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ANOMALOUS SOLUTE TRANSPORT IN AN ELEMENT OF A FRACTURED-POROUS MEDIUM WITH MEMORY EFFECTS

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Fractured porous media (FPM) in theoretical investigations is considered as a system of fractures and adjacent porous blocks (matrix). In such media, solute transport occurs mainly through the fracture system with mass transfer in porous blocks. In this work, the problem of solute transport through an element of FPM is studied, taking into account the memory effects. The medium consists of a single fracture and a porous block (matrix) bordering it. The problem was solved numerically using the finite difference method with Caputo's definition of fractional derivatives. Based on the numerical results, concentration profiles in the fracture and matrix were obtained. The influence of the fractional order of derivatives on the distribution of concentration is shown. The current, total and summing flow rates of the solute from the fracture into the matrix were also determined.

Keywords: diffusion, fractional derivatives, flow rate, fractured-porous medium, matrix.

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1 Introduction

In recent years, energy problems have become a pressing issue all over the world, so much attention is paid to oil and gas production as the main source of energy. Oil and gas are located underground, largely in fractured porous media. Numerous scientific research are being carried out and scientific papers are being published on the transfer of gas and oil in reservoirs. Fractures and fracture networks are major conduits for the transport of hydrothermal fluids, contaminants in water and groundwater systems, and oil and gas in reservoirs [1]. Here another problem arises, when the solute transport through the channels, diffusion transport occurs into the porous medium near the fracture. In [2] the above mentioned problems are modeled based on the physical and mechanical nature of the medium, analytical solution of the problem has been obtained and compared with experimental results. Diffusion of a solute from the fracture to the matrix is an important factor in assessing the transfer of pollutants and assessing the diffusion characteristics of a medium [3, 4].

In laboratory experiments carried out on a small sample taken from a homogeneous porous medium to study the solute transport at different flow velocities, and a model suitable for this medium was created, and the results obtained with its help were compared with the experimental results [5]. The assumption of a linear dependence of the dispersion and flow rates coefficients on the mass flow significantly improves the model and is confirmed when compared with experimental results [5]. This is important not only for studying the transfer of oil and gas, but also the transportation of pollutants in a FPM [6–8].

Underground rocks can be deformed by pressure from overlying rock layers. If a fractured-porous medium is deformable, then oil production in such environments has its specifics. Several works have modeled multiphase filtration processes in deformable porous media. In this case, the density of each phase is assumed to be constant, capillary pressures between phases are neglected, and problems for this model are formulated and solved numerically [9, 10].

Sometimes the results of modeling the solute transport in complex structural media using Fick's law do not agree with experimental data [11, 12], but relatively good results are obtained when modeling such processes using fractional differential equations [13].

Based on the structure and properties of FPM, many problems have been modeled and solved using various methods. Solute transport processes in coaxial cylindrical two-zone inhomogeneous media were modeled by fractional order differential equations and the corresponding problems were solved numerically taking into account the presence of two zones of the medium - micropores and macropores [14].

This paper considers the problem of anomalous solute transport in a FPM element consisting of a single fracture and a matrix [2, 15]. The transport memory effect is taken into account both in the fracture and in the porous block. Fractional time derivatives in the equations appear both in the matrix and in the fracture. Thus, here anomalous phenomena can occur in a fracture and a porous block to varying degrees, which leads to various options for the manifestation of the mutual influence of anomalous phenomena in a fracture and a porous block.

2 Statement of problem

A fracture is considered to be a semi-infinite one-dimensional object. This formulation does not take into account the second dimension of the fracture, namely its width. The porous block occupies a quarter of the entire surface (Fig. 1). Following this formulation, the region $R\{0 \leqslant x < \infty, 0 \leqslant y < \infty\}$. From the $x = 0$ end of the fracture a liquid with concentration c_0 is injected. Let the liquid flow in the fracture at a given constant velocity v. Initially, the fracture and porous block are considered to be filled with pure (no solute) liquid. In a fracture, convective-diffusion solute transport occurs, and only diffusion in a porous block. The transfer process in a porous block and fracture occurs with the manifestation of anomalous phenomena.

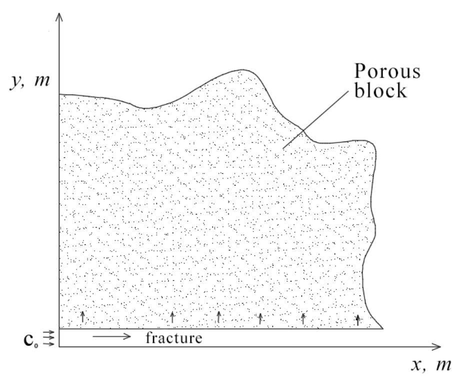

Fig. 1. The element of fractured - porous medium

The equations of solute transport and fluid flow in the FPM element we write in the following form

$$\frac{\partial^{\alpha} c_f}{\partial t^{\alpha}} + v\frac{\partial c_f}{\partial x} = D_f \frac{\partial c_f^2}{\partial x^2} + m_0 D_m \frac{\partial^{1-\gamma}}{\partial t^{1-\gamma}} \left(\frac{\partial c_m}{\partial y}\right)\bigg|_{y=0}, \tag{1}$$

$$\frac{\partial^{\gamma} c_m}{\partial t^{\gamma}} = D_m \frac{\partial^2 c_m}{\partial y^2}, \tag{2}$$

where $c_m = c_m(t,x,y)$ is the solute concentration in the porous block $\mathrm{m}^3/\mathrm{m}^3$; $c_f = c_f(t,x)$ is the solute concentration in the fracture, $\mathrm{m}^3/\mathrm{m}^3$; D_f is diffusion coefficient in the fracture, $\mathrm{m}^2/\mathrm{s}^{\alpha}$; v is velocity of fluid, $\mathrm{m}/\mathrm{s}^{\alpha}$; D_m is coefficient of diffusion in the matrix, m^2/s^{γ}; m_0 is porosity coefficient of the matrix, t is time, $s; x, y$ are coordinates; α, γ is orders of fractional derivatives with respect to time, $(0 < \alpha, \gamma \leqslant 1)$.

It is assumed that up to the boundaries of the matrix $y = \infty$ and $x = \infty$ the concentration front c_m does not reach. Under these conditions, the initial and boundary conditions have the form:

$$c_f(0,x) = 0, c_m(0,x,y) = 0, \tag{3}$$

$$c_f(t,0) = c_0, c_f(t,\infty) = 0, \quad c_0 = \text{ const }, \tag{4}$$

$$c_m(t,x,0) = c_f(t,x), c_m(t,x,\infty) = 0. \tag{5}$$

The problem (1), (2) under conditions (3) - (5) is solved by the finite difference method [16]. To accomplish this, following net is constructed

$$\bar{\omega}_{h_1 h_2 \tau} = \{(t_j, x_i, y_k)\,,\ t_j = \tau j,\ x_i = i h_1,\ y_k = k h_2,$$

$$j = \overline{0, J},\ i = 0, 1, \ldots,\ k = 0, 1, \ldots,\ \tau = T/J\}\,,$$

where h_1 is the net step by the axis x, h_2 is the net step in the direction of y, τ is the time step of the net, T is the maximum time during which the process is studied, J is the number of net intervals in t.

Equations (1), (2) are approximated on the net $\omega_{h_1 h_2 \tau}$. To do this, explicit scheme is used, and fractional derivatives are defined in the sense of Caputo. Consequently, the approximations have the form

$$\begin{gathered}
\frac{1}{\Gamma(2-\alpha)\,\tau^{\alpha}} \left[\sum_{p=0}^{j-2} \left((c_f)_i^{p+1} - (c_f)_i^{p}\right) \cdot \left((j-p+1)^{1-\alpha} - (j-p)^{1-\alpha}\right)\right] + \\
+ \frac{1}{\Gamma(2-\alpha)\,\tau^{\alpha}} \left[(c_f)_i^{j+1} - (c_f)_i^{j}\right] + v\frac{(c_f)_{i+1}^{j} - (c_f)_i^{j}}{h_1} = \\
= \frac{m_0 D_m}{\Gamma(1+\gamma)\,\tau^{1-\gamma} h_2^{\delta}} \cdot \left[\sum_{l=0}^{j-1} \left((c_m)_{i0}^{l+1} - (c_m)_{i0}^{l} - (c_m)_{i1}^{l+1} + (c_m)_{i1}^{l}\right)\right] \cdot \\
\cdot \left((j-l)^{\gamma} - (j-l-1)^{\gamma}\right) + \frac{D_f}{h_1^2} \cdot \left((c_f)_{i-1}^{j} - 2(c_f)_i^{j} + (c_f)_{i+1}^{j}\right),
\end{gathered} \tag{6}$$

$$\frac{1}{\Gamma(2-\gamma)\,\tau^{\gamma}}\left[\sum_{l=0}^{j-2}\left((c_m)_{ik}^{l+1}-(c_m)_{ik}^{l}\right)\cdot\left((j-l+1)^{1-\gamma}-(j-l)^{1-\gamma}\right)\right]+$$
$$+\frac{1}{\Gamma(2-\gamma)\,\tau^{\gamma}}\left[(c_m)_{ik}^{j+1}-(c_m)_{ik}^{j}\right]=$$
$$=\frac{D_m}{h_2^2}\cdot\left((c_m)_{ik-1}^{j}-2(c_m)_{ik}^{j}+(c_m)_{ik+1}^{j}\right), \tag{7}$$

where $(c_f)_i^j, (c_m)_{ik}^j$ are net values of functions $c_f(t,x)$ and $c_m(t,x,y)$ at net points (t_j,x_i) and (t_j,x_i,y_k) accordingly, $\Gamma(\cdot)$ is gamma function.

Net equations (6), (7) are reduced to the following recurrent form

$$(c_f)_i^{j+1}=\frac{m_0 D_m\Gamma(2-\alpha)\tau^{\alpha}}{\Gamma(1+\gamma)\tau^{1-\gamma_h}}\cdot\left[\sum_{l=0}^{j-1}(c_m)_{i0}^{l+1}-(c_m)_{i0}^{l}\right]-$$
$$-\frac{m_0 D_m\Gamma(2-\alpha)\tau^{\alpha}}{\Gamma(1+\gamma)\tau^{1-\gamma_h}}\cdot\left[\sum_{l=0}^{j-1}(c_m)_{i1}^{l+1}-(c_m)_{i1}^{l}\right]\cdot$$
$$\cdot\left((j-l)^{\gamma}-(j-l-1)^{\gamma}\right)-\frac{\nu\Gamma(2-\alpha)\tau^{\alpha}\left((c_f)_{i+1}^{j}-(c_f)_{i}^{j}\right)}{h_1}+ \tag{8}$$
$$+\frac{\Gamma(2-\alpha)\tau^{\alpha}D_f}{h_1^2}\cdot\left((c_f)_{i-1}^{j}-2(c_f)_{i}^{j}+(c_f)_{i+1}^{j}\right)-$$
$$-\sum_{p=0}^{j-2}\left((c_f)_i^{p+1}-(c_f)_i^{p}\right)\cdot\left((j-p+1)^{1-\alpha}-(j-p)^{1-\alpha}\right)-(c_f)_i^j,$$

$$(c_m)_{ik}^{j+1}=(c_m)_{ik}^{j}+\frac{D_m\Gamma(2-\gamma)\tau^{\gamma}}{h_2^2}\cdot\left((c_m)_{i,k-1}^{j}-2(c_m)_{i,k}^{j}+(c_m)_{i,k+1}^{j}\right)-$$
$$-\sum_{l=0}^{j-2}\left((c_m)_{ik}^{l+1}-(c_m)_{ik}^{l}\right)\cdot\left((j-l+1)^{1-\gamma}-(j-l)^{1-\gamma}\right), \tag{9}$$
$$i=\overline{0,I-1},\; j=\overline{0,J-1},\; k=\overline{0,K-1}.$$

The initial and boundary conditions are approximated as

$$(c_f)_i^0=0, \tag{10}$$
$$(c_m)_{ik}^0=0, \tag{11}$$
$$(c_f)_0^j=c_0, \tag{12}$$
$$(c_m)_{i0}^j=(c_f)_i^j, \tag{13}$$
$$(c_f)_I^j=0, \tag{14}$$
$$(c_m)_{iK}^j=0. \tag{15}$$

3 Results

Numerical calculations according to (8) - (15) are carried out using the following initial parameters [2–4]: $c_0 = 0.01,\ m^3/m^3; D_m = 5 \cdot 10^{-6}, m^2/s^\gamma; D_f = 1 \cdot 10^{-5}, m^2/s^\alpha; v = 1 \cdot$ $\cdot\, 10^{-5}, m/s^\alpha; m_0 = 0.2; t = 3600$ s and various γ, α.

The results of numerical calculations are presented in Fig. 2-7.
Fig. 2 compares the concentration surfaces c_m for the anomalous case $\alpha < 1$ with classical case $\alpha = 1$. As α decreases from 1, a slower propagation of concentration surfaces in the porous block is observed. This is a consequence of "slow"diffusion in the fracture with decreasing α from 1 . This slowdown in the process is more clearly visible in sections of the concentration surface c_m at different x (Fig. 3).

In Fig. 4, 5 shown the results at $\alpha = 0.9$ and different γ. As the value of γ decreases, "slow"diffusion occurs in the porous block. This, in turn, leads to a slowdown in the rate of decrease in values of c_f, i.e. as γ decreases, the concentration c_f increases. This can be seen from the graphs in Fig. 5, where one can see an increase in c_f (i.e. c_m at $y = 0$) with a simultaneous slowdown in the distribution of profiles in the direction of y.

Fig. 6, 7 show changes in the relative flow rates of the solute across the common boundary of the media.
The current relative flow rate of solute through $y = 0$ is defined as

$$Q = -\, m_0 D_m \frac{\partial^{1-\gamma}}{\partial t^{1-\gamma}} \left(\frac{\partial c_m}{\partial y} \right) \bigg|_{y=0}. \tag{16}$$

The total Q_{total} and summing Q_{sum} relative flow rates across the border $y = 0$ are also determined

$$Q_{\text{total}} = \int_0^\infty Q dx, \tag{17}$$

$$Q_{\text{sum}} = \int_0^t Q_{\text{total}}\, dt = \int_0^t \int_0^\infty Q dx dt. \tag{18}$$

As can be seen in Fig. 6.7 "slow"diffusion in a fracture leads to a decrease in the current relative flow rate. "Slow"diffusion in a porous block due to the formation of large concentration gradients at the boundary $y = 0$ leads to an increase in current flow rate (Fig. 7). In some cases, a nonmonotonic dependence was obtained for the current relative flow rate. This is due to a change in the total flow rate across the border $y = 0$ due to a change in the concentration gradient. For the summing relative flow rate across the border $y = 0$ a monotonically increasing dependence on time was obtained (Fig. 6,7).

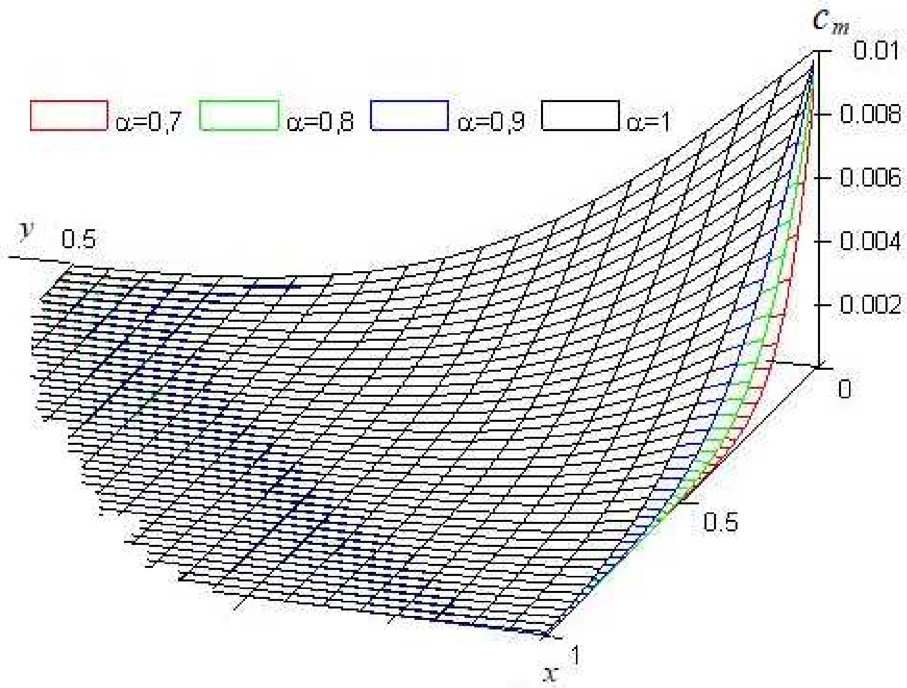

Figure 1 Surface concentration c_m at $t = 3600\,\text{s}$, $\gamma = 1$ and various α

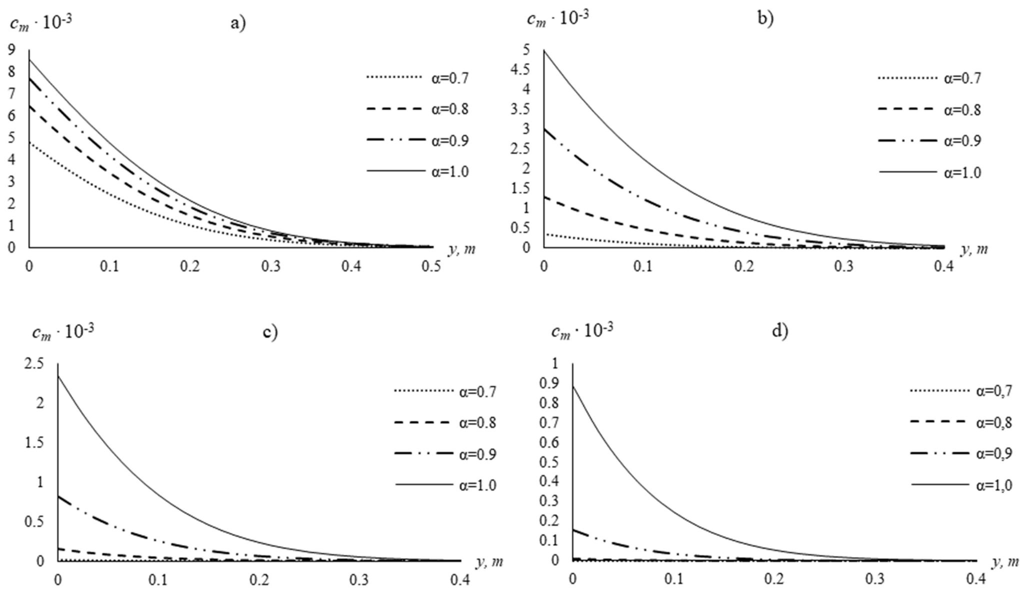

Figure 2 Concentration profiles c_m on sections $x = 0.1\,m$ (a), $x = 0.3\,m$ (b), $x = 0.5\,m$ (c), $x = 0.7\,m$ (d) at $t = 3600\,s$, $\gamma = 1$.

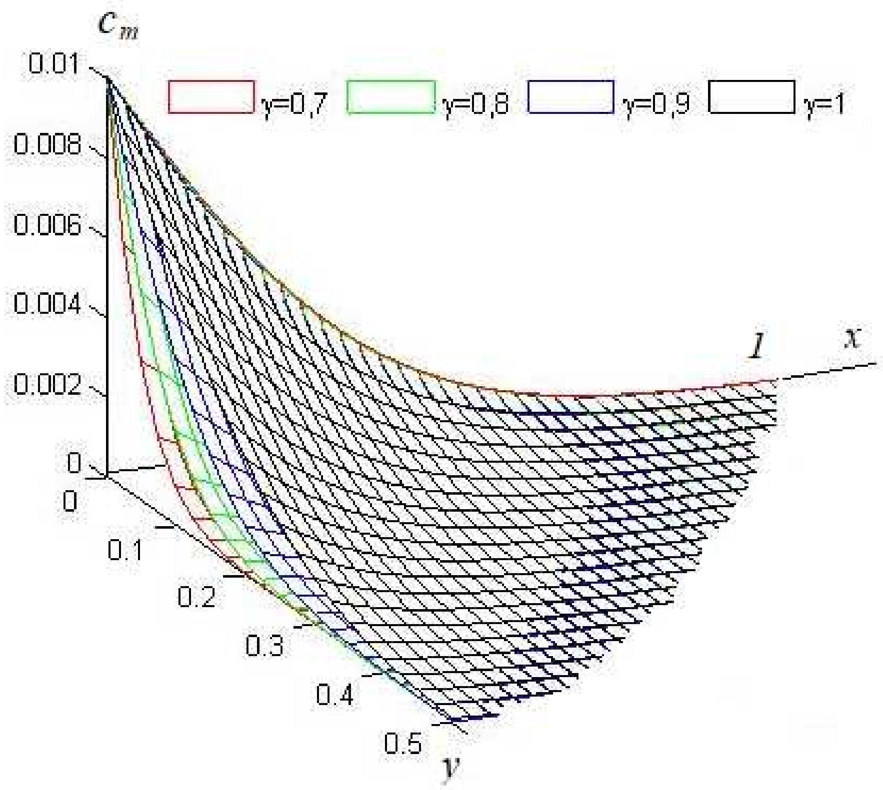

Figure 3 Surface concentration c_m at $t = 3600\,\mathrm{s}$, $\alpha = 0.9$ and various γ

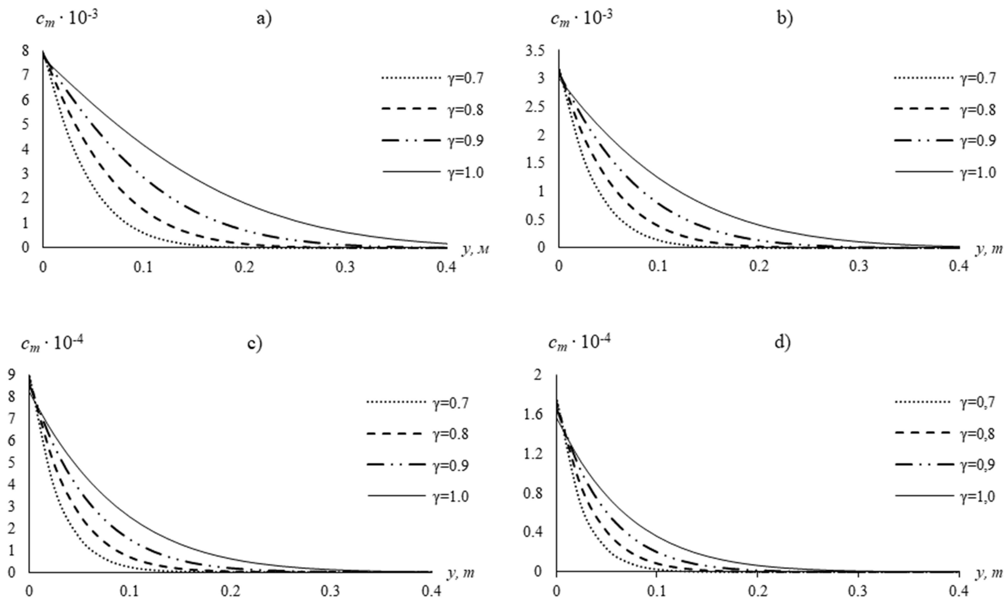

Figure 4 Concentration profiles c_m on sections $x = 0.1\,m$ (a), $x = 0.3\,m$ (b), $x = 0.5\,m$ (c), $x = 0.7\,m$ (d) at $t = 3600\,s$, $\alpha = 0.9$.

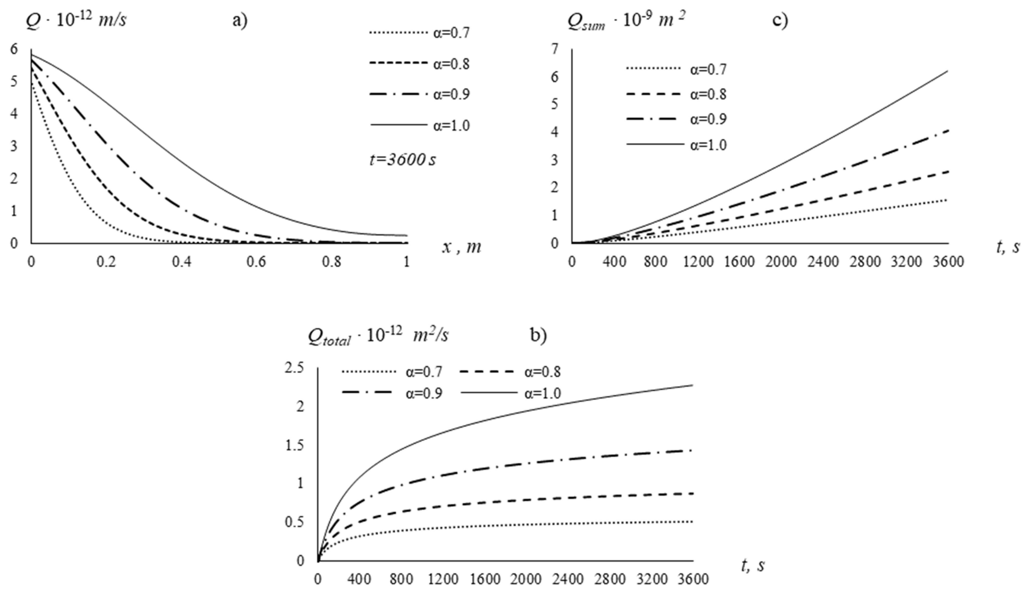

Figure 5 Change in flow rates Q (a) by x, Q_{total} (b), Q_{sum} (c) by t at $\gamma = 1$ and different values of α.

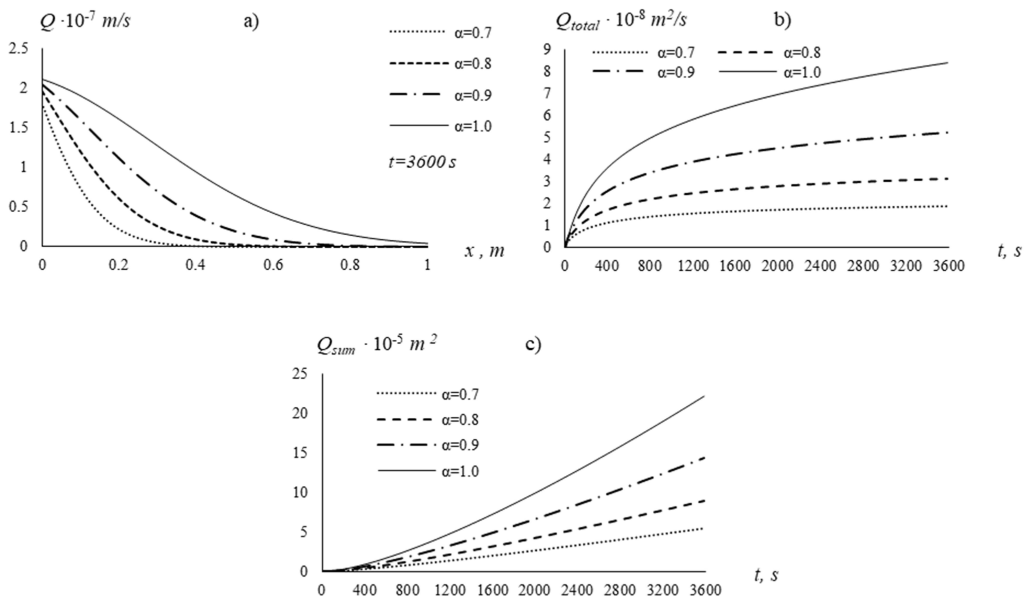

Figure 6 Change in flow rates Q (a) by x, Q_{total} (b), Q_{sum} (c) by t at $\gamma = 0.9$ and different values of α.

4 Conclusion

In the FPM element consisting of one fracture and a porous block (matrix) bordering it, the problem of solute transport was studied taking into account the anomalies of the transport in the fracture and matrix. The classical model of solute transport has been improved using fractional derivatives. A system of differential equations with fractional derivatives is solved numerically and, based on a numerical experiments, the concentration fields in the fracture and in the porous block are determined. Numerical calculations show that changing the concentration profiles of solute concentration in one medium affects the second.

As the fractional order of time derivative in both zones decreases from 1, the diffusion of the solute slows down, i.e., the phenomenon of "slow"diffusion occurs. Slowing down the diffusion process in a porous block leads to an acceleration of the distribution of the solute in the fracture. The slowdown of diffusion in a fracture leads to a similar slowdown in the diffusion process in the porous block.

Since the concentration enters the medium only through a fracture, a flow of substance occurs at the boundary of the two media. Therefore, the concentration gradient at the boundary is different from zero. The dependences of the current, total and summing relative flow rates of the solute are variable spatially and temporally.

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ПЕРЕНОС ВЕЩЕСТВА В ЭЛЕМЕНТЕ ТРЕЩИНОВАТО-ПОРИСТОЙ СРЕДЫ С УЧЕТОМ ЭФФЕКТА ПАМЯТИ

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Трещиновато-пористая среда (ТПС) в теоретических исследованиях рассматривается как система трещин и прилегающих к ним пористых блоков (матрицы). В таких средах транспорт растворенных веществ осуществляется в основном через систему трещин с массопереносом в пористых блоках. В данной работе изучается задача транспорта вещества через элемент трещиновато-пористой среды (ТПС) с учетом эффекта памяти. Среда состоит из одиночной трещины и граничащего с ней пористого блока (матрицы). Задача решаются численно с использованием метода конечных разностей с определением дробных производных по Капуто. На основе численных результатов получены профили концентрации в трещине и матрице. Показано влияние дробного порядка производных на распределение концентрации. Также определены текущие, полные и суммирующие расходы потока вещества из трещины в матрицу.

Ключевые слова: диффузия, дробные производные, массообмен, трещиновато-пористая среда, матрица.

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