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COMPARATIVE ANALYSIS OF UNKNOWN PARAMETER ESTIMATION OF THE GAMMA DISTRIBUTION WITH RIGHT-CENSORED DATA IN INCOMPLETE STATISTICAL MODELS

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In this article, the problem of estimating the parameters of the gamma distribution under censored data conditions in incomplete statistical models is considered. Numerical maximum likelihood methods are analyzed, including the Nelder-Mead and Expectation-Maximization (EM) algorithms, which are applied for estimating the distribution parameters. A comparison of estimation accuracy at different levels of censoring is conducted, allowing the identification of the advantages and limitations of each method. The obtained results show that the EM algorithm provides higher estimation accuracy under censoring conditions, while the Nelder-Mead method demonstrates stable results under full observation. The influence of the proportional hazards model on parameter estimation under dependent censoring is also examined. This study expands the investigation of numerical methods for estimating distribution parameters under incomplete data conditions, offering recommendations on selecting the most effective method depending on the sample characteristics and the level of censoring.

Keywords: Gamma distribution, Nelder-Mead simplex algorithm, Expectation-Maximization, Maximum Likelihood Estimation, Right-censoring, Proportional hazards model.

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1 Introduction

Estimating unknown parameters using statistical methods, such as the Maximum Likelihood Estimation (MLE) method, often requires the use of numerical optimization methods. This is due to the fact that analytical solutions can be complicated by the need for integration and differentiation of distribution functions, especially in the case of censored data [7, 8]. This paper examines the estimation of parameters of gamma distribution under random right censoring. Special attention is given to two cases: when the distribution of the uncensored random variable is independent of the unknown parameter and when it depends on it. This problem is of significant importance in fields such as reliability analysis and life data statistics [4, 6, 8].

To estimate the unknown parameters of gamma distribution, we use the Maximum Likelihood Estimation method combined with the Nelder-Mead and Expectation-Maximization (EM) algorithms. The EM algorithm [7] is particularly useful in conditions of censoring, as it allows for proper handling of incomplete data through iterative approximations of the full likelihood function. The Nelder-Mead algorithm [11], being a direct search method, efficiently minimizes the objective function without the need to calculate gradients, which is especially useful in cases with high-dimensional parameter spaces.

Research in parameter estimation under censoring has been actively developed over the past few decades. The main approaches include maximum likelihood methods [6, 8], Bayesian methods [2, 9], as well as various numerical optimization algorithms [7, 10, 11]. These studies highlight the importance of selecting an appropriate numerical method for parameter estimation under censoring, as well as demonstrate the advantages and limitations of various approaches, especially in the context of random censoring and proportional hazard models [4, 5]. Issues of statistical efficiency in models of random censoring have been investigated in works [1, 2]. Furthermore, in [3], the problem of parameter estimation for the gamma distribution using numerical methods is examined. This paper also discusses parameter estimation for gamma distribution in different models of random censoring using numerical optimization methods. This is confirmed by several studies demonstrating the stability of algorithms under various censoring models, which propose comparative evaluations of numerical algorithms based on fully or partially observed data. The results obtained are presented in both tabular and graphical form for better visual perception.

2 Right-Censoring models

Right censoring occurs when the observed value of a random variable (r.v.) is only known within a certain range, which is common in reliability and survival analysis problems. Let the random variable X have a distribution function $F(X, \theta)$, where θ is an unknown parameter, $\theta \in \Theta$. We observe a sample $C_n = \{(Z_i, \delta_i), 1 \leqslant i \leqslant n\}$, where $Z_i = \min(X_i, T_i)$, and $\delta_i = I(X_i \leqslant T_i)$ is the indicator of the event $\{X_i \leqslant T_i\}$, which determines whether censoring occurred. Here, interpretations for T_i are considered:
T_i - **constant** and T_i -**r.v.**.

Let X have a probability density function $f(X; \theta)$, then the joint probability density function of the r.v. $Z_i = \min(X_i, T_i)$ and $\delta_i = I(X_i \leqslant T_i)$ can be written as follows:

$$L(Z_i; \theta) = \begin{cases} f(Z_i; \theta), & \text{if} \quad \delta_i = 1, \\ 1 - F(Z_i; \theta), & \text{if} \quad \delta_i = 0. \end{cases}$$

Accordingly, the likelihood function is:

$$L(Z^n; \theta) = \prod_{i=1}^{n} \left[f(Z_i; \theta)^{\delta_i} \cdot (1 - F(Z_i; \theta))^{1-\delta_i} \right]. \tag{1}$$

Where $Z^n = ((Z_1, \delta_1), ..., Z_n, \delta_n))$. Natural logarithm function is

$$\ln L(Z^n; \theta) = \sum_{i=1}^{n} \left[\delta_i \ln f(Z_i; \theta) + (1 - \delta_i) \ln(1 - F(Z_i; \theta)) \right]. \tag{2}$$

To simplify computations, the log-likelihood function $l(Z^n; \theta)$ is often used, as it converts products into sums.

$$l(Z^n; \theta) = \ln L(Z^n; \theta). \tag{3}$$

If the likelihood function is differentiable with respect to θ, the MLE $\hat{\theta}$ satisfies the equation:

$$\left. \frac{\partial l(Z^n; \theta)}{\partial \theta} \right|_{\theta = \hat{\theta}} = 0. \tag{4}$$

3 Analytical approach with MLE method for Gamma distribution

The Gamma distribution can be parameterized with shape α and rate β and is denoted as:

$$X \sim \Gamma(\alpha, \beta) = \text{Gamma}(\alpha, \beta).$$

The corresponding probability density function (pdf) in the shape-rate parameterization for the Gamma distribution is:

$$f(x; \alpha, \beta) = \frac{x^{\alpha-1} e^{-\beta x} \beta^{\alpha}}{\Gamma(\alpha)}, \quad \text{for } x > 0, \alpha > 0, \beta > 0. \tag{5}$$

where $\Gamma(\alpha)$ is the gamma function; for all positive integers, $\Gamma(\alpha) = (\alpha - 1)!$.

The cumulative distribution function (cdf) is the regularized Gamma function:

$$F(x; \alpha, \beta) = \int_0^x f(t; \alpha, \beta) dt = \frac{\gamma(\alpha, \beta x)}{\Gamma(\alpha)}. \tag{6}$$

where $\gamma(\alpha, x)$ is the lower incomplete gamma function.

In the context of right-censored data, selecting an appropriate probability distribution is crucial for accurate parameter estimation. Common choices include the Gamma and Exponential distributions, both widely applied in survival analysis and reliability engineering .

Let's consider the Gamma distribution and the case when

$$\alpha = 1, \quad \beta = \frac{1}{\lambda},$$

for simplicity and we can define pdf and cdf:

$$\text{pdf: } f(x; 1, \frac{1}{\lambda}) = \frac{1}{\lambda} e^{-x/\lambda}.$$

$$\text{cdf: } F(x; 1, \frac{1}{\lambda}) = \int_0^x \frac{1}{\lambda} e^{-t/\lambda} dt = 1 - e^{-x/\lambda}.$$

Now, we can take the natural logarithm of the likelihood function:

$$l(X, \lambda) = \sum_{i=1}^{n} [\delta_i \ln f(x_i; \lambda) + (1 - \delta_i) \ln(1 - F(x_i; \lambda))] =$$

$$= \sum_{i=1}^{n} \left[\delta_i \ln \left(\frac{1}{\lambda} e^{-x_i/\lambda} \right) + (1 - \delta_i) \ln \left(1 - (1 - e^{-x_i/\lambda}) \right) \right] =$$

$$= \sum_{i=1}^{n} \left[-\delta_i \ln \lambda - \delta_i \frac{x_i}{\lambda} - (1 - \delta_i) \frac{x_i}{\lambda} \right]. \tag{7}$$

To maximizing the log-likelihood function we should calculate the partial derivatives of the log-likelihood with respect to the parameters:

$$\frac{\partial l(X, \lambda)}{\partial \lambda} = 0.$$

$$\frac{\partial l(X, \lambda)}{\partial \lambda} = \sum_{i=1}^{n} \left(-\delta_i \frac{1}{\lambda} + \delta_i \frac{x_i}{\lambda^2} + (1 - \delta_i) \frac{x_i}{\lambda^2} \right) = 0. \tag{8}$$

Solving for λ, we obtain:

$$\hat{\lambda}_{MLE} = \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} \delta_i}. \tag{9}$$

When $\alpha = 2, \beta = \beta$ we can find out pdf and cdf for the Gamma function:

$$\text{pdf: } f(x;2,\beta) = xe^{-\beta x}\beta^2.$$

$$\text{cdf: } F(x;2,\beta) = \int_0^x \frac{te^{-\beta t}\beta^2}{\Gamma(2)}dt = [\Gamma(\alpha) = (\alpha-1)!, \quad \Gamma(2) = 1] = \int_0^x te^{-\beta t}\beta^2 dt =$$

$$[\text{we take substitute by parts formula}],$$

$$\int udv = uv - \int vdu,$$

$$u = t, \quad dv = e^{-\beta t}dt,$$

$$du = dt, \quad v = -\frac{1}{\beta}e^{-\beta t} =$$

$$= \beta^2 t\left(-\frac{1}{\beta}e^{-\beta t}\right)\Big|_0^x + \beta^2\int_0^x \frac{1}{\beta}e^{-\beta t}dt =$$

$$= -\beta xe^{-\beta x} + \beta\left(-\frac{1}{\beta}e^{-\beta t}\right)\Big|_0^x = 1 - e^{-\beta x}(\beta x + 1). \tag{10}$$

Now, we can find the natural logarithm of the likelihood function using equation (2):

$$l(\beta|X,\delta) = \sum_{i=1}^{n}\left[\delta_i \ln f(x_i;\beta) + (1-\delta_i)\ln(1-F(x_i;\beta))\right] =$$

$$= \sum_{i=1}^{n}\left[\delta_i \ln\left(x_i e^{-\beta x_i}\beta^2\right) + (1-\delta_i)\ln(1-(1-e^{-\beta x_i}(\beta x_i+1)))\right] =$$

$$= \sum_{i=1}^{n}\left[\delta_i \ln x_i - \delta_i\beta x_i + 2\delta_i\ln\beta - \beta x_i + \delta_i\beta x_i + \ln(\beta x_i + 1) - \delta_i\ln(\beta x_i+1)\right] =$$

$$= \sum_{i=1}^{n}\left[\delta_i \ln x_i + 2\delta_i\ln\beta - \beta x_i + \ln(\beta x_i+1) - \delta_i\ln(\beta x_i+1)\right].$$

To maximizing the equation (11), we should calculate the partial derivatives of the log-likelihood with respect to the parameters:

$$\frac{\partial l(\beta|X,\delta)}{\partial\beta} = 0,$$

$$\frac{\partial l(\beta|X,\delta)}{\partial\beta} = \sum_{i=1}^{n}\left(\frac{2\delta_i}{\beta} - x_i + \frac{x_i}{\beta x_i+1} - \frac{\delta_i x_i}{\beta x_i+1}\right) = 0,$$

$$\sum_{i=1}^{n}\left(\frac{2\delta_i}{\beta} - x_i + \frac{x_i}{\beta x_i+1} - \frac{\delta_i x_i}{\beta x_i+1}\right) = 0. \tag{11}$$

After some algebra, we define $\hat{\beta}_{MLE}$:

$$\hat{\beta}_{MLE} = \frac{n\left[\sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n} \delta_i\right]}{2\sum_{i=1}^{n} \delta_i \sum_{i=1}^{n} x_i - \left(\sum_{i=1}^{n} x_i\right)^2 + \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \delta_i x_i}. \tag{12}$$

We use the estimated value $\hat{\beta}_{MLE}$ to find the $\hat{\alpha}_{MLE}$ and go back to find the cdf of the Gamma distribution when $\alpha = \alpha$, $\beta = \hat{\beta}_{MLE}$:

$$F(x; \alpha, \hat{\beta}_{MLE}) = \int_0^x \frac{t^{\alpha-1} e^{-\hat{\beta}_{MLE} t} \hat{\beta}_{MLE}^{\alpha}}{\Gamma(\alpha)} dt, \quad [\Gamma(\alpha) = (\alpha - 1)!]. \tag{13}$$

Generally, due to complexity of the $F(x; \alpha, \hat{\beta}_{MLE})$ integral function for any values of the Gamma distribution parameters α and β, it is common to employ numerical techniques or specialized software tools (including mathematical software or programming libraries like Python, MATLAB, or R) to perform the integration process and derive the cdf, rather than relying on analytical solutions.

4 Numerical methods

The EM algorithm is an iterative method used to estimate the maximum likelihood parameters when the data is incomplete or contains latent variables.

4.1 Expectation-Maximization (EM)

We proceed as follows:

1. Let $k = 0$. Give an initial estimate for θ. Call it $\hat{\theta}^{(k)}$.
2. Given observed data $Z^n = ((Z_1, \delta_1), ..., (Z_n, \delta_n))$ and assuming that $\hat{\theta}^{(k)}$ is correct for the parameter θ, find the conditional density $f(X|Z^n, \hat{\theta}^{(k)})$ for the completion variables.
3. Calculate the conditional expected log-likelihood or **"Q-function"**:

$$Q(\theta|\hat{\theta}^{(k)}) = \mathbb{E}\left[\ln L(X^n; \theta) \mid Z^n, \hat{\theta}^{(k)}\right]. \tag{14}$$

Here, the expectation is with respect to the conditional distribution of X^n given Z^n and $\hat{\theta}^{(k)}$, and thus can be written as:

$$Q(\theta|\hat{\theta}^{(k)}) = \int \ln f(Z^n, X|\theta) \cdot f(X|Z^n, \hat{\theta}^{(k)}) dX. \tag{15}$$

(The integral is high-dimensional and is taken over the space where X^n lives.)

4. Find the θ that maximizes $Q(\theta|\hat{\theta}^{(k)})$. Call this $\hat{\theta}^{(k+1)}$.

Let $k = k + 1$ and return to Step 2.

The EM Algorithm is iterated until the estimate for θ stops changing. Usually, a tolerance ε is set, and the algorithm is iterated until:

$$|\hat{\theta}^{(k+1)} - \hat{\theta}^{(k)}| < \varepsilon. \tag{16}$$

We will show that this stopping rule makes sense in the sense that once that distance is less than ε, it will remain less than ε.

4.2 Nelder-Mead Simplex Algorithm

The Nelder-Mead algorithm is a direct search method that does not require derivative information. It is often used for non-differentiable functions.

1. **Initialize the Simplex.** Define an initial simplex consisting of $m+1$ points (for m - dimensional parameter space):
$$S = \{\theta_1, \theta_2, ..., \theta_{m+1}\}. \tag{17}$$
Evaluate the objective function (negative log-likelihood):
$$L(\theta_i) = -\ln L(Z^n; \theta_i). \tag{18}$$
Sort the points such that:
$$L(\theta_1) \leqslant L(\theta_2) \leqslant ... \leqslant L(\theta_{m+1}). \tag{19}$$
2. **Compute the Centroid.** Compute the centroid of the best m points:
$$\theta_c = \frac{1}{m}\sum_{i=1}^{m} \theta_i. \tag{20}$$
3. **Reflection.** Reflect the worst point θ_{m+1} across the centroid:
$$\theta_r = \theta_c + \alpha(\theta_c - \theta_{m+1}).$$
where α is the reflection coefficient. If $L(\theta_1) \leqslant L(\theta_r) < L(\theta_m)$, accept θ_r.
4. **Expansion (If Reflection is Good).**
If $L(\theta_r) < L(\theta_1)$, compute an expansion:
$$\theta_e = \theta_c + \gamma(\theta_r - \theta_c). \tag{21}$$
where $\gamma > 1$. If $L(\theta_e) < L(\theta_r)$, accept θ_e, otherwise accept θ_r.
5. **Contraction (If Reflection is Bad).**
If $L(\theta_r) \geqslant L(\theta_m)$, perform contraction:
$$\theta_s = \theta_c + \rho(\theta_{m+1} - \theta_c) \tag{22}$$
where $0 < \rho < 1$. If $L(\theta_s) < L(\theta_{m+1})$, accept θ_s, otherwise perform a full shrink:
$$\theta_i = \theta_1 + \sigma(\theta_i - \theta_1) \tag{23}$$
where $0 < \sigma < 1$.
6. **Convergence Check.**
Check if the simplex size is small:
$$\sum_{i=1}^{m+1} ||\theta_i - \theta_c|| < \varepsilon. \tag{24}$$
If the condition is met, return the best parameter $\hat{\theta}$.

5 Numerical Results

As shown in Equation 14 above, finding MLE estimates analytically becomes challenging for the Gamma distribution when right-censored data is present. Therefore, we numerically solved the maximization step in Equation 4 using two numerical methods: the EM algorithm and the Nelder-Mead simplex algorithm. These methods were applied to both

Table 1 Estimation of unknown parameters of the Gamma distribution using two algorithms

Gamma Distribution, Initial Guess $\alpha = 2, \beta = 1$; T_i - constant

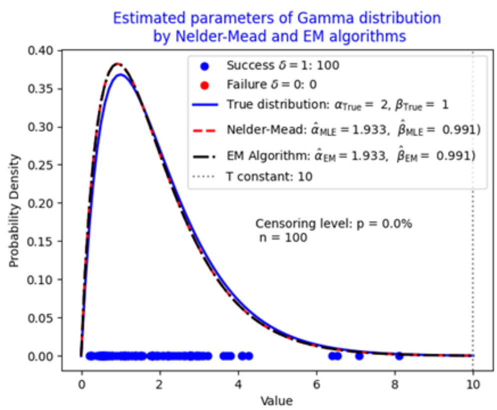
n Sample data	T_i	$p\%$	MLE by EM algorithm	Difference $\Delta_1 = \|\alpha - \hat{\alpha}_{EM}\|$ $\Delta_2 = \|\beta - \hat{\beta}_{EM}\|$	MLE by Nelder-Mead simplex algorithm	Difference $\Delta_1 = \|\alpha - \hat{\alpha}_{NM}\|$ $\Delta_2 = \|\beta - \hat{\beta}_{NM}\|$
100	5	0	$\alpha = 1.933, \beta = 0.991$	0.067, 0.009	$\alpha = 1.933, \beta = 0.991$	0.067, 0.009
100	0.7	25	$\alpha = 2.157, \beta = 0.895$	0.157, 0.105	$\alpha = 1.499, \beta = 1.634$	0.501, 0.634
100	0.35	50	$\alpha = 2.163, \beta = 0.882$	0.163, 0.118	$\alpha = 1.135, \beta = 2.645$	0.865, 1.645
500	5	0	$\alpha = 2.061, \beta = 0.994$	0.061, 0.006	$\alpha = 2.061, \beta = 0.994$	0.061, 0.006
500	0.7	25	$\alpha = 2.112, \beta = 0.904$	0.112, 0.096	$\alpha = 1.595, \beta = 1.618$	0.405, 0.618
500	0.35	50	$\alpha = 2.128, \beta = 0.899$	0.128, 0.101	$\alpha = 1.142, \beta = 2.324$	0.858, 1.324
1000	5	0	$\alpha = 2.019, \beta = 1.005$	0.019, 0.005	$\alpha = 2.019, \beta = 1.005$	0.019, 0.005
1000	0.7	25	$\alpha = 2.119, \beta = 0.913$	0.119, 0.087	$\alpha = 1.623, \beta = 1.527$	0.377, 0.527
1000	0.35	50	$\alpha = 2.142, \beta = 0.918$	0.142, 0.082	$\alpha = 1.201, \beta = 2.197$	0.799, 1.197

the **constant** and **random variable** cases of the censoring distribution, and the results for the first case are presented in Table 1 and Figures A.

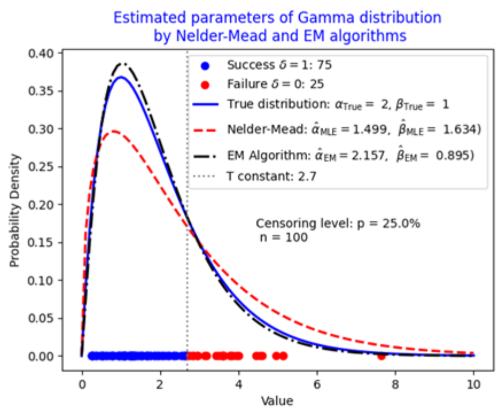
For the Gamma distribution, this table presents the estimates of both the shape α and scale β parameters. Under complete data conditions ($p = 0$), both algorithms perform similarly, as they are solving well-defined optimization problems. However, when the level of censoring increases ($p = 50$), the EM algorithm proves to be more robust, particularly in estimating β, which often exhibits greater sensitivity to data censoring.

In contrast, the Nelder-Mead algorithm struggles with higher deviations in both α and β, particularly due to its reliance on a direct likelihood optimization process that is less effective in handling censoring data points.

Figures A



a



b

Figures A: a and b show the estimated unknown parameters of the Gamma distribution by the Nelder-Mead and EM algorithms with **complete data** and **censoring data**, respectively. In both graphs, the results of the algorithms are represented by the fitted red dashed and black dashed lines, and are compared to the true blue smooth line.

Now we consider T_i as a r.v. denote it as $T_i = Y_i$. Let the d.f. $G(x)$ and probability density function $g(x)$ of the r.v. Y does not depend on the unknown parameter. For instance, $N(0,1)$, Exp(1), and $R[0,1]$. Consider the sample $C^n = \{(Z_i, \delta_i), 1 \leqslant i \leqslant n\}$, where $Z_i = \min(X_i, Y_i)$ and $\delta_i = I(X_i \leqslant Y_i)$;

Accordingly, the likelihood function is:

$$L(Z^{(n)};\theta) = \prod_{i=1}^{n} \left\{ (f(X_i;\theta)(1-G(Y_i))^{\delta_i} \cdot ((1-F(X_i;\theta)) \cdot g(Y_i))^{(1-\delta_i)} \right\} = \\ = \prod_{i=1}^{n} \left\{ f(X_i;\theta)^{\delta_i} (1-F(X_i;\theta))^{(1-\delta_i)} \right\} \cdot \prod_{i=1}^{n} \left\{ (1-G(Y_i))^{\delta_i} g(Y_i)^{(1-\delta_i)} \right\}. \tag{25}$$

Loglikelihood function is:

$$\log L(Z^n;\theta) = \sum_{i=1}^{n} [\delta_i \log f(X_i;\theta) + (1-\delta_i)\log(1-F(X_i;\theta))] + \\ + \sum_{i=1}^{n} \delta_i \log(1-G(Y_i)) + \\ + \sum_{i=1}^{n} (1-\delta_i) \log g(Y_i). \tag{26}$$

Since the distribution function of the random variable Y does not depend on the unknown parameter θ, the maximum likelihood estimation equation (4) remains the same as in the case when T is constant, after differentiation with respect to the parameters in equation (27).

In this study, we explore two estimation methods, focusing on maximizing equation (27) using numerical techniques. The estimation results are compared, and graphs are provided to illustrate their application to the Gamma distribution (see Table 2 and Figures B below).

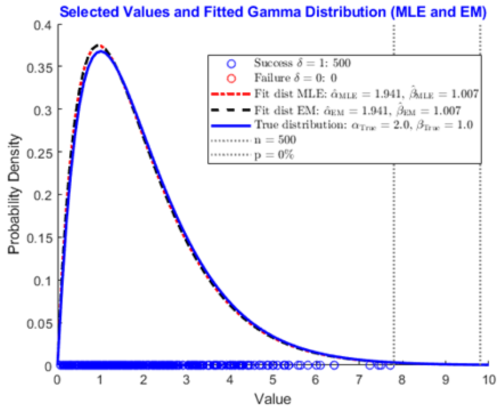
Table 2 Estimation of unknown parameters of the Gamma distribution using two algorithms

Gamma Distribution, Initial Guess $\alpha = 2, \beta = 1$**;**
$Y_i \sim \text{Exp}(1)$, $1 \leqslant i \leqslant n$

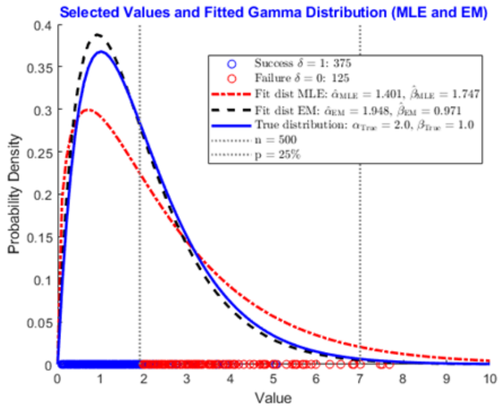
n Sample data	$p\%$	MLE by EM algorithm	Difference $\Delta_1 = \lvert\alpha - \hat{\alpha}_{EM}\rvert$ $\Delta_2 = \lvert\beta - \hat{\beta}_{EM}\rvert$	MLE by Nelder-Mead simplex algorithm	Difference $\Delta_1 = \lvert\alpha - \hat{\alpha}_{NM}\rvert$ $\Delta_2 = \lvert\beta - \hat{\beta}_{NM}\rvert$
100	0	$\alpha = 1.912, \beta = 0.923$	0.088, 0.077	$\alpha = 1.912, \beta = 0.923$	0.088, 0.077
100	25	$\alpha = 1.869, \beta = 0.817$	0.131, 0.183	$\alpha = 1.424, \beta = 1.697$	0.576, 0.697
100	50	$\alpha = 1.724, \beta = 0.809$	0.276, 0.191	$\alpha = 1.026, \beta = 2.847$	0.974, 1.847
500	0	$\alpha = 1.941, \beta = 1.007$	0.059, 0.007	$\alpha = 1.941, \beta = 1.007$	0.059, 0.007
500	25	$\alpha = 1.948, \beta = 0.971$	0.052, 0.029	$\alpha = 1.401, \beta = 1.747$	0.599, 0.747
500	50	$\alpha = 1.889, \beta = 0.902$	0.111, 0.098	$\alpha = 1.104, \beta = 2.753$	0.896, 1.753
1000	0	$\alpha = 1.959, \beta = 0.995$	0.041, 0.005	$\alpha = 1.959, \beta = 0.995$	0.041, 0.005
1000	25	$\alpha = 1.892, \beta = 0.901$	0.108, 0.099	$\alpha = 1.432, \beta = 1.962$	0.568, 0.962
1000	50	$\alpha = 1.824, \beta = 0.848$	0.176, 0.152	$\alpha = 1.116, \beta = 2.514$	0.884, 1.514

The results of the Gamma distribution show that the EM algorithm is better suited for handling random censoring. As the censoring level increases (p = 50), the differences between the true and estimated parameters grow more pronounced for the Nelder-Mead method. However, the EM algorithm effectively handles the random censoring of both the shape α and the scale β, maintaining lower deviations between censoring levels. This reflects its ability to manage the complexity introduced by treating T_i as a random variable, thereby preserving the reliability of its estimates under varying censoring conditions.

Figures B



c



d

Figures B: c and d show the estimated unknown parameters of the Gamma distribution by the Nelder-Mead and EM algorithms with **complete data** and **censoring data**, respectively. In both graphs, the results of the algorithms are represented by the fitted red dashed and black dashed lines, and are compared to the true blue smooth line.

6 Proportional Hazards Model

In survival data statistics and reliability analysis, there is often a need to estimate the parameters of distributions when censored data is present. A particularly important case is informative censoring, where the censoring random variable Y depends on the unknown parameter θ of the observed random variable X. In such situations, the proportional hazards model (PHM) is applied, which establishes a specific relationship between the distributions of X and Y [4, 5].

Assume that in some probability space, there are independent random variables X and Y, having distribution functions $F(x;\theta)$ and $G(x;\theta)$, respectively, both depending on a common parameter $\theta \in \Theta$. Observations are conducted for a sample of n independent experiments, where the sample is presented as $C^{(n)} = \{(Z_i, \delta_i), i = 1, \ldots, n\}$, where:

$$Z_i = \min(X_i, Y_i), \quad \delta_i = I(X_i \leqslant Y_i),$$

and $I(\cdot)$ is the indicator function.

In the proportional hazards model, the following relationship between the distributions of X and Y holds:

$$1 - G(x;\theta) = (1 - F(x;\theta))^{\gamma}, \quad g(x;\theta) \cdot (1 - F(x;\theta)) = \gamma f(x;\theta) \cdot (1 - G(x;\theta)), \tag{27}$$

where $\gamma > 0$ is a known constant characterizing the degree of proportionality between the hazards. For this model, the likelihood function $L(\theta)$ is based on the probabilities of observing each of the possible outcomes (Z_i, δ_i) in the sample $C^{(n)}$. The likelihood function for the entire sample $C^{(n)}$ is expressed as:

$$L(\theta) = \prod_{i=1}^{n} \left[f(Z_i;\theta) \cdot S(Z_i;\theta)^{\gamma}\right]^{\delta_i} \left[\gamma \cdot f(Z_i;\theta) \cdot S(Z_i;\theta)^{\gamma-1}\right]^{1-\delta_i}, \quad (28)$$

where $S(z;\theta) = 1 - F(z;\theta)$.

For convenience in optimization and to simplify computations, the logarithm of the likelihood function is used:

$$\log L(\theta) = \sum_{i=1}^{n} \left[\delta_i \left(\log f(Z_i;\theta) + \gamma \log S(Z_i;\theta)\right) + \right.$$

$$\left. +(1-\delta_i)\left(\log\gamma + \log f(Z_i;\theta) + (\gamma-1)\log S(Z_i;\theta)\right)\right]. \quad (29)$$

In general, finding an analytical solution to the equation (30) may be difficult or impossible, which is why numerical optimization methods are used. To find the maximum of the logarithmic likelihood function, numerical algorithms such as the Nelder-Mead and EM algorithms are applied. The results of the estimates are compared in graphical and tabular way in Gamma distribution (see Table 3 and Figures C below).

Table 3 Estimation of unknown parameters of the Gamma distribution using two algorithms in proportional hazards model

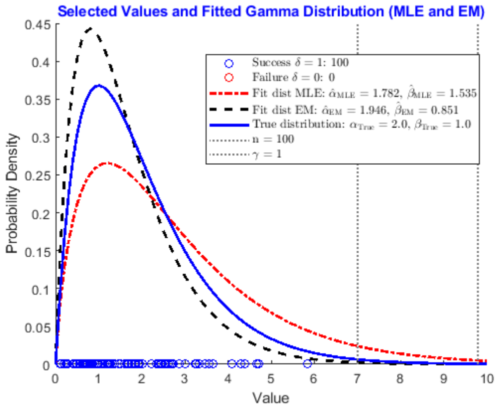
Gamma Distribution, Initial Guess $\alpha = 2, \beta = 1$;
$Y_i \sim \text{Exp}(1)$, $1 \leqslant i \leqslant n$

n Sample data	Censoring level γ	MLE by EM algorithm	Difference $\Delta_1 = \lvert\alpha - \hat{\alpha}_{EM}\rvert$ $\Delta_2 = \lvert\beta - \hat{\beta}_{EM}\rvert$	MLE by Nelder-Mead simplex algorithm	Difference $\Delta_1 = \lvert\alpha - \hat{\alpha}_{NM}\rvert$ $\Delta_2 = \lvert\beta - \hat{\beta}_{NM}\rvert$
100	1	$\alpha = 1.946, \beta = 0.851$	0.054, 0.149	$\alpha = 1.782, \beta = 1.535$	0.218, 0.535
100	0.5	$\alpha = 2.061, \beta = 0.898$	0.061, 0.102	$\alpha = 1.948, \beta = 1.288$	0.052, 0.288
100	1.5	$\alpha = 1.904, \beta = 0.829$	0.096, 0.171	$\alpha = 1.752, \beta = 1.659$	0.248, 0.659
500	1	$\alpha = 1.952, \beta = 0.863$	0.048, 0.137	$\alpha = 1.837, \beta = 1.416$	0.163, 0.416
500	0.5	$\alpha = 2.054, \beta = 0.919$	0.054, 0.081	$\alpha = 1.889, \beta = 1.236$	0.111, 0.236
500	1.5	$\alpha = 1.906, \beta = 0.958$	0.094, 0.042	$\alpha = 1.763, \beta = 1.814$	0.237, 0.814
1000	1	$\alpha = 1.964, \beta = 0.913$	0.036, 0.087	$\alpha = 1.841, \beta = 1.404$	0.159, 0.404
1000	0.5	$\alpha = 2.036, \beta = 1.012$	0.036, 0.012	$\alpha = 1.968, \beta = 1.215$	0.032, 0.215
1000	1.5	$\alpha = 1.921, \beta = 0.897$	0.079, 0.103	$\alpha = 1.798, \beta = 1.602$	0.202, 0.602

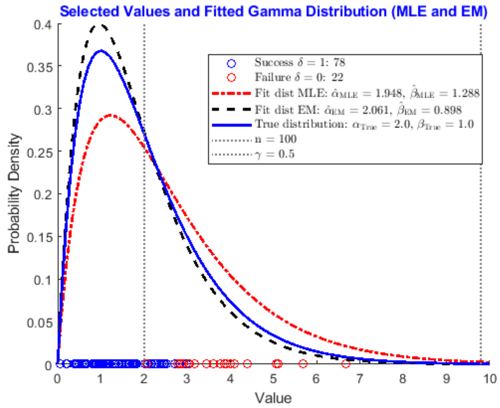
In the Proportional Hazards Model (PHM), where the censoring distribution is influenced by the unknown parameter, the findings reveal the limitations of the Nelder-Mead method when the censoring level γ exceeds 1. The EM algorithm, which iteratively refines estimates using the likelihood function, yields more precise parameter estimates, especially when the censoring variable is dependent on the parameter of interest. While the Nelder-Mead method performs reasonably well at lower levels of censoring, its accuracy declines as γ increases. This is due to the complexity introduced by the proportional hazards structure, which makes direct likelihood maximization less dependable. The EM algorithm's advantage in these situations demonstrates its superior capability in handling dependent censoring effectively.

Figures C: e and f show the estimated unknown parameter of the Exponential distribution by the Nelder-Mead and EM algorithms with **complete data** and **censoring**

Figures C



e



f

data, respectively. In both graphs, the results of the algorithms are represented by the fitted red dashed and black dashed lines, and are compared to the true blue smooth line.

7 General comparisons

Comparison of the obtained estimates when T is constant, r.v., and PHM, with the comparison presented in Table 4, showing differences for the Gamma distribution.

Table 4 Comparison of the differences in results for T_i as a constant and r.v. for Gamma Distribution in three states

Gamma Distribution, Initial Guess $\alpha = 2$, $\beta = 1$;
$Y_i \sim \text{Exp}(1)$, $1 \leqslant i \leqslant n$.

n	$p\%$	Difference EM T_i constant $\Delta_1 = \|\alpha - \hat{\alpha}_{EM}\|$ $\Delta_2 = \|\beta - \hat{\beta}_{EM}\|$	Difference N-M T_i constant $\Delta_1 = \|\alpha - \hat{\alpha}_{NM}\|$ $\Delta_2 = \|\beta - \hat{\beta}_{NM}\|$	Difference EM T_i r.v. $\Delta_1 = \|\alpha - \hat{\alpha}_{EM}\|$ $\Delta_2 = \|\beta - \hat{\beta}_{EM}\|$	Difference N-M T_i r.v. $\Delta_1 = \|\alpha - \hat{\alpha}_{NM}\|$ $\Delta_2 = \|\beta - \hat{\beta}_{NM}\|$	Censoring level γ	Difference EM PHM γ $\Delta_1 = \|\alpha - \hat{\alpha}_{EM}\|$ $\Delta_2 = \|\beta - \hat{\beta}_{EM}\|$	Difference N-M PHM γ $\Delta_1 = \|\alpha - \hat{\alpha}_{NM}\|$ $\Delta_2 = \|\beta - \hat{\beta}_{NM}\|$
100	0	0.067, 0.009	0.067, 0.009	0.088, 0.077	0.088, 0.077	1	0.054, 0.149	0.218, 0.535
100	25	0.157, 0.015	0.501, 0.634	0.131, 0.086	0.576, 0.697	0.5	0.061, 0.102	0.052, 0.288
100	50	0.163, 0.118	0.865, 1.645	0.276, 0.191	0.697, 1.847	1.5	0.096, 0.288	0.248, 0.659
500	0	0.078, 0.013	0.058, 0.017	0.065, 0.059	0.059, 0.059	1	0.038, 0.132	0.198, 0.594
500	25	0.128, 0.019	0.858, 0.011	0.111, 0.098	0.896, 0.747	0.5	0.035, 0.194	0.047, 0.286
500	50	0.161, 0.019	0.832, 1.324	0.094, 0.191	0.948, 1.154	1.5	0.094, 0.244	0.237, 0.657
1000	0	0.019, 0.012	0.019, 0.006	0.005, 0.004	0.005, 0.005	1	0.024, 0.003	0.206, 0.199
1000	25	0.137, 0.023	0.789, 0.756	0.112, 0.056	0.584, 0.884	0.5	0.061, 0.013	0.123, 0.288
1000	50	0.142, 0.082	0.799, 1.197	0.176, 0.152	0.814, 1.514	1.5	0.070, 0.002	0.062, 0.222

The results for the Gamma distribution demonstrate that when T_i is constant, both algorithms show similar accuracy, but as T_i transitions to a random variable or is incorporated within the PHM, the Nelder-Mead method exhibits significantly larger deviations, especially for higher censoring levels. In contrast, the EM algorithm provides more reliable estimates for both α and β parameters, even under complex censoring conditions. This demonstrates the EM algorithm's superior ability to adjust to random censoring and dependent censoring structures in the PHM, making it the more robust choice for parameter estimation in incomplete models.

8 Concluding Remarks

This study presents a comparative analysis of numerical methods for estimating the parameters of the gamma distribution under censoring conditions. It is established that MLE combined with the EM algorithm provides the highest estimation accuracy in random censoring models where analytical solutions are not feasible. At the same time, the Nelder-Mead method demonstrates stable results under full observation, but its accuracy significantly decreases with increasing censoring levels.

Additionally, it is shown that under dependent censoring, standard MLE methods lose efficiency, whereas the EM algorithm remains preferable. This is due to its ability to correctly handle incomplete data by maximizing the expected likelihood function.

The obtained results have practical significance for reliability analysis, biostatistics, and financial mathematics, where estimating distribution parameters in the presence of censored data plays a crucial role. However, open questions remain regarding the comparative efficiency of Bayesian estimation methods under high censoring levels, as well as the application of stochastic approximations and adaptive numerical algorithms to improve estimation accuracy. Future research is planned to explore Bayesian estimation methods under censoring conditions and to extend the analysis to other types of censoring and statistical models.

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СРАВНИТЕЛЬНЫЙ АНАЛИЗ ОЦЕНКИ НЕИЗВЕСТНЫХ ПАРАМЕТРОВ ГАММА-РАСПРЕДЕЛЕНИЯ С ЦЕНЗУРИРОВАННЫМИ СПРАВА ДАННЫМИ В НЕПОЛНЫХ СТАТИСТИЧЕСКИХ МОДЕЛЯХ

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В данной статье рассматривается задача оценивания параметров гамма-распределения в условиях цензурированных данных в неполных статистических моделях. Анализируются методы численного максимального правдоподобия, включая алгоритмы Нелдера-Мида и ожидания-максимизации (EM), применяемые для оценивания параметров распределения. Проведено сравнение точности оценок при различных уровнях цензурирования, что позволило выявить преимущества и ограничения каждого метода. Полученные результаты показывают, что алгоритм EM обеспечивает более высокую точность оценок в условиях цензурирования, в то время как метод Нелдера-Мида демонстрирует стабильные результаты при полном наблюдении. Также рассматривается влияние модели пропорциональных рисков на оценивание параметров в условиях зависимого цензурирования. Статья расширяет исследование численных методов оценивания параметров распределений в условиях неполных данных, предлагая рекомендации по выбору наиболее эффективного метода в зависимости от характеристик выборки и уровня цензурирования.

Ключевые слова: гамма-распределение, симплекс-алгоритм Нелдера-Мида, метод ожидания-максимизации, метод максимального правдоподобия, правоцензурирование, модель пропорциональных рисков.

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