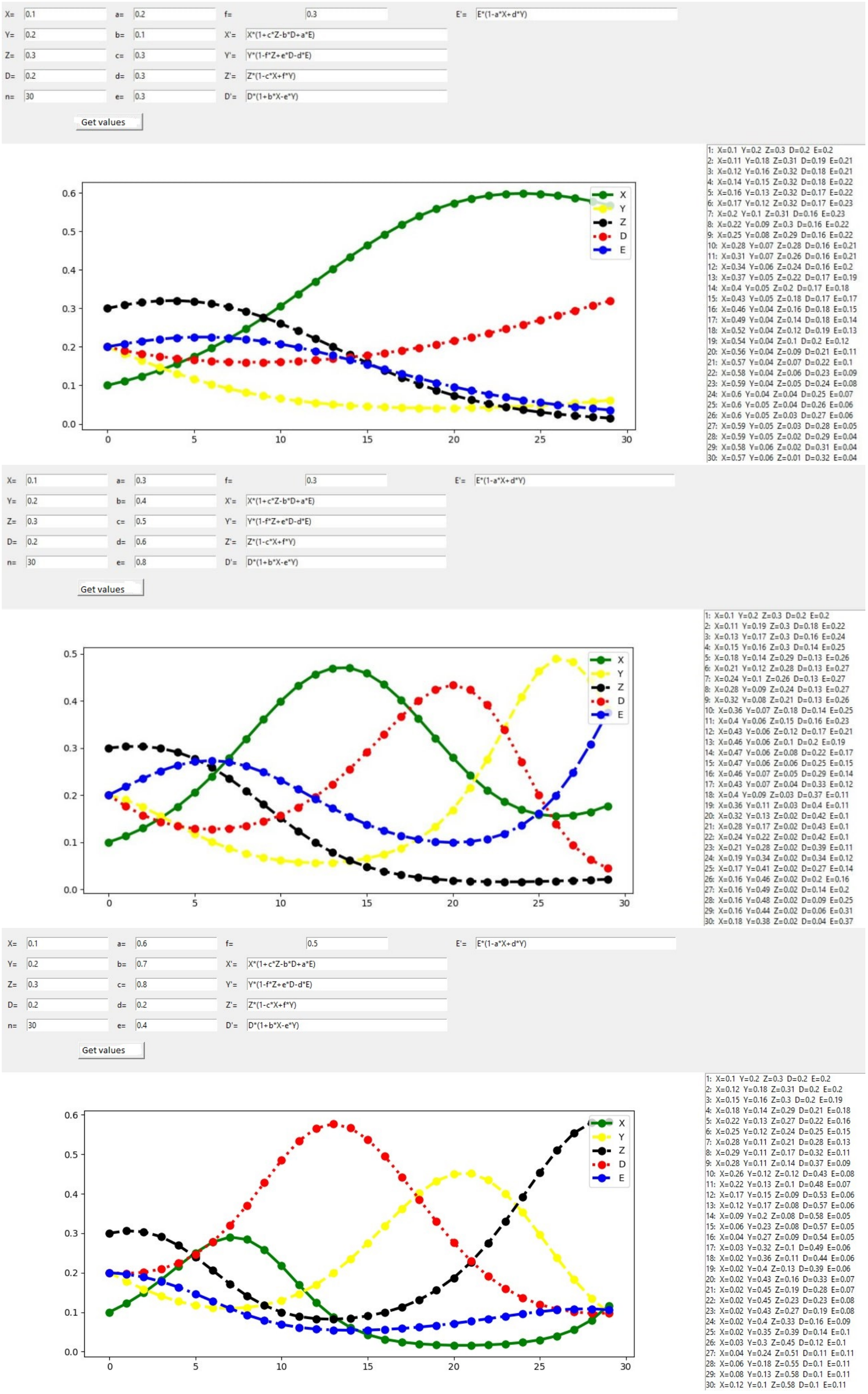

Figure 6 30-day analysis of the spread of two-symptom diseases among the population

4 Conclusion

In [1–3], [6, 7] discrete Lotka-Volterra mappings with skew-symmetric matrices in general position are investigated. It is known [11] that tournaments correspond to a matrix

Table 1. Results of a 30-day numerical analysis of the prevalence of two-symptom diseases among the population

X=0.1, Y=0.2, Z=0.3, D=0.2, E=0.2														
a=0.2, b=0.1, c=0.3, d=0.3, e=0.3, f=0.3					a=0.3, b=0.4, c=0.5, d=0.6, e=0.8, f=0.3					a=0.6, b=0.7, c=0.8, d=0.2, e=0.4, f=0.5				
X	Y	Z	D	E	X	Y	Z	D	E	X	Y	Z	D	E
0.1	0.2	0.3	0.2	0.2	0.1	0.2	0.3	0.2	0.2	0.1	0.2	0.3	0.2	0.2
0.11	0.18	0.31	0.19	0.21	0.11	0.19	0.3	0.18	0.22	0.12	0.18	0.31	0.2	0.19
0.12	0.16	0.32	0.18	0.21	0.13	0.17	0.3	0.16	0.24	0.15	0.16	0.3	0.2	0.19
0.14	0.15	0.32	0.18	0.22	0.15	0.16	0.3	0.14	0.25	0.18	0.14	0.29	0.21	0.18
0.16	0.13	0.32	0.17	0.22	0.18	0.14	0.29	0.13	0.26	0.22	0.13	0.27	0.22	0.16
0.17	0.12	0.32	0.17	0.23	0.21	0.12	0.28	0.13	0.27	0.25	0.12	0.24	0.25	0.15
0.2	0.1	0.31	0.16	0.23	0.24	0.1	0.26	0.13	0.27	0.28	0.11	0.21	0.28	0.13
0.22	0.09	0.3	0.16	0.22	0.28	0.09	0.24	0.13	0.27	0.29	0.11	0.17	0.32	0.11
0.25	0.08	0.29	0.16	0.22	0.32	0.08	0.21	0.13	0.26	0.28	0.11	0.14	0.37	0.09
0.28	0.07	0.28	0.16	0.21	0.36	0.07	0.18	0.14	0.25	0.26	0.12	0.12	0.43	0.08
0.31	0.07	0.26	0.16	0.21	0.4	0.06	0.15	0.16	0.23	0.22	0.13	0.1	0.48	0.07
0.34	0.06	0.24	0.16	0.2	0.43	0.06	0.12	0.17	0.21	0.17	0.15	0.09	0.53	0.06
0.37	0.05	0.22	0.17	0.19	0.46	0.06	0.1	0.2	0.19	0.12	0.17	0.08	0.57	0.06
0.4	0.05	0.2	0.17	0.18	0.47	0.06	0.08	0.22	0.17	0.09	0.2	0.08	0.58	0.05
0.43	0.05	0.18	0.17	0.17	0.47	0.06	0.06	0.25	0.15	0.06	0.23	0.08	0.57	0.05
0.46	0.04	0.16	0.18	0.15	0.46	0.07	0.05	0.29	0.14	0.04	0.27	0.09	0.54	0.05
0.49	0.04	0.14	0.18	0.14	0.43	0.07	0.04	0.33	0.12	0.03	0.32	0.1	0.49	0.06
0.52	0.04	0.12	0.19	0.13	0.04	0.09	0.03	0.37	0.11	0.02	0.36	0.11	0.44	0.05
0.54	0.04	0.1	0.2	0.12	0.36	0.11	0.03	0.04	0.11	0.02	0.4	0.13	0.39	0.06
0.56	0.04	0.09	0.21	0.11	0.32	0.13	0.02	0.42	0.1	0.02	0.43	0.16	0.33	0.07
0.57	0.04	0.07	0.22	0.1	0.28	0.17	0.02	0.43	0.1	0.02	0.45	0.19	0.28	0.07
0.58	0.04	0.06	0.23	0.09	0.24	0.22	0.02	0.42	0.1	0.02	0.45	0.23	0.23	0.08
0.59	0.04	0.05	0.24	0.08	0.21	0.28	0.02	0.39	0.11	0.02	0.43	0.27	0.19	0.08
0.6	0.04	0.04	0.25	0.07	0.19	0.34	0.02	0.34	0.12	0.02	0.4	0.33	0.16	0.09
0.6	0.05	0.04	0.26	0.06	0.17	0.41	0.02	0.27	0.14	0.02	0.35	0.39	0.14	0.1
0.6	0.05	0.03	0.27	0.06	0.16	0.46	0.02	0.2	0.16	0.03	0.3	0.45	0.12	0.1
0.59	0.05	0.03	0.28	0.05	0.16	0.49	0.02	0.14	0.2	0.04	0.24	0.51	0.11	0.11
0.59	0.05	0.02	0.29	0.04	0.16	0.48	0.02	0.09	0.25	0.06	0.18	0.55	0.1	0.11
0.58	0.06	0.02	0.31	0.04	0.16	0.44	0.02	0.06	0.31	0.08	0.13	0.58	0.1	0.11
0.57	0.06	0.01	0.32	0.04	0.18	0.38	0.02	0.04	0.37	0.12	0.1	0.58	0.1	0.11

of this type. In the proposed work, the dynamics of trajectories, internal points of degenerate cases of Lotka-Volterra mappings are investigated, since they can be used to model the course of airborne diseases. In [12, 13], a discrete analogue of the compartmental model based on the degenerate Lotka-Volterra mapping operating in S^4 is proposed. We propose a complete analytical study for a discrete model, which can be built on the basis of a mapping acting in a three-dimensional simplex defined by equalities (6). In addition, a degenerate Lotka-Volterra mapping acting in S^4 is considered.

In the continuous models considered in [16–18], it is important to find the basic reproductive number and, by its value, conclude whether the virus persists in the population or not. We also propose degenerate cases of Lotka-Volterra mappings as discrete analogues. The main difference lies in finding a set of limit points of a positive and negative trajectory, the meaning of which, in turn, lies in the area of the part of the population in which epidemiology begins and, accordingly, where it ends.

Here we have constructed a discrete model based on degenerate Lotka-Volterra mappings corresponding to complete oriented bigraphs. The constructed model describes the evolution of the interaction of two viruses. An example of such viruses is tuberculosis and its co-infections. The main goal is to evaluate the survival of tuberculosis patients with co-infection (viral hepatitis B and C) and receiving multicomponent chemotherapy. We took data from thirty days of these viruses and made a numerical analysis of their course among the population of the Osh region of the Republic of Kyrgyzstan.

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ВЫРОЖДЕННЫЕ ОТОБРАЖЕНИЯ ЛОТКИ-ВОЛЬТЕРРЫ И СООТВЕТСТВУЮЩИЕ ИМ БИГРАФЫ КАК ДИСКРЕТНАЯ МОДЕЛЬ ЭВОЛЮЦИИ ВЗАИМОДЕЙСТВИЯ ДВУХ ВИРУСОВ

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В работе предлагается дискретная модель взаимодействия двух воздушно-капельных вирусов, основанная на операторе Лотки-Вольтерры, действующем в четырёхмерном симплексе. Эта модель описывает развитие эпидемического процесса в замкнутой популяции, разделённой на пять классов: восприимчивые, находящиеся в латентной стадии первого вируса, инфицированные первым вирусом, инфицированные вторым вирусом и выздоровевшие после первого вируса. Математическая структура модели учитывает сложные переходы между состояниями и взаимодействие штаммов, включая случаи коинфекции. Особое внимание уделено анализу множества начальных и конечных состояний болезни, определяемых системой неравенств. В зависимости от параметров модели эти множества могут лежать на различных гранях симплекса, что соответствует различным сценариям начала и окончания эпидемии. Рассматриваются два ключевых эпидемиологических сценария: один с полным выздоровлением после первого вируса, другой — с переходом инфицированных первым вирусом в состояние коинфекции с невозможностью полного выздоровления. Модель применима к анализу коинфекции туберкулёза с вирусными гепатитами B и C и позволяет оценивать влияние различных параметров на выживаемость пациентов при многокомпонентной терапии. В заключении проведён численный эксперимент: представлены траектории, фазовые портреты и 30-дневная динамика распространения заболевания, иллюстрирующие поведение системы при различных начальных условиях и параметрах.

Ключевые слова: отображение Лотки-Вольтерры, симплекс, кососимметричная матрица, выпуклая оболочка, траектория, частично ориентированный граф, биграф.

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